Portfolio Optimisation: Shadow Prices and Fractional Brownian Motion

Christoph Czichowsky

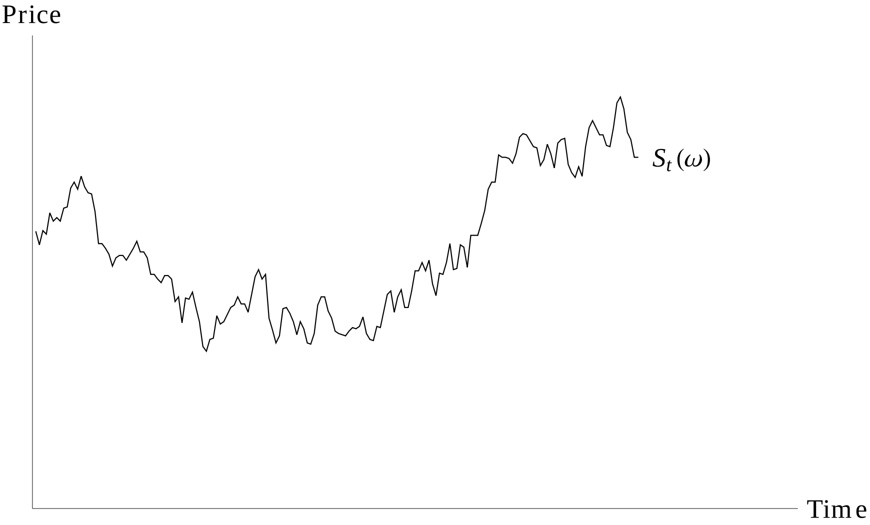
Department of Mathematics
London School of Economics and Political Science

7th General AMaMeF and Swissquote Conference

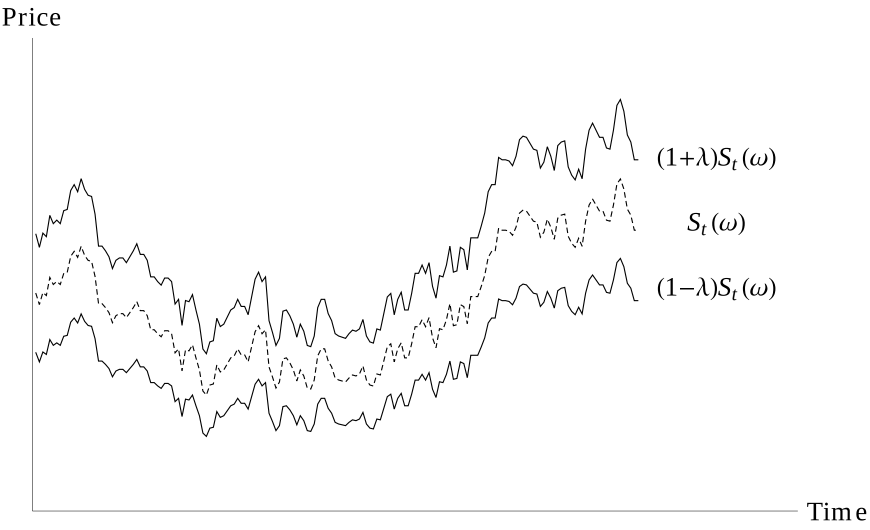
based on joint work(s) with Walter Schachermayer

Outline

Stock price



Bid-ask spread



Overview and comparison

	frictionless markets	markets with transaction costs
trading	buy and sell at **same** price S_t	buy at **higher** ask price $(1+\lambda)S_t$ sell at **lower** bid price $(1-\lambda)S_t$
"no arbitrage" price process	must be a semimartingale + very handy	can also be a **non-semimartingale** − more difficult to handle
critical Hölder exponent	either exactly 1 or exactly 2 − seems restrictive from a statistical point of view	any value in $[1,\infty)$ + more robust
optimal strategies	+ nice results for standard utilities	− hard to compute even for standard utilities and semimartingales
trading volume	− typically infinite, not possible in reality	+ automatically finite
summary	+ typically very handy − not always realistic	− more difficult to handle + more realistic

Financial markets with transaction costs

- Fix a strictly positive càdlàg **stochastic process** $S = (S_t)_{0 \leq t \leq T}$.
- A **self-financing trading strategy** under transaction costs $\lambda \in (0,1)$ is a predictable **finite variation** process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ such that

$$d\varphi_t^0 \leq -(1+\lambda) S_t (d\varphi_t^1)^+ + (1-\lambda) S_t (d\varphi_t^1)^-.$$

- A self-financing strategy φ is **admissible**, if its **liquidation value**

$$\begin{aligned} V_t(\varphi) &:= \varphi_t^0 + (\varphi_t^1)^+ (1-\lambda) S_t - (\varphi_t^1)^- (1+\lambda) S_t \\ &= \varphi_0^0 + \varphi_0^1 S_0 + \int_0^t \varphi_s^1 dS_s - \lambda \int_0^t S_s d|\varphi^1|_s - \lambda S_t |\varphi_t^1| \\ &\geq -M^\varphi \end{aligned}$$

for some $M^\varphi > 0$ simultaneously for all $t \in [0, T]$.

- Denote by $\mathcal{A}^\lambda(x)$ the set of all self-financing and admissible trading strategies under transaction costs λ starting with $(\varphi_0^0, \varphi_0^1) = (x, 0)$.

Utility maximisation under transaction costs

- **Primal problem**: find **optimal trading strategy** $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ to

$$\text{maximise} \quad E\left[U\left(V_T(\varphi)\right)\right] := E\left[U\left(x + \int_0^T \varphi^1_u dS_u - \lambda \int_0^T S_u d|\varphi^1|_u\right)\right].$$

- **Dual problem**: find **optimal λ-consistent price system** $(\widehat{Z}^0, \widehat{Z}^1)$, i.e. local martingales $(Z^0, Z^1) > 0$ such that $\widetilde{S} := \frac{Z^1}{Z^0} \in [(1-\lambda)S, (1+\lambda)S]$, to

$$\text{minimise} \quad E\left[U^*\left(Z^0_T\right) + xZ^0_T\right].$$

- **Lagrange duality**: If $(\widehat{Z}^0, \widehat{Z}^1)$ exists, then $V_T(\widehat{\varphi}) = (U')^{-1}(\widehat{Z}^0_T)$.
- **Technical point**: Solution $(\widehat{Z}^0, \widehat{Z}^1)$ to dual problem is, in general, only a **limit** of consistent price systems, i.e., an **optional strong supermartingale**.

Utility maximisation under transaction costs

- **Primal problem**: find **optimal trading strategy** $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ to

$$\text{maximise} \quad E\left[U\left(V_T(\varphi)\right)\right] := E\left[U\left(x + \int_0^T \varphi_u^1 dS_u - \lambda \int_0^T S_u d|\varphi^1|_u\right)\right].$$

- **Dual problem**: find **optimal λ-consistent price system** $(\widehat{Z}^0, \widehat{Z}^1)$, i.e. local martingales $(Z^0, Z^1) > 0$ such that $\widetilde{S} := \frac{Z^1}{Z^0} \in [(1-\lambda)S, (1+\lambda)S]$, to

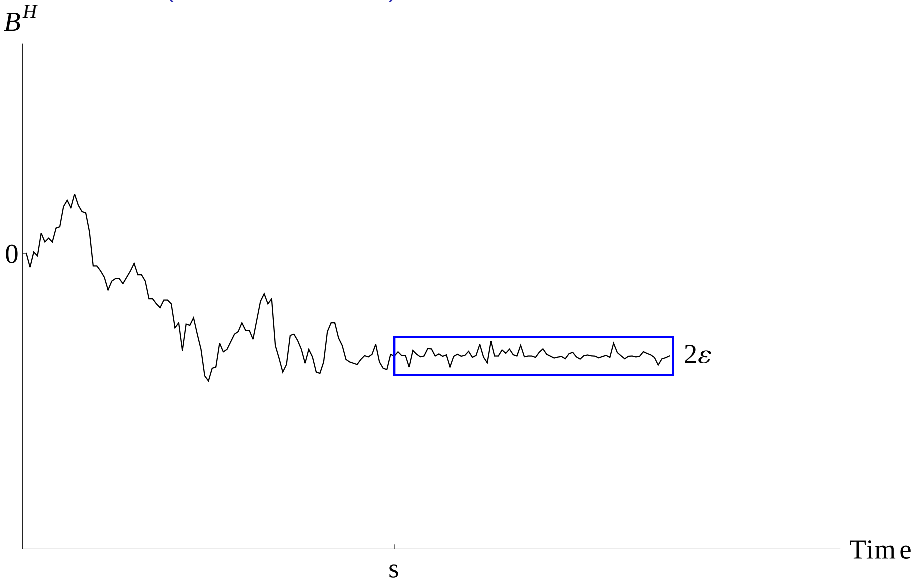
$$\text{minimise} \quad E\left[U^*\left(Z_T^0\right) + xZ_T^0\right].$$

- **Lagrange duality**: If $(\widehat{Z}^0, \widehat{Z}^1)$ exists, then $V_T(\widehat{\varphi}) = (U')^{-1}(\widehat{Z}_T^0)$.
- **Technical point**: Solution $(\widehat{Z}^0, \widehat{Z}^1)$ to dual problem is, in general, only a **limit** of consistent price systems, i.e., an **optional strong supermartingale**.
- Guasoni (2002): In principle, the above allows also to consider **non-semimartingales** for S. — So what about concrete examples?

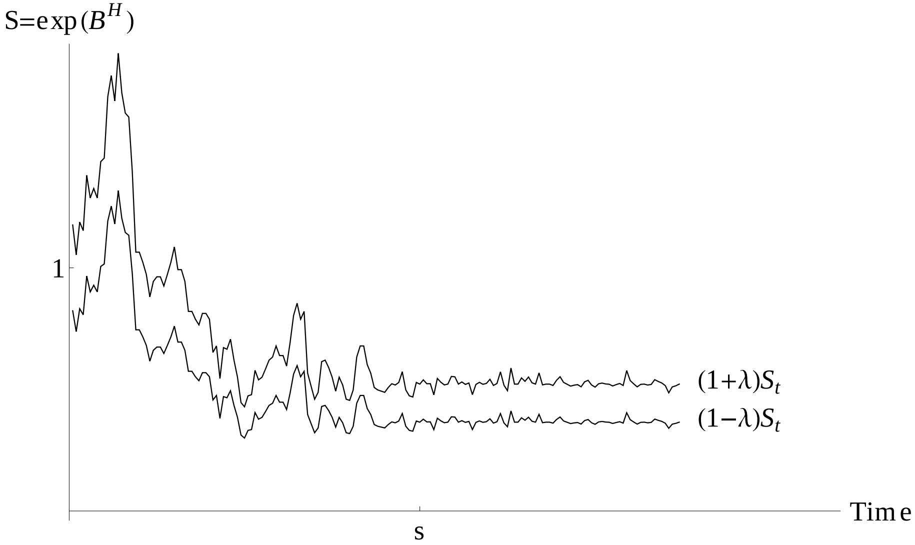
Fractional Brownian motion

- A "nice" class of **Gaussian processes** $B^H = (B_t^H)$ indexed by $H \in (0,1)$.
- Mandelbrot: Natural model for stock prices.
- Critical Hölder exponent is $\frac{1}{H}$ and can therefore take any value in $(1,\infty)$.
- Prime example of non-semimartingales for $H \neq \frac{1}{2}$.
- For frictionless trading, fractional models like the **fractional Black-Scholes model** $S = \exp(B^H)$ admit "arbitrage"; see e.g. Rogers (1997), Cheridito (2003) for explicit constructions.
- Guasoni (2006): The fractional Black-Scholes model is arbitrage-free under transaction costs, as $B^H = \log(S)$ is **sticky**.

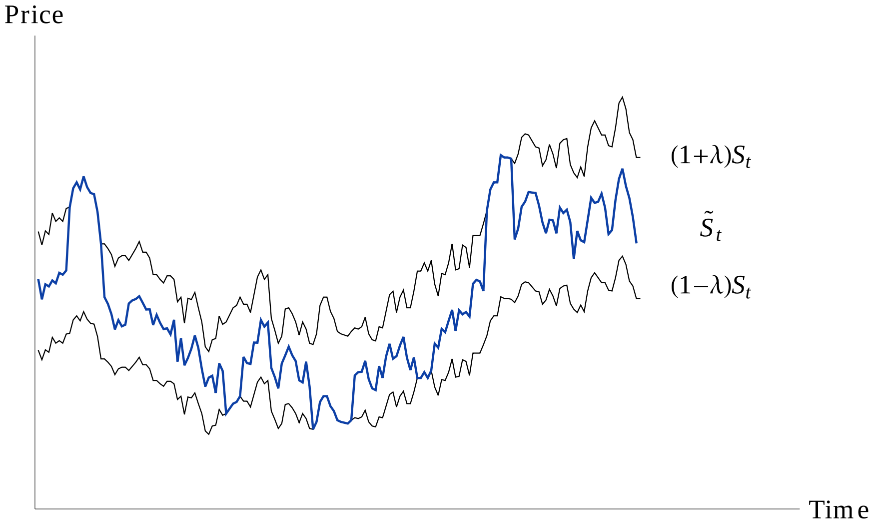
Stickiness (Guasoni 2006)



No arbitrage under transaction costs



Shadow price (Jouini/Kallal, Cvitanić/Karatzas)



Shadow price

Definition

A semimartingale price process $\widehat{S} = (\widehat{S}_t)$ *is a* **shadow price**, *if*

i) $\widehat{S}$ *is valued in the bid-ask spread* $[(1-\lambda)S, (1+\lambda)S]$.

ii) *The solution* $\widehat{\psi}$ *to the* **frictionless** *utility maximisation problem: to*

$$\textit{maximise} \qquad E\big[U\big(V_T(\psi)\big)\big] := E\left[U\left(x + \int_0^T \psi_s d\widehat{S}_s\right)\right]$$

exists.

iii) $\widehat{\psi}$ *is of finite variation and "admissible" under transaction costs.*

iv) $\{d\widehat{\psi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ *and* $\{d\widehat{\psi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.

Then $\widehat{\psi}$ *coincides with the solution* $\widehat{\varphi}$ *under transaction costs.*

Existence of shadow prices?

- Cvitanić/Karatzas (1996): Existence in an Itô process setting, **if** the solution to the dual problem is a local martingale. — Not clear under which conditions this is the case.
  - Kallsen/Muhle-Karbe (2011): finite probability space.
  - Explicit constructions for various concrete problems in the classical(!) Black-Scholes model; Kallsen/Muhle-Karbe (2009),...
  - Beyond the classical Black-Scholes model?
  - C./Deutsch/Forde/Zhang: Construction for geometric Ornstein-Uhlenbeck process.
  - No-shortselling (somewhat different problem); Loewenstein (2001), Benedetti/Campi/Kallsen/Muhle-Karbe (2011).
- No general results that apply to Cvitanić/Karatzas (1996) so far.
- Counter-examples in discrete time:
  - Benedetti/Campi/Kallsen/Muhle-Karbe (2011).
  - C./Muhle-Karbe/Schachermayer (2012).

Outline

Sufficient conditions

Theorem (C./Schachermayer/Yang)

Suppose that

0) $U : (0,\infty) \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$.

i) S *is continuous.*

ii) S *satisfies* **(NUPBR)**.

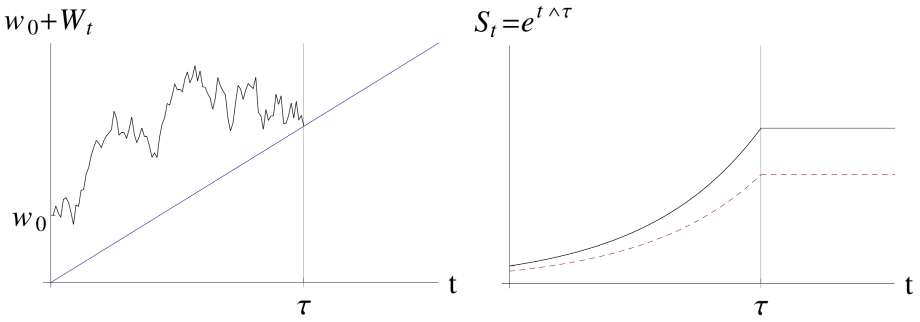
iii) $u(x) := \sup_{\varphi \in \mathcal{A}^\lambda(x)} E\big[U\big(V_T(\varphi)\big)\big] < \infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *is a local martingale and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *a shadow price process.*

- Conditions can be verified without knowing the solution to the dual problem before; compare Cvitanić/Karatzas (1996).
- **Quite sharp:** There exist **counter-examples**, if i) or ii) are not satisfied.
- Condition ii), which implies that S is a **semimartingale**, **cannot** be replaced by the weaker condition that "$\log(S)$ is sticky" typically used for fBm.

Example: S is continuous and sticky

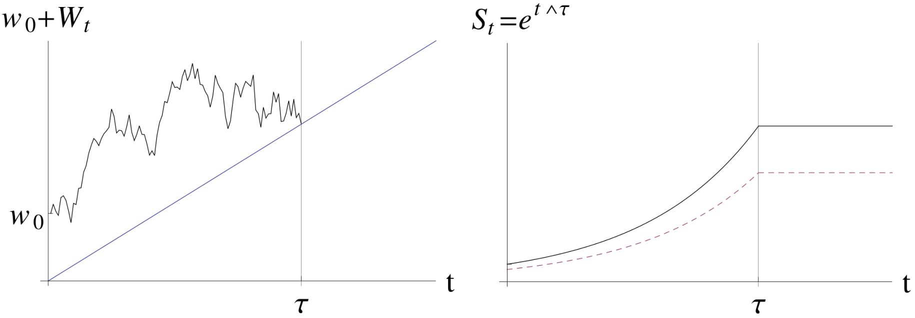
- C./Schachermayer/Yang.



- S admits an **unbounded increasing profit** and hence **no** ELMM.

Example: S is continuous and sticky

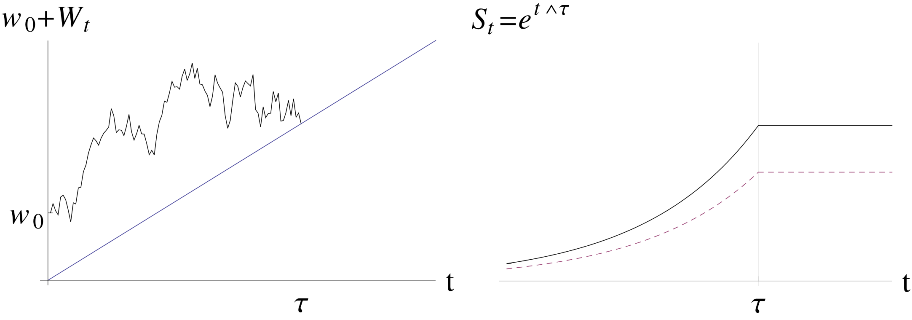
- C./Schachermayer/Yang.



- S admits an **unbounded increasing profit** and hence **no** ELMM.
- **No** solution to **any** frictionless utility maximisation problem.

Example: S is continuous and sticky

- C./Schachermayer/Yang.



- S admits an **unbounded increasing profit** and hence **no** ELMM.
- **No** solution to **any** frictionless utility maximisation problem.
- However, $\log(S)$ is **sticky** and S is arbitrage-free under transaction costs.

Example (cont.)

Proposition (C./Schachermayer/Yang)

There exists a **non-decreasing function** $\ell : [0,\infty) \to [0, \frac{1}{\lambda}]$ *such that the optimal strategy* $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ *to*

$$E\left[\log\left(V_\tau(\varphi)\right)\right] \to \max!, \qquad \varphi \in \mathcal{A}^\lambda(1),$$

is given by the **smallest non-decreasing process** $\widehat{\varphi}^1$ *such that*

i) $d\widehat{\varphi}^0_t = -(1+\lambda)S_t d\widehat{\varphi}^1_t$ *for all* $t \geq 0$.

ii) $\frac{1}{\lambda} \geq \frac{\widehat{\varphi}^1_t S_t}{\widehat{\varphi}^0_t + \widehat{\varphi}^1_t S_t} \geq \ell(w_0 + W_t - t)$ *for all* $t \geq 0$.

Moreover, there exists $\overline{w} \in (0,\infty)$ *such that* $\ell(w) = \frac{1}{\lambda}$ *for all* $w \geq \overline{w}$.

For $w_0 > \overline{w}$, we would therefore have

$$\widehat{S}_t = (1+\lambda)S_t \qquad \text{for all } t \leq \sigma := \inf\{s > 0 \mid w_0 + W_s - s < \overline{w}\}$$

for any candidate shadow price and hence **no shadow price** exists.

Sufficient conditions (cont.)

Theorem (C./Schachermayer/Yang)

Suppose that

0) $U : (0,\infty) \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$.

i) S *is continuous.*

ii) S *satisfies* **no simple arbitrage (NSA)**.

iii) $u(x) := \sup_{\varphi \in \mathcal{A}^\lambda(x)} E\big[U\big(V_T(\varphi)\big)\big] < \infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *is a local martingale and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *a shadow price process.*

- Bender (2012): For continuous S, (NSA) $\iff$ (NOA) and (TWC).

Sufficient conditions (cont.)

Theorem (C./Schachermayer/Yang)

Suppose that

0) $U : (0,\infty) \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$.

i) S *is continuous.*

ii) S *satisfies* **no simple arbitrage (NSA)**.

iii) $u(x) := \sup_{\varphi \in \mathcal{A}^\lambda(x)} E\big[U\big(V_T(\varphi)\big)\big] < \infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *is a local martingale and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *a shadow price process.*

- Bender (2012): For continuous S, (NSA) $\Longleftrightarrow$ (NOA) and (TWC).
- Is (NSA) satisfied for the **fractional Black-Scholes model**?

Sufficient conditions (cont.)

Theorem (C./Schachermayer/Yang)

Suppose that

0) $U : (0, \infty) \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$.

i) S *is continuous.*

ii) S *satisfies* **no simple arbitrage (NSA)**.

iii) $u(x) := \sup_{\varphi \in \mathcal{A}^\lambda(x)} E\big[U\big(V_T(\varphi)\big)\big] < \infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *is a local martingale and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *a shadow price process.*

- Bender (2012): For continuous S, (NSA) $\Longleftrightarrow$ (NOA) and (TWC).
- Is (NSA) satisfied for the **fractional Black-Scholes model**?
- Peyre (2015): $S = \exp(B^H)$ satisfies (TWC).

Sufficient conditions (cont.)

Theorem (C./Schachermayer/Yang)

Suppose that

0) $U : (0, \infty) \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$.

i) S *is continuous.*

ii) S *satisfies* **no simple arbitrage (NSA)**.

iii) $u(x) := \sup_{\varphi \in \mathcal{A}^\lambda(x)} E\left[U\left(V_T(\varphi)\right)\right] < \infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *is a local martingale and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *a shadow price process.*

- Bender (2012): For continuous S, (NSA) $\iff$ (NOA) and (TWC).
- Is (NSA) satisfied for the **fractional Black-Scholes model**?
- Peyre (2015): $S = \exp(B^H)$ satisfies (TWC).
- Condition iii) is satisfied for $U(x) = \frac{x^p}{p}$ for $p \in (-\infty, 0)$.

Sufficient conditions

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$ *and* $\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is locally bounded.*

ii) S *admits a* λ'*-consistent price system* $(\bar{Z}^0, \bar{Z}^1)$ *for* $\lambda' \in [0, \lambda)$ *such that*

iii) $E[U^*(\bar{Z}^0_T)] < +\infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

Sufficient conditions

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$ *and* $\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is locally bounded.*

ii) S *admits a* λ'*-consistent price system* $(\bar{Z}^0, \bar{Z}^1)$ *for* $\lambda' \in [0, \lambda)$ *such that*

iii) $E[U^*(\bar{Z}^0_T)] < +\infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

- Condition 0) implies that $\lim_{y\to\infty} \frac{U^*(y)}{y} = \infty$ and therefore allows to apply the **de la Vallée-Poussin criterion** for uniform integrability.

Sufficient conditions

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$ *and* $\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is locally bounded.*

ii) S *admits a* λ'*-consistent price system* $(\bar{Z}^0, \bar{Z}^1)$ *for* $\lambda' \in [0, \lambda)$ *such that*

iii) $E[U^*(\bar{Z}^0_T)] < +\infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

- Condition 0) implies that $\lim_{y\to\infty} \frac{U^*(y)}{y} = \infty$ and therefore allows to apply the **de la Vallée-Poussin criterion** for uniform integrability.
- In principle, the above allows to consider **non-semimartingales** for S.

Sufficient conditions

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$ *and* $\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is locally bounded.*

ii) S *admits a* λ'*-consistent price system* $(\bar{Z}^0, \bar{Z}^1)$ *for* $\lambda' \in [0, \lambda)$ *such that*

iii) $E[U^*(\bar{Z}^0_T)] < +\infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

- Condition 0) implies that $\lim_{y\to\infty} \frac{U^*(y)}{y} = \infty$ and therefore allows to apply the **de la Vallée-Poussin criterion** for uniform integrability.
- In principle, the above allows to consider **non-semimartingales** for S.
- For the **fractional Black-Scholes model** conditions i) and ii) are satisfied.

Sufficient conditions

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$ *and* $\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is locally bounded.*

ii) S *admits a* λ'*-consistent price system* $(\bar{Z}^0, \bar{Z}^1)$ *for* $\lambda' \in [0, \lambda)$ *such that*

iii) $E[U^*(\bar{Z}^0_T)] < +\infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

- Condition 0) implies that $\lim_{y\to\infty} \frac{U^*(y)}{y} = \infty$ and therefore allows to apply the **de la Vallée-Poussin criterion** for uniform integrability.
- In principle, the above allows to consider **non-semimartingales** for S.
- For the **fractional Black-Scholes model** conditions i) and ii) are satisfied.
- So what about iii) for $U(x) = 1 - e^{-x}$? — Hard to verify directly.

Sufficient conditions (cont.)

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *is bounded from above and satisfies* $\liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is continuous.*

ii) S *is* **sticky**.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

- Conditions i)–ii) are satisfied for the **fractional Black-Scholes model**.

Sufficient conditions (cont.)

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *is bounded from above and satisfies* $\liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is continuous.*

ii) S *is* **sticky**.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

- Conditions i)–ii) are satisfied for the **fractional Black-Scholes model**.
- Proof combines arguments from convex duality with the stickiness condition.

Sufficient conditions (cont.)

Theorem (C./Schachermayer)

Suppose that

0) $U : \mathbb{R} \to \mathbb{R}$ *is bounded from above and satisfies* $\liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1$.

i) S *is continuous.*

ii) S *is* **sticky**.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *exists and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *is a shadow price.*

- Conditions i)–ii) are satisfied for the **fractional Black-Scholes model**.
- Proof combines arguments from convex duality with the stickiness condition.
- By the change of measure $\frac{dP_B}{dP} = \frac{\exp(B)}{E[\exp(B)]}$ the above also gives the existence of **exponential utility indifference prices** for any claim $B \in L^\infty(P)$.

Outline

Future research

- **Future goals**:
  - **Quantitative** results for fractional models.
  - Understand impact of **non-semimartingality** on optimal strategy.
  - Utility-based **pricing** and **hedging** for fractional models.

Future research

- **Future goals**:
  - **Quantitative** results for fractional models.
  - Understand impact of **non-semimartingality** on optimal strategy.
  - Utility-based **pricing** and **hedging** for fractional models.
- For the **fractional Black-Scholes model** $S = \exp(B^H)$ the shadow price is

1) an **Itô process**, i.e.

$$d\widehat{S}_t = \widehat{S}_t \left(\widehat{\mu}_t dt + \widehat{\sigma}_t dW_t \right),$$

2) evolving in the bid-ask spread $\widehat{S} \in [(1-\lambda)S, (1+\lambda)S]$ such that

3) the optimal strategies coincide, i.e. $\widehat{\psi} = \widehat{\varphi}$, and

4) $\{d\widehat{\varphi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ and $\{d\widehat{\varphi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.

Future research

- **Future goals**:
  - **Quantitative** results for fractional models.
  - Understand impact of **non-semimartingality** on optimal strategy.
  - Utility-based **pricing** and **hedging** for fractional models.
- For the **fractional Black-Scholes model** $S = \exp(B^H)$ the shadow price is
  1) an **Itô process**, i.e.

$$d\widehat{S}_t = \widehat{S}_t \left(\widehat{\mu}_t dt + \widehat{\sigma}_t dW_t \right),$$

  2) evolving in the bid-ask spread $\widehat{S} \in [(1-\lambda)S, (1+\lambda)S]$ such that
  3) the optimal strategies coincide, i.e. $\widehat{\psi} = \widehat{\varphi}$, and
  4) $\{d\widehat{\varphi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ and $\{d\widehat{\varphi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.
- **Basic idea:** Combine 1)–4) with results for utility maximisation for **Itô processes** to describe optimal strategy $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ more explicitly.

Future research

- **Future goals**:
  - **Quantitative** results for fractional models.
  - Understand impact of **non-semimartingality** on optimal strategy.
  - Utility-based **pricing** and **hedging** for fractional models.
- For the **fractional Black-Scholes model** $S = \exp(B^H)$ the shadow price is

  1) an **Itô process**, i.e.

$$d\widehat{S}_t = \widehat{S}_t \left(\widehat{\mu}_t dt + \widehat{\sigma}_t dW_t \right),$$

  2) evolving in the bid-ask spread $\widehat{S} \in [(1-\lambda)S, (1+\lambda)S]$ such that

  3) the optimal strategies coincide, i.e. $\widehat{\psi} = \widehat{\varphi}$, and

  4) $\{d\widehat{\varphi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ and $\{d\widehat{\varphi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.
- **Basic idea:** Combine 1)–4) with results for utility maximisation for **Itô processes** to describe optimal strategy $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ more explicitly.
- This then also gives results for **exponential utility indifference pricing** by comparing **two** shadow prices given by the Itô processes $\widehat{S}^B$ and $\widehat{S}$.

Future research

- **Future goals**:
  - **Quantitative** results for fractional models.
  - Understand impact of **non-semimartingality** on optimal strategy.
  - Utility-based **pricing** and **hedging** for fractional models.
- For the **fractional Black-Scholes model** $S = \exp(B^H)$ the shadow price is
  1) an **Itô process**, i.e.
  $$d\widehat{S}_t = \widehat{S}_t \left(\widehat{\mu}_t dt + \widehat{\sigma}_t dW_t \right),$$
  2) evolving in the bid-ask spread $\widehat{S} \in [(1-\lambda)S, (1+\lambda)S]$ such that
  3) the optimal strategies coincide, i.e. $\widehat{\psi} = \widehat{\varphi}$, and
  4) $\{d\widehat{\varphi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ and $\{d\widehat{\varphi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.
- **Basic idea:** Combine 1)–4) with results for utility maximisation for **Itô processes** to describe optimal strategy $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ more explicitly.
- This then also gives results for **exponential utility indifference pricing** by comparing **two** shadow prices given by the Itô processes $\widehat{S}^B$ and $\widehat{S}$.
- Importance: Superreplication price is too high by face-lifting theorems.

Summary

Sufficient conditions for existence of shadow prices:

1) S is continuous and satisfies (NSA) $U : (0, \infty) \to \mathbb{R}$. **Quite sharp.**

2) S is locally bounded and admits a $CPS^{\lambda'}$ $(\bar{Z}^0, \bar{Z}^1)$ for $\lambda' \in [0, \lambda)$ satisfying $E[U^*(\bar{Z}^0_T)] < \infty$ for $U : \mathbb{R} \to \mathbb{R}$.

3) S is continuous and sticky for $U : \mathbb{R} \to \mathbb{R}$ bounded from above.

Counter-examples for $U : (0, \infty) \to \mathbb{R}$:

- S is continuous and sticky are **not** sufficient.

Fractional Brownian motion:

- Existence of shadow price for bounded power and exponential utility.
- Shadow price is Itô process.
- Exploit connection to frictionless markets to obtain quantitative results.

Thank you for your attention!

http://www.maths.lse.ac.uk/Personal/christoph

Talk based on

C. Czichowsky and W. Schachermayer.
Strong supermartingales and limits of non-negative martingales.
Preprint, 2013. To appear in *The Annals of Probability*.

C. Czichowsky and W. Schachermayer.
Duality theory for portfolio optimisation under transaction costs.
Preprint, 2014. To appear in *The Annals of Applied Probability*.

C. Czichowsky, W. Schachermayer, and J. Yang.
Shadow prices for continuous price processes.
Preprint, 2014. To appear in *Mathematical Finance*.

C. Czichowsky and W. Schachermayer.
Portfolio optimisation beyond semimartingales: shadow prices and fractional Brownian motion.
Preprint, 2015.