Contingent claim evaluation in the setting of stochastic mortality and interest rates with dependence

Griselda Deelstra

Université Libre de Bruxelles

Joint work with

Martino Grasselli (Universita di Padova and De Vinci Finance Lab, Paris la Defense)

& Christopher Van Weverberg (ULB)

7th General AMaMeF and Swissquote Conference
7-10 September, 2015
EPFL, Lausanne

Plan

Plan

1. Introduction
2. General affine framework and change of measure
3. Some insurance products in the general framework
4. Some particular models
5. Numerical illustrations
6. Conclusions

Introduction

- A large number of life insurance products such as annuities include interest rate and mortality risks.
- Nowadays, it is widely admitted that mortality intensities behave in a stochastic way (see e.g. Milevsky and Promislow [2001], Dahl [2004], and Biffis [2005]). Theoretically, both interest and mortality rates are positive.
- Even when stochastic mortality models showed up during the 90's, the actuarial community made the assumption that mortality risk is independent of interest risk. The assumption that interest risk and mortality risk are independent may seem acceptable in the short term. However, in the long term, it seems intuitive that demographic changes can affect the economy. Several papers show that interest risk and mortality risk are not necessarily independent (see e.g. Favero et al. [2011], Maurer [2014], Dacorogna and Cadena [2015], Nicolini [2004]).
- Jalen and Mamon [2009] and Liu et al. [2014] studied the dependence between the mortality and the interest rates in a Gaussian framework: increasing linear correlations imply in this model increasing prices for Guaranteed Annuity Options (GAOs).

Goal

- We investigate the consequences of a dependence structure between mortality risk and interest rate risk on the pricing of insurance contingent claims.
- We consider a general affine framework like in Keller-Ressel and Mayerhofer [2015]. In particular, we look at the following models:
  - the multi-CIR model
  - the Wishart model.
- Wishart processes have been first defined by Bru [1991] and are introduced in finance by Gouriéroux and Sufana [2003, 2011]. They represent a matrix extension of the square-root model that allows for a non trivial correlation between the diagonal terms which are by definition positive (see e.g. Cuchiero et al. [2011] for a complete characterization).
- We study the impact of the dependence on the prices of some insurance contracts such as indexed annuities and Guaranteed Annuity Options (GAOs).

Plan

Model

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where the probability $\mathbb{Q}$ is a risk-neutral probability measure.

Let

- $\mathcal{R}_t$ be the filtration generated by the interest rate process
- $\mathcal{M}_t$ the filtration generated by the mortality process
- $\mathcal{F}_t := \mathcal{R}_t \vee \mathcal{M}_t$ the sigma-algebra generated by $\mathcal{R}_t \cup \mathcal{M}_t$
- $\tau_M(x)$ the random variable corresponding to the future lifetime of an individual aged x at time 0

We will denote by $\mu(x, x+t)$ the force of mortality of an individual at time t with age $x+t$. We assume that the dynamics of the force of mortality $\mu(x, x+t)$ are given by

$$\mu(x, x+t) = \bar{\mu}^x + \langle M^x, X_t \rangle, \tag{1}$$

and the dynamics of interest rate r_t by

$$r_t = \bar{r} + \langle R, X_t \rangle \tag{2}$$

with X being either a classical affine process with state spaces $\mathbb{R}_+^m \times \mathbb{R}^n$ or an (affine) Wishart process on the state space S_d^+ (the cone of positive semidefinite symmetric $d \times d$ matrices).

Survival benefit

The probability of survival up to time $T \geq t$ on the set $\{\tau_M(x) > t\}$ is given by

$$\mathbb{Q}(\tau_M(x) > T|\mathcal{M}_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu(x,x+s)ds}|\mathcal{M}_t\right].$$

In the following, if there is no confusion about the age x, we will denote $\mu(x, x+t)$ by μ_t and we will omit the superscript or argument x in the notations of $\bar{\mu}$, M and τ_M.

We are interested in calculating the value at time t of an insurance contingent claim paying a single benefit C_T upon survival of the insured at time T which we denote by $SB_t(C_T; T)$ as in Biffis [2005].

Using the risk-neutral pricing approach, this basic insurance contract has the following value at time t

$$\begin{aligned} SB_t(C_T; T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} 1_{\{\tau_M > T\}} C_T|\mathcal{F}_t\right] \\ &= 1_{\{\tau_M > t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s+\mu_s)ds} C_T|\mathcal{F}_t\right] \\ &= 1_{\{\tau_M > t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds} C_T|\mathcal{F}_t\right]. \end{aligned} \quad (3)$$

Change of probability measure

We will define the probability measure $\mathbb{Q}^{T,\mu}$ with the Radon-Nikodym derivative of $\mathbb{Q}^{T,\mu}$ with respect to $\mathbb{Q}$ as (see also Liu et al. [2014])

$$\frac{d\mathbb{Q}^{T,\mu}}{d\mathbb{Q}} := \zeta_T = \frac{e^{-\int_0^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}\right]},$$

and we define for $t \leq T$

$$\zeta_t^T = \mathbb{E}^{\mathbb{Q}}[\zeta_T|\mathcal{F}_t]. \quad (4)$$

Therefore, using Bayes' rule, for any $\mathcal{F}_T$-measurable random variable C_T:

$$SB_t(C_T; T) = 1_{\{\tau_M > t\}}\tilde{P}(t,T)\mathbb{E}^{\mathbb{Q}^{T,\mu}}\left[C_T|\mathcal{F}_t\right]. \quad (5)$$

with

$$\tilde{P}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\langle R+M, X_s\rangle ds}\right].$$

$\tilde{P}(t,T)$ denotes the price at time t of a *survival zero-coupon bond* (SZCB hereafter) with maturity T (for an insured of age x at time 0).

Plan

Life annuity in the unified approach

A pre-numerando (i.e. payments are made at the beginning of period) life annuity starting at time T is an insurance product that pays out a monetary unit at each date $T, T+1, T+2, \ldots$ until the death of the insured. Therefore, the price at time T of a pre-numerando life annuity starting at time T with yearly payments of one unit for a person aged x at time 0 is given by

$$
\begin{aligned}
\ddot{a}_x(T) &= \sum_{j=0}^{\omega-(x+T)} \tilde{P}(T, T+j) \\
&= \sum_{j=0}^{\omega-(x+T)} \mathbb{E}_T^{\mathbb{Q}} \left[e^{-\int_T^{T+j} (\bar{r}+\bar{\mu}) + \langle (R+M), X_s \rangle ds} \right] \\
&= \sum_{j=0}^{\omega-(x+T)} e^{-(\bar{r}+\bar{\mu})(j)} e^{-\Phi_{(0,v)}(j,R+M) - \langle \Psi_{(0,v)}(j,R+M), X_T \rangle}
\end{aligned}
$$

where $\Phi_{(0,v)}(j, R+M)$ and $\Psi_{(0,v)}(j, R+M)$ are solutions to generalized Riccati equations as in e.g. Keller-Ressel and Mayerhofer [2015] and where ω is the largest possible survival age.

Indexed annuity in the unified approach

We now consider a (pre-numerando) T_1-years deferred life annuity which turns out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured. We denote this T_1-years deferred indexed annuity by $SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1)$. Therefore

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} SB_0(1+\gamma r_h; h) \quad (6)$$

$$= \sum_{h=T_1}^{\omega-x-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h (r_s+\mu_s)ds}(1+\gamma r_h)\right]. \quad (7)$$

We provide two different approaches for evaluating this indexed annuity product in the setting of unified affine approach:

- a change of measure approach
- the Fourier method or Duffie et al. [2000] method

Proposition

The present value of a T_1-years deferred life annuity which pays out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\left\langle R, \mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h]\right\rangle\right) \quad (8)$$

$$= \sum_{h=T_1}^{\omega-x-1} \Big((1+\gamma\bar{r})\tilde{P}(0,h) + \gamma e^{-(\bar{r}+\bar{\mu})h} L^0_\nu(0,h,-(R+M),0,\nu R)\Big) \quad (9)$$

where L^0_ν denotes the derivative wrt $\nu \in \mathbb{R}$ at $\nu = 0$ of the following function

$$L(t,T,\theta_1,\theta_2,\nu\theta_3) = \mathbb{E}^{\mathbb{Q}}_t\left[e^{\int_t^T \langle\theta_1, X_u\rangle du + \langle(\theta_2+\nu\theta_3), X_T\rangle}\right], \quad (10)$$

with $(t,T,\theta_1,\theta_2,\nu\theta_3) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}^{3d}$ for which the transform is well-defined.

Proof

i) Change of measure approach: If we use the measure $\mathbb{Q}^{T,\mu}$ associated with the survival bond as numéraire, then we can express the indexed annuity as follows

$$\begin{aligned} SB_0(\ddot{a}_{x+T_1}(1+\gamma r);\, T_1) &= \sum_{h=T_1}^{\omega-x-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h (r_s+\mu_s)ds}(1+\gamma r_h)\right] \\ &= \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\mathbb{E}^{\mathbb{Q}^{h,\mu}}\left[1+\gamma r_h\right] \\ &= \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\left\langle R, \mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h]\right\rangle\right) \end{aligned}$$

where we used the equation (2) in the last equality.

ii) Duffie et al. [2000] method:
Thanks to the particular structure of our payoff we are able to apply the method of Duffie et al. [2000]. If we denote by L the function

$$L(t,T,\theta_1,\theta_2,\nu\theta_3) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \langle\theta_1,X_u\rangle du + \langle(\theta_2+\nu\theta_3),X_T\rangle}\right], \tag{11}$$

then this moment generating funcion is well-known. By computing the derivative of (11) with respect to ν and evaluating this derivative at $\nu = 0$, we get

$$\partial_\nu L(t,T,\theta_1,\theta_2,\nu\theta_3)\big|_{\nu=0} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \langle\theta_1,X_u\rangle du + \langle\theta_2,X_T\rangle)}\langle\theta_3,X_T\rangle\right].$$

Simple calculations lead to the following equality

$$\begin{aligned} SB_0(\ddot{a}_{x+T_1}(1+\gamma r);T_1) &= \sum_{h=T_1}^{\omega-x-1} e^{-(\bar{r}+\bar{\mu})h}\left((1+\gamma\bar{r})\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h\langle R+M,X_s\rangle ds}\right]\right. \\ &\quad \left. + \gamma\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h\langle R+M,X_s\rangle ds}\langle R,X_h\rangle)\right]\right). \end{aligned}$$

We notice that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h \langle R+M, X_s\rangle ds}\right] = e^{(\bar{r}+\bar{\mu})h}\tilde{P}(0,h)$$

and that, by adopting the notation of Chiarella et al. [2014]

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h \langle R+M, X_s\rangle ds}\langle R, X_h\rangle\right] &= \partial_\nu L(0,h,-(R+M),0,\nu R)\big|_{\nu=0} \\ &= L_\nu^0(0,h,-(R+M),0,\nu R). \qquad (12)\end{aligned}$$

Therefore, the indexed annuity has the following pricing formula

$$\begin{aligned}SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) &= \sum_{h=T_1}^{\omega-x-1}\Big((1+\gamma\bar{r})\tilde{P}(0,h) \\ &\quad + \gamma e^{-(\bar{r}+\bar{\mu})h} L_\nu^0(0,h,-(R+M),0,\nu R)\Big).\end{aligned}$$

Guaranteed Annuity Options (GAO) in the unified approach

We consider a GAO giving to the policyholder the right to choose at time T between an annual payment of g where g is a fixed rate called the guaranteed annuity rate or a cash payment equal to the capital 1 (see e.g. Pelsser [2003], Liu et al. [2013, 2014] and Zhu and Bauer [2011]).
At time T the value of the GAO is given by

$$\begin{aligned} V(T) &= \max(g\ddot{a}_x(T), 1) \\ &= 1 + g\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right). \end{aligned}$$

Applying the risk neutral valuation procedure, we can write the value of the optional part of a GAO entered by an x-year policyholder at time $t = 0$ as

$$C(0,x,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T (r_s+\mu_s)ds} g\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right)\right]; \tag{13}$$

and when using the probability measure $\mathbb{Q}^{T,\mu}$ defined in (4), by

$$C(0,x,T) = g\tilde{P}(0,T)\mathbb{E}^{\mathbb{Q}^{T,\mu}}\left[\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right)\right]. \tag{14}$$

$\longrightarrow$ Monte-Carlo simulations since similar to basket options.

Plan

Gaussian model of Liu et al. (2014)

The dynamics of the force of mortality μ_t of an individual at time t with age $x+t$ are given by

$$d\mu_t = c\mu_t dt + \xi dZ_t \tag{15}$$

and the dynamics of interest rate r_t by

$$dr_t = a(b - r_t)dt + \sigma dW_t^1 \tag{16}$$

with $Z_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$ where W_t^1 and W_t^2 are independent standard Brownian motions.

The authors show numerically that increasing linear correlations imply in this model increasing prices for Guaranteed Annuity Options (GAOs)

Multidimensional CIR processes

In this subsection, we model X by an n-dimensional affine process whose independent components evolve according to the CIR risk neutral dynamics

$$dX_{it} = k_i(\theta_i - X_{it})dt + \sigma_i\sqrt{X_{it}}dW^{\mathbb{Q}}_{it}, \quad i = 1, .., n. \tag{17}$$

The price of the SZCB $\tilde{P}(t, T)$ can be easily derived in this framework:

$$\begin{aligned}\tilde{P}(t,T) &= \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T(\bar{r}+\bar{\mu})+\langle(R+M),X_s\rangle ds}\right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)}\prod_{i=1}^n \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T(R_i+M_i)X_{is}ds}\right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)}\prod_{i=1}^n e^{-\phi_i(T-t,R_i+M_i)-\psi_i(T-t,R_i+M_i)X_{it}} \end{aligned} \tag{18}$$

where ϕ_i and ψ_i are solution of the Riccati equations (Duffie et al. [2000]).

Indeed, ϕ_i and ψ_i are the following solutions of the Riccati equations (Duffie et al. [2000]).

$$\psi_i(\tau, u_i) = \frac{2u_i}{\zeta(u_i)+k_i} - \frac{4u_i\zeta(u_i)}{\zeta(u_i)+k_i}\frac{1}{(\zeta(u_i)+k_i)\exp[\zeta(u_i)\tau]+\zeta(u_i)-k_i}, \quad (19)$$

$$\begin{aligned}\phi_i(\tau, u_i) &= -\frac{k_i\theta_i}{\sigma_i^2}[\zeta(u_i)+k_i]\tau + \frac{2k_i\theta_i}{\sigma_i^2}\log[(\zeta(u_i)+k_i)\exp[\zeta(u_i)\tau]+\zeta(u_i)-k_i] \\ &\quad -\frac{2k_i\theta_i}{\sigma_i^2}\log(2\zeta(u_i)), \quad (20)\end{aligned}$$

where $\zeta(u_i) = \sqrt{k_i^2 + 2u_i\sigma_i^2}$.

Indexed annuity in the multi-CIR case

Proposition (Change of measure approach)

In the multidimensional CIR model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured can be written as follows:

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\sum_{i=0}^{n} R^i \mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_{ih}]\right)$$

where the expectation of X_{ih} under the measure $\mathbb{Q}^{T,\mu}$ is given by

$$\mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_{ih}] = \left(x_i e^{\frac{\sigma_i^2}{k_i\theta_i u_i}\phi_i(h,u_i)} + k_i\theta_i\int_0^h \exp\left(\frac{\sigma_i^2}{k_i\theta_i u_i}\phi_i(h-s,u_i)+k_i s\right)ds\right) \times \exp\left(-k_i h\right).$$

Proposition (Fourier approach)

In the multidimensional CIR model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by

$$\begin{aligned} SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) &= \sum_{h=T_1}^{\omega-x-1} \Big((1+\gamma\bar{r})\tilde{P}(0,h) \\ &\quad + \gamma e^{-(\bar{r}+\bar{\mu})h} L^0_\nu(0,h,-(R+M),0,\nu R) \Big) \end{aligned}$$

where

$$L^0_\nu(t,T,\theta_1,\theta_2,\nu\theta_3) = \left[\sum_{i=1}^{n} \partial_\nu \widetilde{L}_i(t,T,\theta_{i1},\theta_{i2},\nu\theta_{i3}) \prod_{\substack{j=1 \\ j\neq i}}^{n} \widetilde{L}_j(t,T,\theta_{j1},\theta_{j2},\nu\theta_{j3}) \right] \Bigg|_{\nu=0},$$

with

$$\widetilde{L}_k(t,T,\theta_{k1},\theta_{k2},\nu\theta_{k3}) = \mathbb{E}^{\mathbb{Q}}_t \left[e^{\int_t^T \theta_{k1} X_{ku} du + (\theta_{k2}+\nu\theta_{k3}) X_{kT}} \right].$$

The Wishart Case

In this section we assume that the affine process $(X_t)_{t\geq 0}$ is a d-dimensional Wishart process. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ and a $d \times d$ matrix Brownian motion W (i.e. a matrix whose entries are independent Brownian motions), the Wishart process X_t (without jumps) is defined as the solution of the $d \times d$-dimensional stochastic differential equation

$$dX_t \quad = \quad (\beta Q^\top Q + HX_t + X_t H^\top)dt + \sqrt{X_t}dW_t Q + Q^\top dW_t^T \sqrt{X_t}, \quad t \geq 0,$$

where $X_0 = x \in S_d^+$, $\beta \geq d-1$, $H \in M_d$ (the set of real $d \times d$ matrices) and $Q \in GL_d$ (the set of invertible real $d \times d$ matrices). In the case where $\beta \geq d+1$, the process takes values in S_d^{++}, i.e. the interior of the cone of positive semidefinite symmetric $d \times d$ matrices denoted by S_d^+ (see e.g. Cuchiero et al. [2011] and Da Fonseca et al. [2013]).

The price of a SZCB in the Wishart case can be derived as follows

$$\begin{aligned}\tilde{P}(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\mathrm{Tr}((R+M)X_s)ds}\right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)}e^{-\phi(T-t,R+M)-\mathrm{Tr}[\psi(T-t,R+M)X_t]}, \qquad (21)\end{aligned}$$

where ϕ and ψ solve the following system of ODE's ($\tau = T-t$)

$$\begin{cases}\frac{\partial\phi}{\partial\tau} = Tr[\beta Q^\top Q\psi(\tau, R+M)], \\ \phi(0, R+M) = 0, \\ \frac{\partial\psi}{\partial\tau} = \psi(\tau, R+M)H + H^\top\psi(\tau, R+M) - 2\psi(\tau, R+M)Q^\top Q\psi(\tau, R+M) + R + M, \\ \psi(0, R+M) = 0.\end{cases}$$

As proposed in Grasselli and Tebaldi [2008] and Da Fonseca et al. [2008], matrix Riccati equations can be linearized:

$$\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau)\end{pmatrix} = \exp\left(\tau\begin{pmatrix} H & 2Q^\top Q \\ R+M & -H^\top\end{pmatrix}\right)$$

It turns out that

$$\begin{aligned}\psi(\tau, R+M) &= A_{22}^{-1}(\tau)A_{21}(\tau) \\ \phi(\tau, R+M) &= \frac{\beta}{2}\left(\log(\det(A_{22}(\tau))) + \tau Tr[H^\top]\right).\end{aligned}$$

We first derive the dynamics of the Wishart process under the measure $\mathbb{Q}^{T,\mu}$. To do that, we find the dynamics of the SZCB price $\tilde{P}(t,T)$ that can be found by applying the Ito's lemma to the expression (21):

$$\begin{aligned}\frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} &= (\bar{r}+\bar{\mu}-\mathrm{Tr}[(R+M)X_t])dt - \mathrm{Tr}[\psi(\tau,R+M)\sqrt{X_t}dW_tQ] \\ &\quad - \mathrm{Tr}[\psi(\tau,R+M)Q^\top(dW_t)^\top\sqrt{X_t}].\end{aligned}$$

Girsanov's theorem gives the link between the Brownian motions under $\mathbb{Q}^{T,\mu}$ and $\mathbb{Q}$:

$$dW_t^{\mathbb{Q}^{T,\mu}} = dW_t^{\mathbb{Q}} + \sqrt{X_t}\psi(T-t,R+M)Q^\top dt.$$

As a consequence, the dynamics of X_t under $\mathbb{Q}^{T,\mu}$ are given by

$$\begin{aligned}dX_t &= \beta Q^\top Q dt + \left(H - Q^\top Q\psi(\tau,R+M)\right)X_t dt + X_t\left(H^\top - \psi(\tau,R+M)Q^\top Q\right) \\ &\quad + \sqrt{X_t}\left(dW_t^{\mathbb{Q}^{T,\mu}}\right)Q + Q^\top\left(dW_t^{\mathbb{Q}^{T,\mu}}\right)^\top\sqrt{X_t}.\end{aligned}$$

Using the distribution of a Wishart process with time-varying linear drift (see Kang and Kang [2013]) leads to a first expression of the indexed annuity.

Proposition (Change of measure approach)

In the Wishart model, the present value of a T_1-years deferred life annuity which pays out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r);\, T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma n\,\mathrm{Tr}[RV(0)]+\gamma\,\mathrm{Tr}\left[R\tilde{\Psi}(0)^\top x\tilde{\Psi}(0)\right]\right),$$

where $V(t)$ and $\tilde{\Psi}(t)$ are solutions of the following system of ordinary differential equations

$$\begin{aligned}\frac{d}{dt}\tilde{\Psi}(t) &= -\widetilde{H}(h-t,R+M)^\top\tilde{\Psi}(t),\\ \frac{d}{dt}V(t) &= -\tilde{\Psi}(t)^\top Q^\top Q\tilde{\Psi}(t),\end{aligned}$$

with $\widetilde{H}(h-t,R+M) = H - Q^\top Q\psi(h-t,R+M)$ and terminal conditions $\tilde{\Psi}(T) = I_d$ and $V(T)=0$.

Proposition (Fourier approach)

In the Wishart model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \dots$ upon survival of the insured is given by

$$SB_0(\ddot{a}_{x+t}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \Big((1+\gamma\bar{r})\tilde{P}(0,h) + \gamma e^{-(\bar{r}+\bar{\mu})h} \left(\mathrm{Tr}(a^0_\nu(h)X_0) + c^0_\nu(h)\right) e^{\mathrm{Tr}(a^0(h)X_0)+c^0(h)} \Big)$$

where (see e.g. Chiarella et al. [2014])

$$\begin{cases} a^0(h) = A_{22}(h)^{-1}A_{21}(h) \\ c^0(h) = -\frac{1}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2 \log(A_{22}(h))\right] - \frac{h}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2 H^\top\right] \\ a^0_\nu(h) = -(A_{22}(h))^{-1}RA_{12}(h)a^0(h) + (A_{22}(h))^{-1}RA_{11}(h) \\ c^0_\nu(h) = -\frac{1}{2}\,\mathrm{Tr}\left(\beta D_{\log,\,A_{22}(h)}(RA_{12}(h))\right) \end{cases}$$

with

$$\begin{pmatrix} A_{11}(h) & A_{12}(h) \\ A_{21}(h) & A_{22}(h) \end{pmatrix} = \exp\left(h \begin{pmatrix} H & 2Q^\top Q \\ R+M & -H^\top \end{pmatrix} \right),$$

and where $D_{\log,\,A}(E)$ represents the Fréchet derivative of the logarithm function computed for the matrix A in the direction $E \in M_n$.

Plan

Multidimensional CIR processes

We consider a 3-dimensional affine positive process, having independent components $X_t = (X_{1t}, X_{2t}, X_{3t})$ ruled by the dynamics

$$dX_{it} = k_i(\theta_i - X_{it})dt + \sigma_i \sqrt{X_{it}} dW_{it}$$

under $\mathbb{Q}$.

We assume that the interest rate process $(r_t)_{t \geq 0}$ and the mortality process $(\mu_t)_{t \geq 0}$ are described by

$$r_t = \bar{r} + X_{1t} + X_{2t}, \quad \mu_t = \bar{\mu} + m_2 X_{2t} + m_3 X_{3t},$$

with $\bar{r}$, $\bar{\mu}$, m_2 and m_3 constants.

In our illustration the coefficient m_2 is fixed in the experiments and m_3 is chosen such that

$$\mathbb{E}^{\mathbb{Q}}[\mu_T] = C_x(T), \tag{23}$$

meaning that the expectation of the mortality is fixed to a level $C_x(T)$ corresponding to the mortality rate, predicted by Gompertz-Makeham model, at age $x + T$ for an individual aged x at time 0.

The linear pairwise correlation between $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$, denoted by ρ_t, is given by

$$\rho_t = \frac{m_2\sigma_2^2 X_{2t}}{\sqrt{\sigma_1^2 X_{1t} + \sigma_2^2 X_{2t}}\sqrt{m_2^2\sigma_2^2 X_{2t} + m_3^2\sigma_3^2 X_{3t}}}. \tag{24}$$

The parameters of the insurance products are given in the following table. We choose $\bar{r} = -0.12332$ and $\bar{\mu} = 0$. The expected value in (23) is fixed to the level $C_{50}(15) = 0.014$.

Product	Parameters	
GAO	$g = 0.111$	$T = 15$
Indexed annuity	$\gamma = 0.06$	$T_1 = 15$

Table: Parameter values of the insurance contracts.

CIR process	Parameters			
X_1	$k_1 = 0.3731$	$\theta_1 = 0.074484$	$\sigma_1 = 0.0452$	Initial value : 0.0510234
X_2	$k_2 = 0.011$	$\theta_2 = 0.245455$	$\sigma_2 = 0.0368$	Initial value : 0.0890707
X_3	$k_3 = 0.01$	$\theta_3 = 0.0013$	$\sigma_3 = 0.0015$	Initial value : 0.0004

Table: Parameter values of the 3-dimensional CIR process.

Sensitivity study with respect to the correlation between mortality and interest rates in the multi-CIR model

m_2	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.1	-0.7196	0.2682815 (0.0010126)	0.2674552 (0.0006967)	5.8269507
-0.01	-0.4128	0.3105420 (0.0012352)	0.3098179 (0.0008260)	6.1072984
-0.001	-0.0516	0.3141705 (0.0012639)	0.3138940 (0.0008408)	6.1387679
0.001	0.0520	0.3161186 (0.0012735)	0.3161355 (0.0008454)	6.1458521
0.01	0.4355	0.3193589 (0.0012892)	0.3202536 (0.0008618)	6.1781468
0.1	0.7310	0.3796493 (0.0016278)	0.3787803 (0.0010536)	6.5415269

Table: Fair values for the GAO and the indexed annuity.

- Increasing linear correlation implies increasing prices (like in the Gaussian framework).

Wishart model

We assume that X follows a Wishart process. We recall that the mortality process $(\mu_t)_{t \geq 0}$ and the interest rate process $(r_t)_{t \geq 0}$ are modeled by

$$r_t = \bar{r} + \mathrm{Tr}(RX_t), \qquad \mu_t = \bar{\mu} + \mathrm{Tr}(MX_t), \qquad t \geq 0. \tag{25}$$

In the following, we will make the particular choice of R and M :

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{26}$$

In order to treat the most simply possible case. For this choice, the stochastic correlation between $(r_t)_{t \geq 0}$ and $(\mu_t)_{t \geq 0}$ is given by

$$\rho_t = \frac{(Q_{11}Q_{12} + Q_{22}Q_{21})X_t^{12}}{\sqrt{(Q_{11}^2 + Q_{21}^2)X_t^{11}(Q_{22}^2 + Q_{12}^2)X_t^{22}}}. \tag{27}$$

Example 1

In this experiment, we consider the following Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & -0.0006 \\ -0.06 & 0.006 \end{pmatrix},$$

$$X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}, \quad \beta = 3, \quad \bar{r} = 4, \quad \bar{\mu} = 0.$$

X_0^{12}	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.002	0.4894936	0.2448621 (0.0003981)	0.2451137 (0.0002435)	5.7801950
-0.0015	0.3671202	0.2437137 (0.0004092)	0.2443471 (0.0002408)	5.7729164
-0.0005	0.1223734	0.2436714 (0.0004018)	0.2437706 (0.0002430)	5.7583871
0	0	0.2431196 (0.0004078)	0.2435689 (0.0002410)	5.7511364
0.0005	-0.1223734	0.2424844 (0.0004001)	0.2429534 (0.0002398)	5.7438950
0.0015	-0.3671202	0.2412104 (0.0004056)	0.2420545 (0.0002440)	5.7294398
0.002	-0.4894936	0.2411214 (0.0004041)	0.2417495 (0.0002440)	5.7222261

Table: Fair values for the GAO and the indexed annuity.

- Increasing linear correlation implies increasing prices.

Example 2

In this second experiment, we consider the following Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & 0.0006 \\ 0.06 & 0.006 \end{pmatrix},$$

$$X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}, \quad \beta = 3, \quad \bar{r} = 4, \quad \bar{\mu} = 0.$$

X_0^{12}	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.002	-0.4894936	0.1994176 (0.0005877)	0.1993275 (0.0003667)	5.2104471
-0.0015	-0.3671202	0.1990714 (0.0005945)	0.1987619 (0.0003767)	5.2045963
-0.0005	-0.1223734	0.1988364 (0.0006011)	0.1986171 (0.0003681)	5.1929144
0	0	0.1984553 (0.0005948)	0.1977835 (0.0003701)	5.1870834
0.0005	0.1223734	0.1984125 (0.0005943)	0.1976614 (0.0003675)	5.1812590
0.0015	0.3671202	0.1982640 (0.0005990)	0.1969242 (0.0003690)	5.1696300
0.002	0.4894936	0.1979702 (0.0005998)	0.1964036 (0.0003824)	5.1638254

Table: Fair values for the GAO and the indexed annuity.

- Increasing linear correlation implies decreasing prices.

Example 3

In this third experiment, we consider the following Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & Q_{12} \\ Q_{12} & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & 0.001 \\ 0.001 & 0.001 \end{pmatrix}, \quad \beta = 3,$$

Q_{12}	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.01	-0.2942210	0.2952542 (0.0008533)	0.2953898 (0.0007196)	6.6586982
-0.006	-0.2447468	0.3385183 (0.0006100)	0.3373131 (0.0005179)	7.0908734
-0.002	-0.1099389	0.3511130 (0.0005080)	0.3512829 (0.0003793)	7.1946104
0.002	0.1099389	0.3296504 (0.0006325)	0.3285171 (0.0004363)	6.9353738
0.006	0.2447468	0.2799801 (0.0008396)	0.2788112 (0.0006115)	6.3815167
0.01	0.2942210	0.2176668 (0.0010351)	0.2159984 (0.0007818)	5.6571110

Table: Fair values for the GAO and the indexed annuity in the Wishart specification, Example 3.

- No monotone relation between correlation and prices (unlike the previous cases).

Plan

- The value of a GAO cannot always be explained only in terms of the initial pairwise linear correlation, e.g. not in advanced affine models (such as the Wishart one).
- This fact has important consequences for risk management in the presence of a dependence which cannot be estimated from datasets.
  - If the prices would increase with the (initial) linear correlation coefficient as in the Gaussian framework as in Liu et al. [2014], the multi-CIR model or the Wishart specifications in Example 1, then the most risk-averse way to choose a correlation when pricing a GAO, when selling a life insurance with such an imbedded option, would be to take the linear correlation coefficient equal to 1. Then the seller is sure to have no underestimation of the price of the GAO.
  - However, in Example 2 in the Wishart case, prices decrease with increasing initial linear correlation and therefore, one would have to follow the opposite rule in that situation.
  - Example 3 in the Wishart case leads to the prices which are not monotone with respect to the correlation, but which seem to lead to the highest prices for zero correlation. Therefore, in this situation, choosing zero correlation might be the appropriate risk-averse choice.
- It is clear that the dependence between mortality and interest rates has an important implication on the pricing of insurance products and that several behaviors are possible

Thank you for your attention

E. Biffis. Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics and Economics*, 37(3):443–468, 2005.

M.-F. Bru. Wishart processes. *Journal of Theoretical Probability*, 4:725–751, 1991.

C. Chiarella, J. Da Fonseca, and M. Grasselli. Pricing range notes within Wishart affine models. *Insurance: Mathematics and Economics*, 58:193–203, 2014.

C. Cuchiero, D. Filipovic, E. Mayerhofer, and J. Teichmann. Affine processes on positive semidefinite matrices. *The Annals of Applied Probability*, 21(2): 397–463, 2011.

J. Da Fonseca, M. Grasselli, and C. Tebaldi. A Multifactor Volatility Heston Model. *Quantitative Finance*, 8(6):591–604, 2008.

J. Da Fonseca, A. Gnoatto, and M. Grasselli. A flexible matrix Libor model with smiles. *Journal of Economic Dynamics and Control*, 37(4):774–793, 2013.

M. Dacorogna and M. Cadena. Exploring the dependence between mortality and market risks. *SCOR Papers*, 2015.

M. Dahl. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. *Insurance: Mathematics and Economics*, 35(1):113–136, 2004.

D. Duffie, J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376, 2000.

C.A. Favero, A.E. Gozluklu, and A. Tamoni. Demographic trends, the dividend-price ratio, and the predictability of long-run stock market returns. *Journal of Financial and Quantitative Analysis*, 46(5):1493–1520, 2011.

C. Gouriéroux and R. Sufana. Wishart Quadratic Term Structure Models. Working paper, available on SSRN at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=757307, 2003.

C. Gouriéroux and R. Sufana. Discrete time Wishart term structure models. *Journal of Economic Dynamics and Control*, 35(6):815–824, 2011.

M. Grasselli and C. Tebaldi. Solvable Affine Term Structure Models. *Mathematical Finance*, 18:135–153, 2008.

L. Jalen and R. Mamon. Valuation of contingent claims with mortality and interest rate risks. *Math. Comput. Model.*, 49:1893–1904, 2009.

C. Kang and W. Kang. Exact Simulation of Wishart Multidimensional Stochastic Volatility Model. Working paper, 2013. URL `http://arxiv.org/abs/1309.0557v1`.

M. Keller-Ressel and E. Mayerhofer. Exponential Moments of Affine Processes. *Annals of Applied Probability*, 25(2):714–752, 2015.

X. Liu, R. Mamon, and H. Gao. A comonotonicity-based valuation method for guaranteed annuity options. *J. Computational Applied Mathematics*, 250: 58–69, 2013.

X. Liu, R. Mamon, and H. Gao. A generalized pricing framework addressing correlated mortality and interest risks: a change of probability measure approach. *Stochastics An International Journal of Probability and Stochastic Processes*, 86(4):594–608, 2014.

T.A. Maurer. Asset pricing implications of demographic change. Working paper, 2014.

M. Milevsky and S. Promislow. Mortality derivatives and the option to annuitise. *Insurance: Mathematics and Economics*, 29(3):299–318., 2001.

E. Nicolini. Mortality, interest rates, investment, and agricultural production in 18th century england. *Explorations in Economic History*, 41(2):130–155, 2004.

A. Pelsser. Pricing and hedging guaranteed annuity options via static option replication. *Insurance: Mathematics and Economics*, 33:283–296, 2003.

N. Zhu and D. Bauer. Applications of forward mortality factor models in life insurance practice. *The Geneva Papers on Risk and Insurance*, 36:567–594, 2011.