Polynomial Chaos Expansion approach to Financial Modeling *AMaMeF 2015*

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Overview

1. General properties of Polynomial Chaos Expansion
   - Theoretical framework of PCE-approximation
   - Convergence properties and general features
   - Numerical computation of PCE
2. PCE-approximation of geometric Brownian motion
   - From the governing SDE to PCE-approximation
   - PCE-approximation vs analytical solution: statistics, quantiles and distributions
3. PCE-approximation of Vasicek interest rate model
4. PCE-approximation of CIR interest rate model
5. Concluding remarks

Univariate decomposition: settings

The univariate Polynomial Chaos Expansion (PCE) is a technique that recovers a given finite second order random variable $Y \in L^2(\Omega, \Sigma, \mathbb{P})$ by means of a linear combination of orthogonal polynomials $\{\Psi_i(\xi)\}_{i\in\mathbb{N}}$. This set is called the Generalized Polynomial Chaos (gPC) basis.

Definition

Let us consider a real-valued scalar random variable Y, defined on $(\Omega, \Sigma, \mathbb{P})$. It belongs to the Hilbert space $L^2(\Omega, \Sigma, \mathbb{P})$ whenever

$$\mathbb{E}\left[Y^2\right] = \int_\Omega \left(Y(\omega)\right)^2 d\mathbb{P}(\omega) < +\infty$$

Moreover this Hilbert space is endowed with a scalar product

$$\langle X, Y\rangle_{\mathbb{P}} = \mathbb{E}\left[XY\right] = \int_\Omega X(\omega)Y(\omega)d\mathbb{P}(\omega)$$

Univariate decomposition: basic random variables

The entry of each polynomial that belongs to gPC basis is a real-valued scalar random variable ξ, defined on $(\Omega, \Sigma, \mathbb{P})$, thus

$$\xi : \Omega \longrightarrow D \subseteq \mathbb{R}$$

Moreover

- ξ it has finite moments of all orders
- ξ it is a continuous random variable, i.e. its cumulative density function $F(x) = \mathbb{P}(\xi \leq x)$ is continuous

Examples: $\mathcal{N}(0, 1/2)$, $\mathcal{U}(-1, 1)$ and Exp(1).

Remark

In further discussion we set $\xi \sim \mathcal{N}(0, 1/2)$ since the differential of the Wiener process dW_t, which appears in governing SDE for interest rate models, is normally distributed.

Univariate decomposition: random variables involved

Since the aim is to decompose a $Y \in L^2(\Omega, \Sigma, \mathbb{P})$ into a basis $\{\Psi_i(\xi)\}_{i\in\mathbb{N}}$, Y has to be "compatible" with the random variable ξ.

This is a matter of matching σ-algebras, indeed it is enough to prove that Y is $\sigma(\xi)$-measurable (where $\sigma(\xi)$ is the σ-algebra generated by ξ).

Theorem

Fixed two measurable functions $Y : \Omega \to \mathbb{R}$ and $\xi : \Omega \to D \subseteq \mathbb{R}$, Y is $\sigma(\xi)$-measurable if there exists some Borel measurable map $g : D \to \mathbb{R}$ such that

$$Y = g \circ \xi$$

Thus the PCE-approximation is reasonable only for each random variable $Y \in L^2(\Omega, \sigma(\xi), \mathbb{P})$

Univariate decomposition: definition of the gPC

The polynomials of the gPC are orthogonal with respect to the measure $\mathbb{P}$, thus for two natural indexes $i \neq j$

$$\delta_{ij} \langle \Psi_i, \Psi_j \rangle_{\mathbb{P}} = \int_\Omega \Psi_i(\xi(\omega))\Psi_j(\xi(\omega))\, d\mathbb{P}(\omega) = \int_D \Psi_i(x)\Psi_j(x)w(x)dx$$

Indeed the random variable $\xi: \Omega \to D$ induces a measure on dF_ξ on the image space $(D, \mathcal{B}(D))$, in particular $dF_\xi = w(x)dx$ where $w(x)$ is the probability density function of ξ.
Thus in our setting $\xi \sim \mathcal{N}(0, 1/2)$, then $D = \mathbb{R}$ and $\{\Psi_i(\xi)\}_{i\in\mathbb{N}}$ reads as the class of **Hermite polynomials**.

$$\begin{cases} \Psi_0(x) & = 1 \\ \Psi_1(x) & = 2x \\ \Psi_2(x) & = 4x^2 - 2 \\ \Psi_3(x) & = 8x^3 - 12x \\ & \vdots \end{cases}$$

Univariate Polynomial Chaos Expansion

Once the basic random variable ξ is set, the associated gPC $\{\Psi_i(\xi)\}_{i\in\mathbb{N}}$ is detected.

Then the univariate Polynomial Chaos Expansion of any $Y \in L^2(\Omega, \sigma(\xi), \mathbb{P})$ is

$$Y = g(\xi) = \sum_{i=0}^{+\infty} c_i \Psi_i(\xi) \,, \qquad c_i = \frac{\mathbb{E}\left[Y\Psi_i\right]}{\mathbb{E}\left[\Psi_i^2\right]} = \frac{\langle Y, \Psi_i \rangle_{\mathbb{P}}}{\|\Psi_i\|_{\mathbb{P}}^2} \,,\ \forall\ i \in \mathbb{N} \,,$$

hence the truncation of the PCE at degree N is

$$Y^{(N)} = \sum_{i=0}^{N} c_i \Psi_i(\xi)$$

Remark

In what follows Y reads as the equity model, resp. Interest rate model at fixed time $T > 0$.

Properties

- **Spectral convergence** in mean square sense ($\mathbb{E}\left[(\cdot)^2\right] = \|\cdot\|_{\mathbb{P}}^2$) of the truncated PCE $Y^{(N)}$ to Y

$$\left\| Y - Y^{(N)} \right\|_{\mathbb{P}}^2 \longrightarrow 0 \, , \; N \to +\infty \, ,$$

since $\{\Psi_i(x)\}_{i\in\mathbb{N}}$ is a maximal Hilbert basis in $L^2(D, w(x)dx)$.

- **Computation of statistics** of Y by means of coefficients of the PCE

$$\mathbb{E}\left[Y\right] = \int_\Omega Y(\omega)dP(\omega) = \int_\Omega g(\xi(\omega))\Psi_0(\xi(\omega))dP(\omega) = c_0$$

$$\mathrm{Var}\left[Y\right] = \mathbb{E}\left[Y^2\right] - c_0^2 = \sum_{i=0}^{+\infty} c_i^2 \|\Psi_i\|_{\mathbb{P}}^2 - c_0^2 = \sum_{i=1}^{+\infty} c_i^2 \|\Psi_i\|_{\mathbb{P}}^2$$

- **Sampling of truncated PCE** $\{Y_l^{(N)}\}_{l=1}^L$: compute a sampling of size L of *basic random variable* $\{\xi_l\}_{l=1}^L$, then

$$Y_l^{(N)} = \sum_{i=0}^{p} c_i \Psi_i(\xi_l) \, , l = 1, \dots, L \, ,$$

Non-Intrusive Spectral Projection (NISP)

Non-Intrusive Spectral Projection (NISP) method computes numerically the coefficients of PCE of Y. Its **main steps** are

- define Y as the output of a process $\mathcal{M}(\xi, \Theta)$. Θ gathers the *characterizing parameters* (e.g. of the interest rate model)
- Set the truncated PCE for the output, namely

$$Y^{(N)} = \sum_{i=0}^{N} c_i \Psi_i(\xi) \qquad c_i = \frac{\mathbb{E}\left[\mathcal{M}\Psi_i\right]}{\mathbb{E}\left[\Psi_i^2\right]} = \frac{\langle \mathcal{M}, \Psi_i \rangle_{\mathbb{P}}}{\|\Psi_i\|_{\mathbb{P}}^2}$$

Remark: $Y^{(N)}$ converges to $\mathcal{M}(\xi, \Theta)$ as $N \to +\infty$.

- Compute the coefficients

$$\langle \mathcal{M}, \Psi_i \rangle_{\mathbb{P}} = \int_D \underbrace{\mathcal{M}(x)\Psi_i(x)}_{g_i(x)} w(x)dx \approx \sum_{j=1}^{N_Q} g_i(x_j) w_j$$

by Gaussian quadrature formulas, quadrature nodes $\{x_j\}_{j=1}^{N_Q}$ being particular realizations of the basic random variable ξ and we set $N_Q = N$ for simplicity.

Geometric Brownian Motion

Detect the PCE-approximation of the Geometric Brownian Motion (gBm) at time $T > 0$ using its governing equation, namely

$$dS_t = rS_t d_t + \sigma S_t dW_t \ ,$$

where $r, \sigma \in \mathbb{R}^+$, gathered into $\Theta = (r, \sigma)$, and S_0 is the starting value.

Required steps:

- **Define a process** $\mathcal{M}(\xi, \Theta)$ representing S_T at a fixed end time T, by the property of Brownian motion: $W_T = \sqrt{2T}\xi$, with $\xi \sim \mathcal{N}(0, 1/2)$
- **Compute the PCE-coefficients** by NISP to approximate $\mathcal{M}(\xi, \Theta)$
- **Detects statistics** of S_T and *post-process analysis*: distribution, quantiles,...

Geometric Brownian Motion: definition of $\mathcal{M}(\xi, \Theta)$

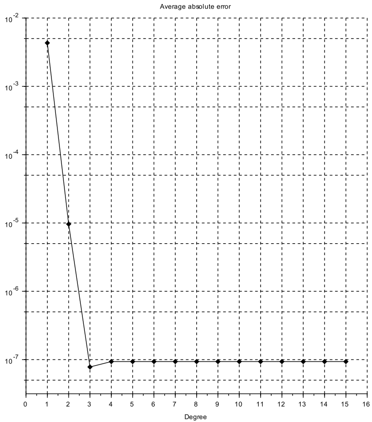
Doss Theory gives useful theoretical results to express S_T as a function of W_T, employing only information available form the governing SDE.

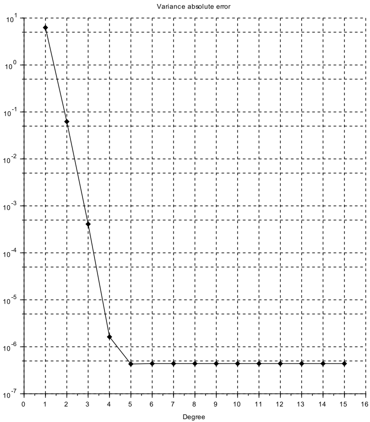
1. Let $H(x, y)$ be a real-valued scalar function satisfying the following ODE for each fixed $x \in \mathbb{R}$
$$\begin{cases} \frac{\partial H(x,y)}{\partial y} = \sigma H(x, y) \\ H(x, 0) = x \end{cases},$$
in such case $H(x, y) = xe^{\sigma y}$.

2. Integrate, from 0 up to time T, the following ODE, path-wise defined
$$\begin{cases} \dot{D}(\omega) = \exp\left\{-\sigma W_t(\omega)\right\}\Big(rH(D(\omega), W_t(\omega)) - \frac{\sigma^2}{2}H(D(\omega), W_t(\omega))\Big)\ ,\ t > 0 \\ D(\omega) = S_0\ ,\ t = 0 \end{cases},$$
since $W_t = \sqrt{2t}\xi$, D_T can be expressed as a function of the basic random variable ξ. **Numerical solver** $\rightarrow$ *adaptive Runge-Kutta* method of order 4

3. The geometric Brownian motion at time T can be expressed as
$$S_T = H(W_T, D_T) = \sqrt{2T}\xi e^{\sigma D_T} = \mathcal{M}(\xi, \Theta)$$

We compare the accuracy of PCE approximation in detecting the mean and the variance, for different set of volatility $\sigma = 15\%$, $S_0 = 100$, $r = 3\%$ and $T = 1$.





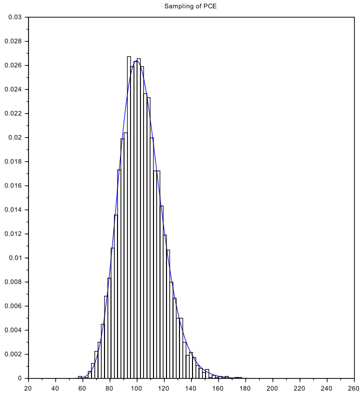
Geometric Brownian Motion: computation results

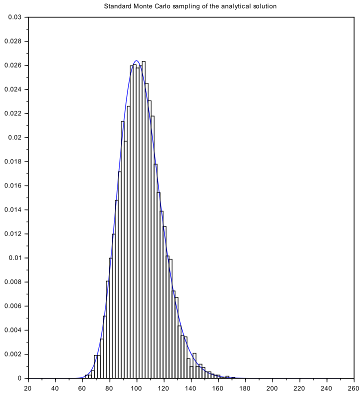
Degree	Average Error	Var Error	Average relative error	Var relative error
1	4.3211e-03	6.2663e+00	4.1934e-05	2.5934e-02
2	9.6288e-06	6.2400e-02	9.3443e-08	2.5825e-04
3	7.8038e-08	4.1041e-04	7.5732e-10	1.6986e-06
4	9.3644e-08	1.6273e-06	9.0877e-10	6.7351e-09
5	9.3664e-08	4.3078e-07	9.0896e-10	1.7829e-09
6	9.3664e-08	4.3922e-07	9.0896e-10	1.8178e-09
7	9.3664e-08	4.3925e-07	9.0896e-10	1.8179e-09
8	9.3664e-08	4.3925e-07	9.0896e-10	1.8179e-09
9	9.3664e-08	4.3925e-07	9.0896e-10	1.8179e-09
10	9.3664e-08	4.3925e-07	9.0896e-10	1.8179e-09
11	9.3664e-08	4.3925e-07	9.0896e-10	1.8179e-09
12	9.3663e-08	4.3925e-07	9.0895e-10	1.8179e-09
13	9.3664e-08	4.3925e-07	9.0895e-10	1.8179e-09
14	9.3664e-08	4.3925e-07	9.0895e-10	1.8179e-09
15	9.3664e-08	4.3925e-07	9.0895e-10	1.8179e-09

Table : Absolute error of the average and the variance of PCE approximation of gBm at time $T = 1$ (whose parameters are $r = 3\%$, $\sigma = 15\%$ and starting value $S_0 = 100$)

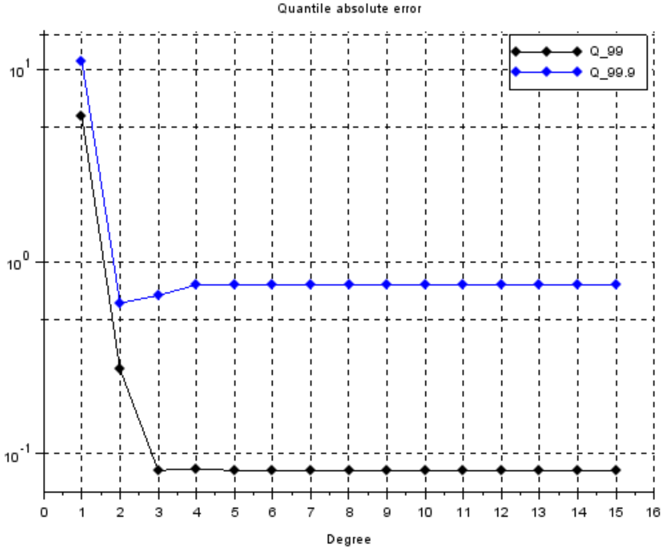
Geometric Brownian Motion: computation results

Analytical probability density function of gBM compared with Monte Carlo sampling of the PCE-approximation (left), and Analytical probability density function and Monte Carlo sampling of the analytical gBM





Geometric Brownian Motion: 99 % and 99.9 % Quantile



Vasicek Interest rate model

Detect the PCE-approximation of the Vasicek Interest rate model at time $T > 0$ using its governing equation, namely

$$dR_t = (\alpha - \beta R_T)d_t + \sigma dW_t \ ,$$

where $\alpha, \beta, \sigma \in \mathbb{R}^+$, gathered into $\Theta = (\alpha, \beta, \sigma)$, and R_0 represents the starting value.

Required steps:

- **Define a process** $\mathcal{M}(\xi, \Theta)$ representing S_T at a fixed end time T, moreover $W_T = \sqrt{2T}\xi$, since $\xi \sim \mathcal{N}(0, 1/2)$
- **Compute the PCE-coefficients** by means of NISP approach, approximating $\mathcal{M}(\xi, \Theta)$
- **Detects statistics** of R_T and its **distribution**.

Vasicek Interest rate model: definition of $\mathcal{M}(\xi, \Theta)$

Again exploiting Doss Theory we express R_T as a function of W_T using only information coming from the governing SDE.

1. Let $H(x, y)$ be a real-valued scalar function satisfying the following ODE for each fixed $x \in \mathbb{R}$

$$\begin{cases} \frac{\partial H(x,y)}{\partial y} = \sigma \\ H(x,0) = x \end{cases}$$

   thus $H(x, y) = \sigma y + x$.

2. Let us integrate path-wise up to time T the following ODE

$$\begin{cases} \dot{D}(\omega) = \alpha - \beta H(D(\omega), W_t(\omega)) & t > 0 \\ D(\omega) = R_0 & t = 0 \end{cases},$$

   since $W_t = \sqrt{2t}\xi$, D_T can be expressed as a function of the basic random variable ξ, and we use **RK4**.

3. The Vasicek Interest rate model at time T can be expressed as

$$R_T = H(W_T, D_T) = \sigma D_T + W_T = \mathcal{M}(\xi, \Theta)$$

Vasicek Interest rate model: computation results

The benchmark is the analytical solution of Vasicek Interest rate model, namely

$$R_T = e^{-\beta T} R_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta T}\right) + \sigma e^{-\beta T} \int_0^T e^{\beta s} dW(s) \;, \tag{1}$$

therefore R_T is normally distributed with mean, resp. variance, given by

$$\mathbb{E}\left[R_T\right] = R_0 e^{-\beta T} + \frac{\alpha}{\beta}(1 - e^{-\beta T}) \;,$$
$$\mathrm{Var}\left[R_T\right] = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta T}) \,,$$

then we compare such values with the statistics obtained by PCE for, e.g., $\sigma = 15\%$, $R_0 = 110$, $\alpha = 0.1$, $\beta = 0.2$ and $T = 1$.

Vasicek Interest rate model: computation results

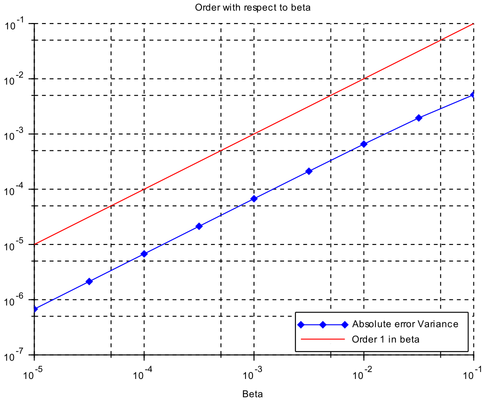
Degree	Average Error	Var Error	Average relative error	Var relative error
0	8.2289e-08	1.8544e-02	9.1279e-10	1.0000e+00
1	8.2303e-08	1.2490e-03	9.1295e-10	6.7349e-02
2	8.2312e-08	1.2490e-03	9.1305e-10	6.7349e-02
3	8.2348e-08	1.2490e-03	9.1345e-10	6.7349e-02
4	8.2321e-08	1.2490e-03	9.1315e-10	6.7349e-02
5	8.2349e-08	1.2490e-03	9.1346e-10	6.7349e-02
6	8.2328e-08	1.2490e-03	9.1322e-10	6.7349e-02
7	8.2308e-08	1.2490e-03	9.1300e-10	6.7349e-02
8	8.2327e-08	1.2490e-03	9.1321e-10	6.7349e-02
9	8.2300e-08	1.2490e-03	9.1291e-10	6.7349e-02
10	8.2298e-08	1.2490e-03	9.1289e-10	6.7349e-02
11	8.2352e-08	1.2490e-03	9.1349e-10	6.7349e-02
12	8.2311e-08	1.2490e-03	9.1303e-10	6.7349e-02
13	8.2364e-08	1.2490e-03	9.1362e-10	6.7349e-02
14	8.2321e-08	1.2490e-03	9.1315e-10	6.7349e-02
15	8.2357e-08	1.2490e-03	9.1355e-10	6.7349e-02

Table : Absolute error of the average and the variance of $R_T^{(N)}$ at time $T = 1$, with respect to the Vasicek model parameters $\alpha = 0.1$, $\beta = 0.2$, $\sigma = 15\%$.

Vasicek Interest rate model: computation results

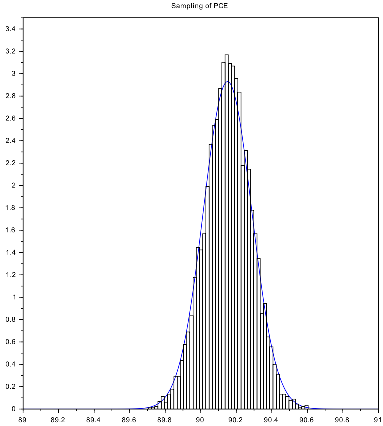
By computing a first order expansion in β, centered at $\beta = 0$, of the variance of the Vasicek process, resp. of $\mathcal{M}(\xi, \Theta)$, by Doss Theory, we obtain an **absolute error linear proportional to** β and since $Y^{(N)} \to \mathcal{M}$, the PCE-approximation error of the variance shares the same feature.

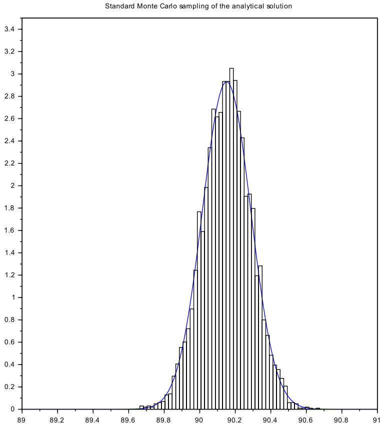
The computations are made for $\beta = 10^{[-5:0.5:-1]}$.



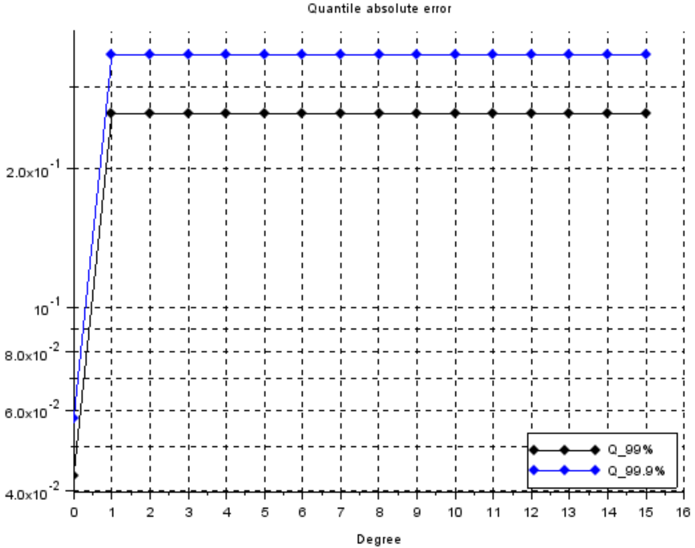
Vasicek Interest rate model: computation results

Left plot: sampling of size 5000 of the PCE-approximation.
Right plot: sampling of size 5000 of the Vasicek interest rate model





Vasicek Interest rate model: computation results



CIR Interest rate model

Approximate the CIR interest rate model at time $T > 0$ by means of PCE, using its governing equation, namely

$$dR_t = (\alpha - \beta R_t)\, dt + \sigma \sqrt{R_t} dW_t \ ,$$

where $\alpha, \beta, \sigma \in \mathbb{R}^+$, gathered into $\Theta = (\alpha, \beta, \sigma)$, and We work with a transformed process: $Y_t = \sqrt{R_t}$ that satisfies the following SDE

$$dY_t = \frac{1}{2Y_t} \left(\alpha - \beta Y_t^2 - \frac{1}{4}\sigma^2 \right) dt + \frac{1}{2}\sigma dW_t \ ,$$

in order to simplify Doss Theory application, since the volatility function is constant,

Remark

The parameters α, β, σ are chosen in order to get a non-centered chi-square random variable with $q = 3$ degree of freedom, namely $\alpha = \frac{3}{4}\sigma^2$.

CIR Interest rate model

Again by Doss Theory, we define

1. $H(x, y)$ as a real-valued scalar function satisfying the following ODE for each fixed $x \in \mathbb{R}$
$$\begin{cases} \frac{\partial H(x,y)}{\partial y} = \frac{\sigma}{2} \\ H(x, 0) = x \end{cases},$$
thus $H(x, y) = \frac{\sigma}{2} y + x$.

2. Let us integrate, form 0 up to time T the following ODE, path-wise defined
$$\begin{cases} \dot{D}(\omega) = \frac{1}{2H(D(\omega), W_t(\omega))} \left(\alpha - \beta \left(H(D(\omega), W_t(\omega)) \right)^2 - \frac{1}{4}\sigma^2 \right) & , t > 0 \\ D(\omega) = R_0 & , t = 0 \end{cases},$$
$W_t = \sqrt{2t}\xi \rightarrow D_T$ is a function of the basic random variable ξ, again we use **RK4**

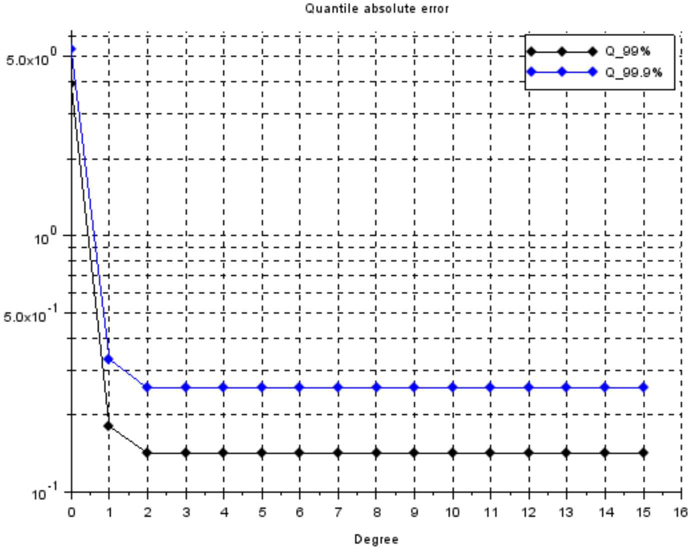
3. The CIR interest rate model at time T can be expressed as
$$R_T = Y_T^2 = (H(W_T, D_T))^2 = \left(\frac{\sigma}{2} D_T + W_T \right)^2 = \mathcal{M}(\xi, \Theta)$$

CIR Interest rate model: computation results $\sigma = 15\%$

Degree	Average Error	Var Error	Average relative error	Var relative error
0	9.2721e-03	2.7353e+00	1.2572e-04	1.0000e+00
1	6.2470e-04	1.8449e-01	8.4691e-06	6.7449e-02
2	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
3	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
4	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
5	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
6	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
7	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
8	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
9	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
10	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
11	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
12	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
13	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
14	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02
15	6.2470e-04	1.8434e-01	8.4691e-06	6.7395e-02

Table : Absolute error of the average and the variance of $R_T^{(p)}$ at time $T = 2$, with respect to the CIR parameters $\alpha = 0.005625$, $\beta = 0.2$, $\sigma = 15\%$.

CIR Interest rate model: computation results $\sigma = 15\%$



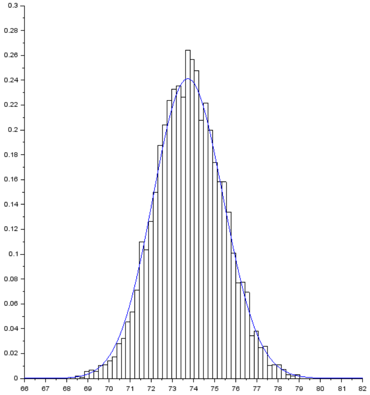
CIR Interest rate model: computation results $\sigma = 15\%$

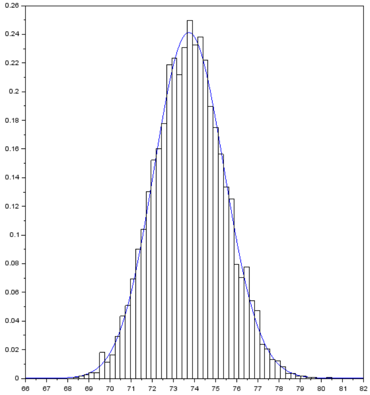
Degree of PCE	$\epsilon_{99\%}$	$\epsilon_{99.9\%}$
0	3.9106e+00	5.3240e+00
1	1.8048e-01	3.3009e-01
2	1.4218e-01	2.5449e-01
3	1.4218e-01	2.5449e-01
4	1.4218e-01	2.5449e-01
5	1.4218e-01	2.5449e-01
6	1.4218e-01	2.5449e-01
7	1.4218e-01	2.5449e-01
8	1.4218e-01	2.5449e-01
9	1.4218e-01	2.5449e-01
10	1.4218e-01	2.5449e-01
11	1.4218e-01	2.5449e-01
12	1.4218e-01	2.5449e-01
13	1.4218e-01	2.5449e-01
14	1.4218e-01	2.5449e-01
15	1.4218e-01	2.5449e-01

Table : Absolute errors of the two quantiles $\hat{Q}_{99\%}$ and $\hat{Q}_{99.9\%}$ of the PCE approximation of the CIR Interest rate model with $T = 2$, $\alpha = 0.046875$, $\beta = 0.2$, $\sigma = 0.25$ and $R_0 = 110$.

CIR Interest rate model: computation results $\sigma = 15\%$

Left plot: sampling of size 5000 of the PCE-approximation.
Right plot: sampling of size 5000 of the CIR interest rate model



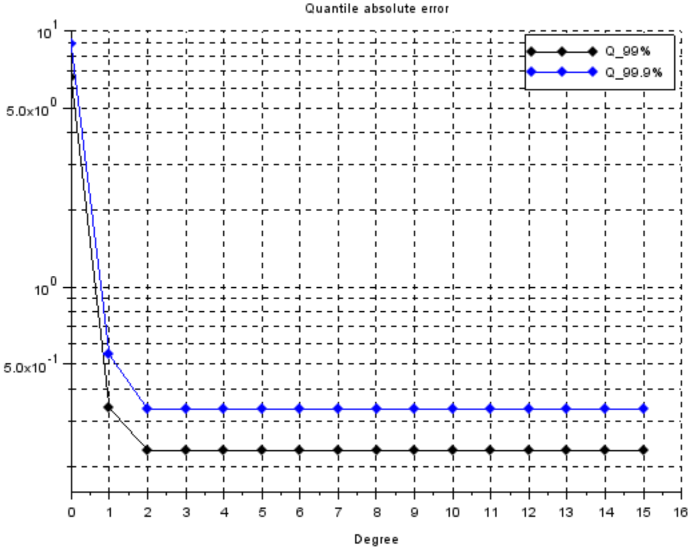


CIR Interest rate model: computation results $\sigma = 25\%$

Degree	Average Error	Var Error	Average relative error	Var relative error
0	2.5756e-02	7.6005e+00	3.4906e-04	1.0000e+00
1	1.7373e-03	5.1400e-01	2.3537e-05	6.7627e-02
2	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
3	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
4	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
5	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
6	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
7	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
8	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
9	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
10	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
11	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
12	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
13	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
14	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02
15	1.7373e-03	5.1284e-01	2.3537e-05	6.7475e-02

Table : Absolute error of the average and the variance of $R_T^{(p)}$ at time $T = 2$, with respect to the CIR parameters $\alpha = 0.046875$, $\beta = 0.2$, $\sigma = 25\%$.

CIR Interest rate model: computation results $\sigma = 25\%$



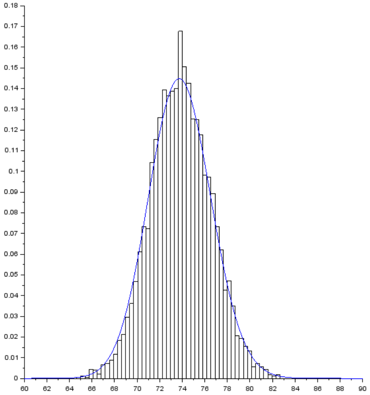
CIR Interest rate model: computation results $\sigma = 25\%$

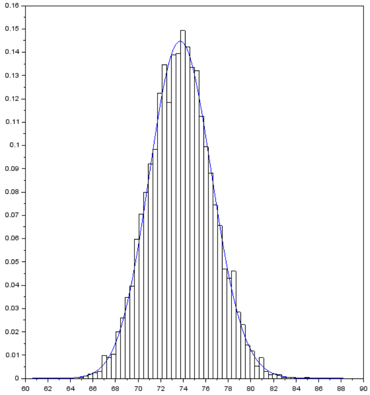
Degree of PCE	$\epsilon_{99\%}$	$\epsilon_{99.9\%}$
0	6.5644e+00	8.8778e+00
1	3.3747e-01	5.4443e-01
2	2.3108e-01	3.3443e-01
3	2.3108e-01	3.3443e-01
4	2.3108e-01	3.3443e-01
5	2.3108e-01	3.3443e-01
6	2.3108e-01	3.3443e-01
7	2.3108e-01	3.3443e-01
8	2.3108e-01	3.3443e-01
9	2.3108e-01	3.3443e-01
10	2.3108e-01	3.3443e-01
11	2.3108e-01	3.3443e-01
12	2.3108e-01	3.3443e-01
13	2.3108e-01	3.3443e-01
14	2.3108e-01	3.3443e-01
15	2.3108e-01	3.3443e-01

Table : Absolute errors of the two quantiles $\hat{Q}_{99\%}$ and $\hat{Q}_{99.9\%}$ of the PCE approximation of the CIR Interest rate model with $T = 2$, $\alpha = 0.046875$, $\beta = 0.2$, $\sigma = 0.25$ and $R_0 = 110$.

CIR Interest rate model: computation results $\sigma = 25\%$

Left plot: sampling of size 5000 of the PCE-approximation.
Right plot: sampling of size 5000 of the CIR interest rate model



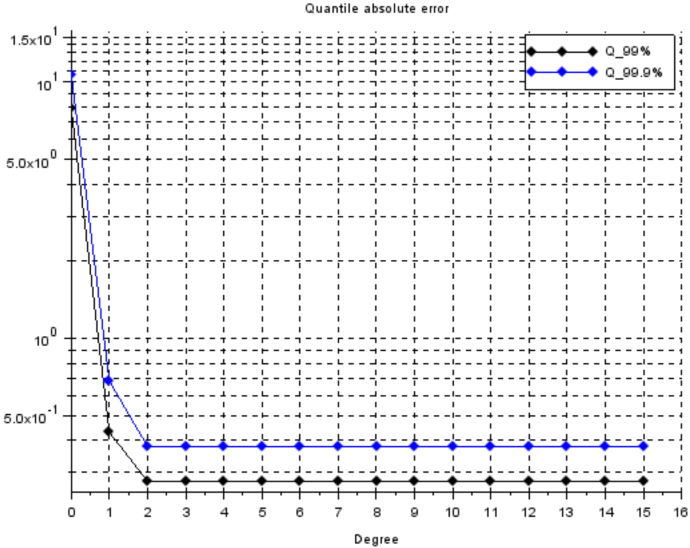


CIR Interest rate model: computation results $\sigma = 30\%$

Degree	Average Error	Var Error	Average relative error	Var relative error
0	3.7089e-02	1.0947e+01	5.0250e-04	1.0000e+00
1	2.5035e-03	7.4166e-01	3.3903e-05	6.7748e-02
2	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
3	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
4	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
5	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
6	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
7	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
8	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
9	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
10	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
11	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
12	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
13	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
14	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02
15	2.5035e-03	7.3927e-01	3.3903e-05	6.7530e-02

Table : Absolute error of the average and the variance of $R_T^{(p)}$ at time $T = 2$, with respect to the CIR parameters $\alpha = 0.0675$, $\beta = 0.2$, $\sigma = 30\%$.

CIR Interest rate model: computation results $\sigma = 30\%$



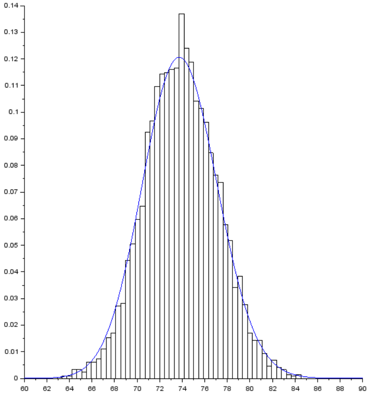
CIR Interest rate model: computation results $\sigma = 30\%$

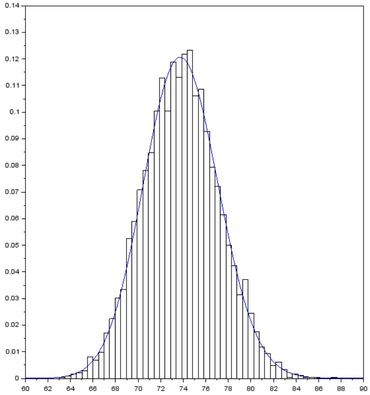
Degree of PCE	$\epsilon_{99\%}$	$\epsilon_{99.9\%}$
0	7.9085e+00	1.0687e+01
1	4.3007e-01	6.8095e-01
2	2.7688e-01	3.7857e-01
3	2.7688e-01	3.7857e-01
4	2.7688e-01	3.7857e-01
5	2.7688e-01	3.7857e-01
6	2.7688e-01	3.7857e-01
7	2.7688e-01	3.7857e-01
8	2.7688e-01	3.7857e-01
9	2.7688e-01	3.7857e-01
10	2.7688e-01	3.7857e-01
11	2.7688e-01	3.7857e-01
12	2.7688e-01	3.7857e-01
13	2.7688e-01	3.7857e-01
14	2.7688e-01	3.7857e-01
15	2.7688e-01	3.7857e-01

Table : Absolute errors of the two quantiles $\hat{Q}_{99\%}$ and $\hat{Q}_{99.9\%}$ of the PCE approximation of the CIR Interest rate model at time $T = 2$. The parameters are set as $\alpha = 0.0675$, $\beta = 0.2$, $\sigma = 0.30$ and $R_0 = 110$.

CIR Interest rate model: computation results $\sigma = 30\%$

Left plot: sampling of size 5000 of the PCE-approximation.
Right plot: sampling of size 5000 of the CIR interest rate model





Efficiency vs Accuracy

As **last comparison** we compute the error of the average and variance of the CIR model and relative computational time costs for: PCE, Monte Carlo (MC) and Quasi Monte Carlo (QMC) methods

- MC standard error is computed as well as the QMC absolute error w.r.t analytical CIR average and variance, using $M = \{2^8, \ldots, 2^{15}\}$ independent solutions of the CIR-models at time T
- The absolute error of PCE is computed and compared
- The computational time costs are normalized with respect to the highest data among MC, QMC and PCE $\rightarrow$ we consider **relative computational time costs** which are more effective than the absolute ones

Efficiency vs Accuracy

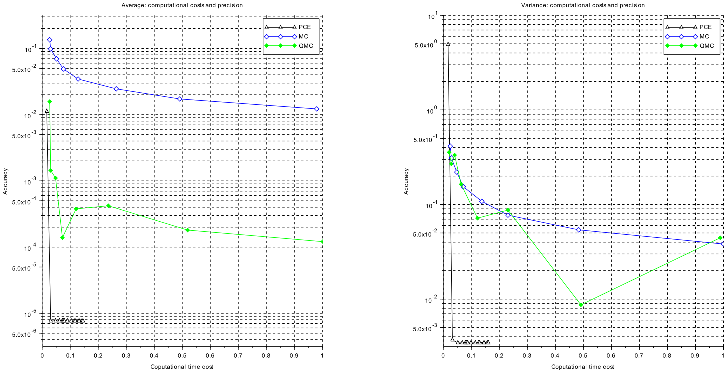


Figure : Semilogy scale plot comparing accuracy and computational time costs for the average (left) and variance (right) of the CIR-model at $T = 2$ by PCE, MC and QMC (number of realizations $M = \{2^8, \dots, 2^{15}\}$), with $\alpha = 0.005625$, $\beta = 0.002$ and $\sigma = 15\%$.

Efficiency vs Accuracy

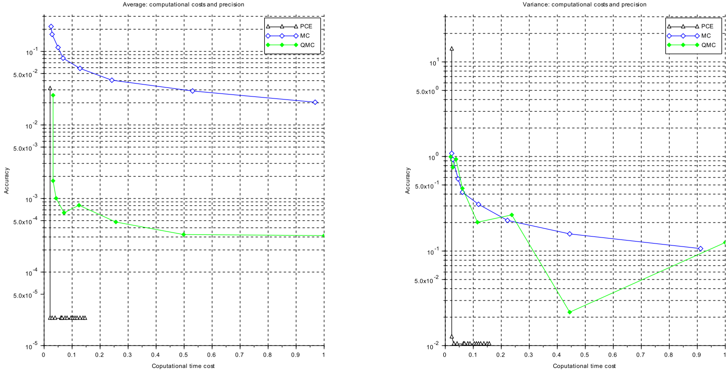


Figure : Semilogy scale plot comparing accuracy and computational time costs for the average (left) and variance (right) of the CIR-model at $T = 2$ by PCE, MC and QMC ($M = \{2^8, \dots, 2^{15}\}$), with $\alpha = 0.046875$, $\beta = 0.002$ and $\sigma = 25\%$.

Efficiency vs Accuracy

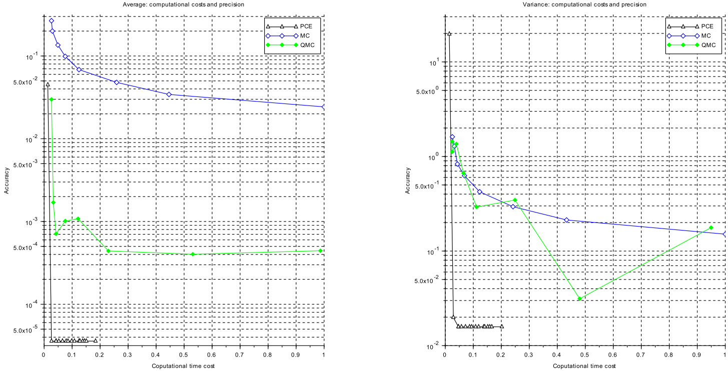


Figure : Semilogy scale plot comparing accuracy and computational time costs for the average (left) and variance (right) of the CIR-model at $T = 2$ by PCE, MC and QMC ($M = \{2^8, \ldots, 2^{15}\}$), with $\alpha = 0.0675$, $\beta = 0.002$ and $\sigma = 30\%$.

Conclusions

- PCE approach and NISP methods have been successfully used to obtain efficient, NOT time consuming and robust numerical approximations for some relevant financial models
- Doss theory is required to link SDE and PCE computations, using information available from the governing SDE
- The comparison, in terms of accuracy and efficiency, between PCE and standard methods as MC and QMC, points out
    - 80% **lower computational** time cost of PCE w.r.t. MC and QMC
    - **Better accuracy**: 10^{-5} for PCE, resp 10^{-4} for MC-QMC for average, and 10^{-2} for PCE, resp 10^{-1} for MC-QMC for variance error

References and further developments

- LDP, M.Bonollo, G.Pellegrini, Polynomial Chaos Expansion approach to interest rate models, Journal of Probability and Statistics, 2015
- LDP, M.Bonollo, G.Pellegrini, A computational spectral approach to interest rate models, SSRN, July 2015

- Application of PCE approximation approach to stochastic volatility models, e.g., **CEV, SABR, Chen**, etc.


THANKS
FOR YOUR ATTENTION
AND
...ANY QUESTION?