The Fourier estimator of the stochastic leverage effect

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Stylized facts

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Recent empirical works as Yu, 2005*, Bollerslev et al,* 2006*, Carr and Wu,* 2007*, however, emphasized that the leverage is time variant, asymmetric and can be itself considered as a stochastic process.*

Data generating process

The log-price and the volatility processes follow

$$\begin{cases} dp(t) & = \sigma(t)dW(t) + a(t)dt \\ d\sigma^2(t) & = \gamma(t)dZ(t) + b(t)dt, \end{cases} \tag{1}$$

where $W(t)$ and $Z(t)$ are Brownian motions correlated by means of the process $\rho(t) \in [-1, 1]$ and defined on a probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions.
$\mathcal{F} = (\mathcal{F}_t)_{t\in[0,2\pi]}$ is the usual augmentation of the natural filtration generated by W and Z.
We refer to the process $\gamma^2(t)$ as the variance of the volatility.

Assumptions

The processes that appear in (1) satisfy:

- **H1** $a(t)$, $b(t)$, $\sigma(t)$, $\gamma(t)$, $\rho(t)$ are continuous on $[0, 2\pi]$ and adapted to the filtration $\mathcal{F}$ with values in $\mathbb{R}$,
- **H2** $\forall p \geq 1$

$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |a(t)|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{t\in[0,2\pi]} |b(t)|^p\Big] < \infty,$$
$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\sigma(t)|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\gamma(t)|^p\Big] < \infty,$$
$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\rho(t)|^p\Big] < \infty,$$

Assumptions

- **H3** $\forall p \geq 1$, the processes $a(t), b(t), \sigma(t), \gamma(t) \in \mathbb{D}^{1,p}$ and

$$\mathbb{E}\Big[\sup_{s,t\in[0,2\pi]}\Big|\mathcal{D}_s a(t)\Big|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{s,t\in[0,2\pi]}\Big|\mathcal{D}_s b(t)\Big|^p\Big] < \infty,$$

$$\mathbb{E}\Big[\sup_{s,t\in[0,2\pi]}\Big|\mathcal{D}_s \sigma(t)\Big|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{s,t\in[0,2\pi]}\Big|\mathcal{D}_s \gamma(t)\Big|^p\Big] < \infty,$$

where $\mathbb{D}^{1,p}$ is the Sobolev space of the generalized derivative in the sense of Malliavin and $\mathcal{D}$ is the Malliavin derivative.

Leverage Effect

The leverage process $\eta(t)$ is defined as

$$< dp(t), d\sigma^2(t) >= \eta(t)dt,$$

we are interested in estimating the integrated quantity

$$\hat{\eta} = \int_0^{2\pi} \eta(t)dt.$$

Non-parametric leverage estimations

- Let X and σ^2 be respectively the return and the volatility process defined as Itô semimartingales on $[0, T]$.
  Denote $\Delta X_s = X_s - X_{s-}$ as the jump size of X at time s. The quadratic variation of X and σ^2 is

$$[X, \sigma^2]_T = < X, \sigma^2 >_T + \sum_{s \leq T} \Delta X_s \Delta \sigma^2_s$$

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- In a semi-martingale model set up, Bandi and Renó, 2012, Cuchiero and Teichmann, 2013, Mykland and Wang, 2012, Veraart and Veraart, 2012, and Aït-Sahalia et al, 2014, have proposed non-parametric estimator of the integrated leverage effect all based on a **pre-estimation of the spot volatility**.

The Fourier Transform Method

- We define an estimator of the continuous quadratic variation between the return and its own volatility using only a **pre-estimation of the Fourier coefficients of the volatility**.

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- We define an estimator of the continuous quadratic variation between the return and its own volatility using only a **pre-estimation of the Fourier coefficients of the volatility**.
- Shifting the analysis of the leverage effect in the frequency domain allows to use the same estimation procedure in the case of irregular trading (non-equidistant observations of the price) and microstructure noise contaminations without ”any manipulation“ of the original data set.

Bohr convolution product

Given two functions Φ and Ψ on the integers $\mathbb{Z}$, we say that the Bohr convolution product exists if the following limit exists for all integers h

$$(\Phi * \Psi)(h) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|l|\leq N} \Phi(l)\Psi(h-l).$$

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For a fixed h, defining $\Phi(l) := c(l, d\sigma^2)$ and $\Psi(h-l) := c(h-l, dp)$

$$c(h,\eta) = \lim_{N\to\infty} \frac{2\pi}{2N+1} \sum_{|l|\leq N} c(l; d\sigma^2) c(h-l, dp) \quad (2)$$

where the convergence is attained in probability (Malliavin and Mancino, 2009).

Discrete sampling

We assume to observe $p(t)$ in $[0, 2\pi]$ at a discrete unevenly spaced grids

$$\mathcal{S}_n := \{0 = t_{0,n} \leq t_{1,n} \leq \ldots \leq t_{k_n,n} = 2\pi\}, \quad \text{for all } i = 0, \ldots, k_n \text{ and } k_n \leq n.$$

We define $\tau(n) := \max_{i=0,\ldots,k_n-1} |t_{i+1,n} - t_{i,n}|$ and the discrete observed return as

$$\delta_{i,n}(p) = p(t_{i+1,n}) - p(t_{i,n}), \quad \text{for all} \quad i = 0, ..., k_n - 1$$

By means of the classical definition of the discrete Fourier transform, we estimate $c(s, dp)$ as

$$c_n(s; dp) = \frac{1}{2\pi} \sum_{i=0}^{k_n-1} \mathrm{e}^{-\mathrm{i} s t_{i,n}} \delta_{i,n}(p).$$

for any integer s such that $|s| \leq 2M + N$.

Discrete sampling

- **Step** 1: pre-estimation of the Fourier coefficients of the volatility obtained by means of the estimator defined in Malliavin and Mancino, 2002.

$$c_{n,M}(l;\sigma^2) = \frac{2\pi}{2M+1} \sum_{|s|\leq M} c_n(s;dp)c_n(l-s;dp)$$

  for any integer l such that $|l| \leq 2N$.

- **Step** 2: we get the estimators of the Fourier coefficients of the leverage process for any integer h such that $|h| \leq N$

$$c_{n,N,M}(h;\eta) = \frac{2\pi}{2N+1} \sum_{|l|\leq N} \mathrm{i}l c_{n,M}(l;\sigma^2)c_n(h-l;dp) \tag{3}$$

Integrated estimator

$$
\begin{aligned}
\hat{\eta}_{n,N,M} &= 2\pi c_{n,N,M}(0;\eta) \\
&= \sum_{j=0}^{k_n-1} \sum_{j'=0}^{k_n-1} \sum_{j''=0}^{k_n-1} D_M(t_{j,n} - t_{j',n}) D'_N(t_{j',n} - t_{j'',n}) \delta_{j,n}(p) \delta_{j',n}(p) \delta_{j'',n}(p) \\
&= \sum_{j \neq j', j' \neq j''} D_M(t_{j,n} - t_{j',n}) D'_N(t_{j',n} - t_{j'',n}) \delta_{j,n}(p) \delta_{j',n}(p) \delta_{j'',n}(p) \\
&\qquad + \sum_{j = j', j' \neq j''} D'_N(t_{j',n} - t_{j'',n}) \delta^2_{j,n}(p) \delta_{j'',n}(p) \qquad (4)
\end{aligned}
$$

where $D_M(t) = \frac{1}{2M+1} \sum_{|s| \leq M} \mathrm{e}^{\mathrm{i}st}$ is the rescaled Dirichlet kernel and D'_N its first derivative.

Integrated estimator

Theorem

We assume that the assumption (H)

$$\frac{N^2}{M} \to 0, \quad and \quad M\tau(n) \to a$$

with $a \in (0, \frac{1}{2})$ *hold true as* $n, N, M \to \infty$ *and* $\tau(n) \to 0$. *Then*

$$\hat{\eta}_{n,N,M} \xrightarrow{\mathbb{P}} \int_0^{2\pi} \eta(t)dt.$$

Integrated estimator

Theorem (C. 2015)

We assume that hypotheses (H) and the following relations

$$\frac{N^3}{M} \to 0 \quad \text{and} \quad M\tau(n) \to a \tag{5}$$

with $a \in (0, \frac{1}{2})$ *hold true as* $n, N, M \to \infty$ *and* $\tau(n) \to 0$*. Then*

$$\sqrt{N}\left(\hat{\eta}_{n,N,M} - \hat{\eta}\right) \xrightarrow{st} \int_0^{2\pi} \sqrt{\varphi(s)}dW'(s)$$

where

$$\varphi(s) = \pi(\sigma^2(s)\gamma^2(s) + \eta^2(s)) + \frac{\pi}{2}\sigma^2(0)\gamma^2(s) + \frac{1}{4}(\sigma^2(2\pi) - \sigma^2(0))^2\sigma^2(2\pi)$$

and $W'(s)$ *is a Brownian motion defined on an extension of the original probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and independent of the original* σ*-algebra* $\mathbb{F}$*.*

Data sets

a) We simulate second-by-second return and variance paths over a daily trading period of $T = 6$ hours, for a total of 250 trading days and $n = 21600$ observations per day.

$$(GH): \begin{cases} dp(t) & = \sigma(t)dX(t) \\ dX(t) & = \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t) \\ d\sigma^2(t) & = \kappa(\beta - \sigma^2(t))dt + \sigma^2\sigma(t)dW_1(t), \end{cases}$$

and the infinitesimal variation of $\rho(t)$ is given by

$$d\rho(t) = ((2\xi - \eta) - \eta\rho(t))dt + \theta\sqrt{(1+\rho(t))(1-\rho(t))}dW_0,$$

where η, ξ and θ are positive constants and W_0 is a Brownian motion. The processes $W_0(t)$, $W_1(t)$ and $W_2(t)$ are assumed to be independent, Verrart and Verrart, 2012.

Data sets

b) We generate non-equidistant observations of the logarithmic price by means of a $B(\alpha, \beta)$ distribution with parameters $\alpha = 1$ and $\beta = 3$ chosen to model the time between trades. Also for this data sets we simulate a total of 250 trading days and $n = 21600$ observations per day.

c)-d)

$$\widetilde{p}(t) = p(t) + \zeta(t), \tag{6}$$

where $p(t)$ is the efficient log price in equilibrium defined in GH-model and $\zeta(t)$ is the microstructure noise. The random shocks ζ are Gaussian i.i.d. and independent from p.

Generalized Heston model

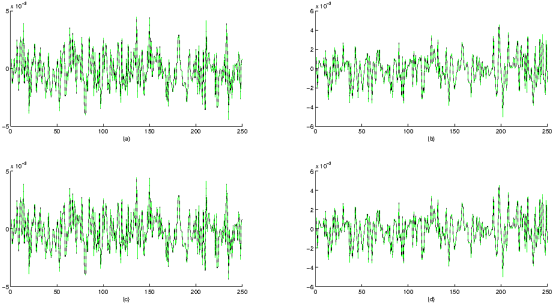

Figure: Real integrated leverage $\hat{\eta}$ (black solid line) and Fourier estimator of the leverage effect $\hat{\eta}_{n,N,M}$ (green dashed line) for the GH-model data set across 250 days: (a) no-noise, equidistant observations (b) no-noise, non-equidistant observations (c) noise, equidistant observations (d) noise, non-equidistant observations.

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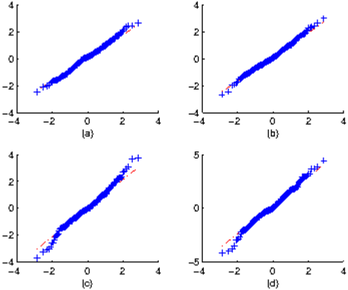

Figure: Quantile-quantile plot of the sample quantiles of the standardized estimation errors of the Fourier estimator of the leverage effect versus the theoretical quantiles of a normal distribution for the GH-model data set: (a) no-noise, equidistant observations (b) no-noise, non-equidistant observations (c) noise, equidistant observations (d) noise, non-equidistant observations.

Conclusions

- In a continuous semi-martingale set up, we have defined a novel estimator of the integrated leverage effect using solely a pre-estimation of the Fourier coefficients of the volatility process.
- The estimator is consistent and asymptotically normal distributed.
- The asymptotic rate of the central limit theorem as long as the finite sample performance of the estimator depend on the growth rate between the parameters M and N respectively the number of the Fourier coefficients of the return and the volatility process to include in the estimation procedure.
- On the finite sample, the Fourier estimator of the leverage effect is accurate in the presence of non-equidistant observations of the price process and microstructure noise contaminations.

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