Affine multiple yield curve models

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Multiple yield curves

- The Libor/Euribor rate for some interval $[T, T + \delta]$, where the tenor δ is typically 1W, 1M, 2M, 3M, 6M or 12M, is the underlying of basic interest rate products, such as FRAs, swaps, caps/floors, swaptions...
- Since the last crisis, due to credit and liquidity risk in the interbank sector, Libor/Euribor rates are considered risk-free any longer.
- Emergence of spreads in fixed income markets, notably spreads between Libor rates and Overnight Indexed Swap (OIS) rates.

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$\Rightarrow$ Interest rate models for multiple yield curves are needed!

An (incomplete) overview of modeling approaches in the literature

- Multiple yield curve modeling:
  - Short rate approaches:
    Kijima et al. (2009), Kenyon (2010), Filipović and Trolle (2013), Morino and Runggaldier (2014), Graselli and Miglietta (2015);
  - LIBOR market model approaches:
    Mercurio (2010,...,2013), Grbac et al. (2015);
  - HJM approaches:
    Moreni and Pallavicini (2010), Pallavicini and Tarenghi (2010), Fujii et al. (2010,2011), Crépey et al. (2012,2015), C. Fontana and Gnoatto (2015).

  Recent books: Henrard (2014) and Grbac & Runggaldier (in preparation).
- Related approaches from the credit risk (defaultable bond) literature:
  Jarrow and Turnbull (1996), Douady and Jeanblanc (2002).

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- The multi-curve setting and our modeling approach
  - OIS rates and FRA rates.
  - A multi-curve extension of classical short rate modeling via multiplicative spreads.

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  - the OIS short rate and
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- Pricing and calibration within this model class
- Relation to other multi-curve models based on affine processes

OIS rates - the “risk-free” world

- Overnight Indexed Swap (OIS): a swap with a fixed leg versus a floating leg, given by a geometric average of the Eonia rates (rates for overnight borrowing in the EU)
- OIS rates are the market quotes for the fixed leg rates and are regarded as the best proxy for risk-free rates.

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- OIS rates are the market quotes for the fixed leg rates and are regarded as the best proxy for risk-free rates.
- By bootstrapping techniques, OIS rates allow to recover (at time t)
  - (risk-free) OIS zero-coupon bond prices: $T \mapsto B(t,T)$;
  - (risk-free) simply compounded OIS forward rates

$$T \mapsto L_t^D(T, T+\delta) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right).$$

  $L_t^D(T, T+\delta)$: “pre-crisis” simply compounded forward Libor rate. At the short end $T = t$, we call $L_T^D(T, T+\delta)$ simply compounded OIS spot rate.

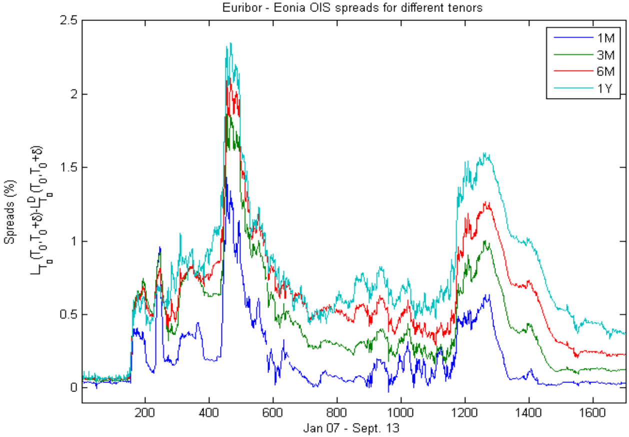
- We denote by $B_t = \exp(\int_0^t r_s ds)$(risk free) OIS bank account with OIS short rate $r_t := \lim_{T \downarrow t} \frac{\log B(t,T)}{T-t}$, which will serve as numéraire.

Libor rates and FRAs - the "risky" world

- $L_T(T, T+\delta)$: Libor rate at time T with maturity $T+\delta$;
  - trimmed average of rates at which interbank term deposits of length δ are being offered by one prime bank to another.
- FRA (Forward Rate Agreement): derivative exchanging at maturity $T+\delta$ the prevailing Libor rate $L_T(T, T+\delta)$ against a fixed rate K, i.e., its payoff at time $T+\delta$ is given by $L_T(T, T+\delta) - K$.
- Among all financial contracts written on Libor rates, FRAs can be rightfully considered (due to the simplicity of their payoff) as the most fundamental instruments and are also liquidly traded on the derivatives' market, especially for short maturities.
- FRAs are traded between fully collateralized counterparties $\Rightarrow$ market prices are so called "clean prices".

Euribor - OIS Eonia spreads

Additive spreads between Libor (Euribor) and simply compounded OIS spot rates $L_T(T, T+\delta) - L_T^D(T, T+\delta)$ from Jan. 2007 to Sept. 2013 for $\delta \in \{1/12, 3/12, 6/12, 1\}$.



The financial market

As basic traded assets in the market, we consider for all maturities (up to a finite time horizon $\mathbb{T}$)

- OIS zero-coupon bonds;
- FRA contracts for a family of tenors $\{\delta_1, \dots, \delta_m\}$.

- The FRA contracts are added to the market composed of all risk-free zero coupon bonds, because they cannot be perfectly replicated by the latter any longer.
- We suppose the existence of a risk neutral measure $\mathbb{Q}$ under which the OIS zero coupon bonds and FRA contracts denominated in units of the OIS bank account are martingales.
- This is a sufficient condition that guarantees "no asymptotic free lunch with vanishing risk" (see C., Klein and Teichmann (2015)), extending the notion "no free lunch with vanishing risk" to markets with uncountably many assets.

FRA rates

Definition (FRA rate)

$L_t(T, T+\delta) :=$ fair *FRA rate* at time t for the interval $[T, T+\delta]$:
rate K fixed at time t such that the value of the FRA becomes null:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{T+\delta}}\Big(L_T(T, T+\delta) - K\Big)\Big|\mathcal{F}_t\right] \stackrel{!}{=} 0.$$

where $\mathbb{Q}$ denotes a risk neutral measure (for the market with the OIS bank account (B_t) as numéraire.)

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By Bayes' formula: $L_t(T, T+\delta) = \mathbb{E}_{\mathbb{Q}^{T+\delta}}\left[L_T(T, T+\delta)|\mathcal{F}_t\right]$ where $\mathbb{Q}^{T+\delta}$ denotes a $(T+\delta)$-forward measure with $B(\cdot, T+\delta)$ as numéraire.

$\Rightarrow$ $\big(L_t(T, T+\delta)\big)_{t\in[0,T]}$ is a $\mathbb{Q}^{T+\delta}$-martingale, for all $T \geq 0$.

What are the quantities to be modeled?

The modeling approach we pursue can be subsumed under spot quantity modeling and can be viewed as multi-curve extension of classical short rate modeling.

Goal: simultaneous dynamic modeling of

1. the risk-free OIS short rate $(r_t)_{t\in[0,\mathbb{T}]}$;
2. Libor rates $(L_t^\delta(t, t+\delta))_{t\in[0,\mathbb{T}]}$ for $\delta \in \{\delta_1, \dots, \delta_m\}$.

In view of ensuring the following order relations

- $L_t(t, t+\delta_i) \geq L_t^D(t, t+\delta_i)$, for all $i = 1, \dots, m$;
- $L_t(t, t+\delta_i) \geq L_t(t, t+\delta_j)$ if $\delta_i > \delta_j$,

and tractability considerations, we model spreads, namely ...

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Multiplicative spot spreads between (normalized) Libor rates and (normalized) simply comp. OIS spot rates

$$S^\delta(t,t) := \frac{1+\delta L_t(t,t+\delta)}{1+\delta L_t^D(t,t+\delta)}, \qquad \text{for } \delta \in \{\delta_1, \dots, \delta_m\}.$$

Spot multi-curve models and multiplicative forward spreads

We consider the following type of models:

Definition

A spot multi-curve model consists of

1. a model for the (risk-free) OIS short rate r;
2. a model for the multiplicative spot spreads $\{(S^{\delta_i}(t,t))_{t\in[0,\mathbb{T}]}, i \in \{1,\dots,m\}\}$.

In such models the multiplicative forward spreads between (normalized) FRA rates and (normalized) simply comp. OIS forward rates defined as

$$S^{\delta}(t,T) := \frac{1+\delta L_t(T,T+\delta)}{1+\delta L_t^D(T,T+\delta)}, \qquad \text{for } \delta \in \{\delta_1,\dots,\delta_m\}.$$

(and then in turn the FRA rates $L_t^{\delta}(T,T+\delta)$) can be obtained as follows.

Multiplicative forward spreads as conditional expectations of spot spreads

Proposition

Let $\mathbb{Q}$ denote a risk neutral measure. Then the multiplicative forward spreads between (normalized) FRA rates and (normalized) simply comp. OIS forward rates $S^{\delta}(t, T)$ are given by

$$S^{\delta}(t, T) = \mathbb{E}^{\mathbb{Q}^T}[S^{\delta}(T, T)|\mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{Q}}[S^{\delta}(T, T)e^{-\int_t^T r_s ds}|\mathcal{F}_t]}{B(t, T)},$$

where $\mathbb{Q}^T$ denotes the T-forward measure, i.e., $\frac{d\mathbb{Q}^T}{d\mathbb{Q}}|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t B(0,T)}$.

Proof.

By the definition of the FRA rates, $(S^{\delta}(t, T))_{t\in[0,T]}$ is a $\mathbb{Q}^T$-martingale for every $\delta \in \{\delta_1, \ldots, \delta_m\}$ and $T \in [0, \mathbb{T}]$. $\square$

Preliminaries on affine processes - Notation

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq \mathbb{T}}, \mathbb{Q})$: filtered probability space where the measure $\mathbb{Q}$ is a risk neutral measure.
- D: closed convex subset of $\mathbb{R}^d$ (with scalar product $\langle \cdot, \cdot \rangle$);
- $\mathcal{X} = (\mathcal{X}_t)_{0\leq t\leq \mathbb{T}}$: adapted stochastic process on D with $\mathcal{X}_0 = x$.

We associate to the process $\mathcal{X}$ and the state space D the set

$$\mathfrak{U}_{\mathcal{X}} := \left\{ \zeta \in \mathbb{C}^d \,\middle|\, \mathbb{E}\left[e^{\langle \zeta, \mathcal{X}_t \rangle}\right] < \infty, \forall t \in [0, \mathbb{T}] \right\}$$

Definition of affine processes

Definition

The time-homogeneous conservative Markov Process $\mathcal{X}$ is called affine if

A1) it is stochastically continuous, i.e., the transition kernels satisfy, for every $t \geq 0$ and $x \in D$, $\lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly on D;

A2) its Fourier-Laplace transform has exponential-affine dependence on the initial state, i.e., there exists functions $\Phi : \mathbb{R}_+ \times \mathfrak{U}_{\mathcal{X}} \to \mathbb{C}$ and $\Psi : \mathbb{R}_+ \times \mathfrak{U}_{\mathcal{X}} \to \mathbb{C}^d$ such that, $\forall x \in D$ and $(t, u) \in \mathbb{R}_+ \times \mathfrak{U}_{\mathcal{X}}$

$$\mathbb{E}_x^{\mathbb{Q}}\left[e^{\langle u, \mathcal{X}_t \rangle}\right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\Phi(t,u) + \langle \Psi(t,u), x \rangle}.$$

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Remark

The characteristic exponents Φ and Ψ are solutions of certain generalized Riccati ODEs which are in a one-to-one correspondence with the semimartingale characteristics of X which are affine functions of X.

Definition of affine spot multi-curve models

- In the sequel X is an affine process with state space D_X on $(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,\mathbb{T}]}, \mathbb{Q})$ where $\mathbb{Q}$ is a risk neutral measure.

Definition

A spot multi-curve model is called affine if

1. the OIS short rate is an affine function in X, i.e., $r_t = l + \langle \lambda, X_t \rangle$ for $t \in [0, \mathbb{T}]$ and for some $l \in \mathbb{R}$ and $\lambda \in \mathbb{R}^d$;
2. the logarithm of the multiplicative spreads is an affine function in X, i.e., $\log(S^{\delta_i}(t,t)) = c_i + \langle u_i, X_t \rangle$ for $t \in [0, \mathbb{T}]$, $i \in \{1, \dots, m\}$ and for scalars $c_1, \dots c_m$ and vectors $u_1, \dots, u_m \in \mathbb{R}^d$.

The vectors $u_0 = 0, u_1, \dots, u_m \in \mathbb{R}^d$ are required to satisfy $(u_i, 1) \in \mathfrak{U}_{(X,Z)}$, for all $i = 0, 1, \dots, m$, where $\mathfrak{U}_{(X,Z)}$ is the set defined for the affine process $\mathcal{X} = (X, Z)$ with state space $D = D_X \times \mathbb{R}$, where

$$Z_t := -\log(B_t) = -\int_0^t r_s ds = -\int_0^t l + \langle \lambda, X_s \rangle ds.$$

Exponentially affine bond prices and spreads

Proposition (C.,Fontana, Gnoatto (2015))

A spot multi-curve model is affine if and only if it generates exponentially affine OIS bond prices and multiplicative forward spreads, i.e.

(i) *the prices of the OIS zero coupon bonds are given by*

$$B(t,T) = \exp\big(\mathcal{A}^0(T-t) + \langle \mathcal{B}^0(T-t), X_t\rangle\big), \qquad 0 \le t \le T < \mathbb{T},$$

for some differentiable functions $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$, and

(ii) *the multiplicative spreads are given by*

$$S^{\delta_i}(t,T) = \exp\big(\mathcal{A}^i(T-t) + \langle \mathcal{B}^i(T-t), X_t\rangle\big), \qquad 0 \le t \le T < \mathbb{T},$$

for some differentiable functions $\mathcal{A}^i(\cdot)$ and $\mathcal{B}^i(\cdot)$, for all $i = 1, \dots, m$.

Exponentially affine bond prices and spreads

Proposition (cont.)

In that case the functions $\mathcal{A}^i(\cdot)$ and $\mathcal{B}^i(\cdot)$, for all $i = 0, 1, \ldots, m$ are given by

$$\mathcal{A}^0(\cdot) = \phi(\cdot, 0, 1), \qquad \mathcal{B}^0(\cdot) = \psi(\cdot, 0, 1)$$

$$\mathcal{A}^i(\cdot) = c_i + \phi(\cdot, u_i, 1) - \phi(\cdot, 0, 1) \qquad \mathcal{B}^i(\cdot) = \psi(\cdot, u_i, 1) - \psi(\cdot, 0, 1),$$

where ϕ and ψ are the characteristic exponents of $\mathcal{X} = (X, Z)$, i.e. in this case

$$\mathbb{E}_{x,z}\left[e^{\langle \zeta, X_t \rangle + \eta Z_t}\right] = e^{\phi(t,u,v) + \langle \psi(t,u,v), x \rangle + \eta z}.$$

Exponentially affine bond prices and spreads

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Proof.

If the spot multi-curve model is affine, we have for the OIS bonds

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} | \mathcal{F}_t\right] = \exp\left(\phi(T-t, 0, 1) + \langle \psi(T-t, 0, 1), X_t\rangle\right).$$

Exponentially affine bond prices and spreads

Proof cont.

Concerning the spreads, we have

$$S^{\delta_i}(t,T) = \frac{\mathbb{E}^{\mathbb{Q}}[e^{c_i + \langle u_i, X_T\rangle} e^{-\int_t^T r_s ds} | \mathcal{F}_t]}{B(t,T)}$$
$$= \exp\left(c_i + \phi(T-t, u_i, 1) - \phi(T-t, 0, 1)\right)$$
$$\times \exp\left(\langle \psi(T-t, u_i, 1) - \psi(T-t, 0, 1), X_t\rangle\right).$$

Conversely, if a spot multi-curve model generates exponentially affine OIS bond prices and spreads, the affine property follows easily from the definition of the short rate and by setting $t = T$ in case of the spreads. $\square$

Ordered spreads

Proposition (C., Fontana, Gnoatto (2015))

Let the affine process X be of form $X = (X^0, Y)$ with state space $D_X = D_{X^0} \times C$ where C denotes some cone.

- *Suppose that $c^i \geq 0$ and that vectors u^i are of the form $u^i = (0, v^i)$ with $v^i \in C^*$, where C^* denotes the dual cone. Then $S^{\delta_i}(t, T) \geq 1$ for all $t \leq T$, $T \leq \mathbb{T}$, $i \in \{1, \dots, m\}$.*
- *Moreover, if additionally $c^1 \leq c^2 \cdots c^m$ and $v_1 \preceq v_2 \preceq \cdots v_m$, where $\preceq$ denotes the partial order on C^*, then we have $S^{\delta_1}(t, T) \leq \ldots \leq S^{\delta_m}(t, T)$ for all $t \leq T$, $T \leq \mathbb{T}$.*

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Proof.

The assertion is simply a consequence of the fact that $(S^{\delta_i}(t,T))_t$ are $\mathbb{Q}^T$ martingales given by $S^{\delta_i}(t,T) = \mathbb{E}^{\mathbb{Q}^T}[S^{\delta_i}(T,T)|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^T}[e^{c_i + \langle v_i, Y_T \rangle}|\mathcal{F}_t]$ and the monotonicity of the conditional expectation. $\square$

A deterministic shift extension - Definition

The formulation of an affine multi-curve model does not automatically fit the observed market prices of linear interest rate derivatives. $\Rightarrow$ Deterministic shift extension (Brigo and Mercurio (2001)) for an exact fit to the initially observed term structures.

Definition

For an affine spot multi-curve model (with OIS short rate r and multiplicative spreads $S^{\delta_i}(t,t)$ $i \in \{1,\dots,m\}$) we define a deterministic shift extension via

1. a (new) OIS short rate $\bar{r}$ given by,
$$\bar{r}_t = r_t + \varphi^0(t) = l + \langle \lambda, X_t \rangle + \varphi^0(t), \quad t \in [0, \mathbb{T}],$$
where φ^0 is a deterministic function such that $\int_0^{\mathbb{T}} |\varphi^0(t)| dt < \infty$.
2. (new) multiplicative spreads $\bar{S}^{\delta_i}(t,t)$ given by
$$\log(\bar{S}^{\delta_i}(t,t)) = \log(S^{\delta_i}(t,t)) + \varphi^i(t) = c_i + \langle u_i, X_t \rangle + \varphi^i(t), \quad t \in [0, \mathbb{T}],$$
where φ^i, $i \in \{1,\dots,m\}$ are deterministic functions.

A deterministic shift extension - Proposition

Proposition (C., Fontana, Gnoatto (2015))

Consider a deterministic shift extension of an affine spot multi-curve model. Then it provides an exact fit to the initially observed term structures if and only if the family of functions $(\varphi^0, \varphi^1, \ldots, \varphi^m)$ *satisfies*

$$\begin{aligned} \varphi^0(t) &= f_0^M(t) - f_0(t), && t \in [0, \mathbb{T}]; \\ \varphi^i(t) &= \log S^{M,\delta_i}(0,t) - \log S^{\delta_i}(0,t) && t \in [0, \mathbb{T}],\ i = 1, \ldots, m, \end{aligned}$$

where $f_0^M(t)$ *(*$f_0(t)$ *resp.) and* $S^{M,\delta_i}(0,t)$ *(*$S^{\delta_i}(0,t)$ *resp.) denote the instantaneous forward rate and multiplicative forward spreads of the market data (original model resp.). In that case, bond prices and multiplicative spreads in the deterministic shift extended model can be represented as*

$$\bar{B}(t,T) = \frac{B^M(0,T)}{B^M(0,t)} \frac{B(0,t)}{B(0,T)} B(t,T), \quad \bar{S}^{\delta_i}(t,T) = \frac{S^{M,\delta_i}(0,T)}{S^{\delta_i}(0,T)} S^{\delta_i}(t,T).$$

A deterministic shift extension - Remark

Remark (Ordering of spreads)

Assuming that the spreads $S^{\delta_i}(t,T)$ generated by the original model are greater than one and ordered, it then holds that

(i) *if* $S^{M,\delta_i}(0,t) \geq S^{\delta_i}(0,t)$ *for all* $t \in [0,\mathbb{T}]$, *then* $\bar{S}^{\delta_i}(t,T) \geq 1$ *for all* $0 \leq t \leq T \leq \mathbb{T}$;

(ii) *for any* $i,j \in \{1,\dots,m\}$ *with* $i<j$, *if*

$$\log S^{M,\delta_j}(0,t) - \log S^{M,\delta_i}(0,t) \geq \log S^{\delta_j}(0,t) - \log S^{\delta_i}(0,t)$$

for all $t \in [0,\mathbb{T}]$, *then* $\bar{S}^{\delta_j}(t,T) \geq \bar{S}^{\delta_i}(t,T)$, *for all* $0 \leq t \leq T \leq \mathbb{T}$.

Pricing of linear products

The prices of all linear interest rate products (i.e., without optionality features) such as

- Forward rate agreements
- Interest rate swaps
- Basis swaps

can be directly expressed in terms of the basic quantities $B(t, T)$ and $S^{\delta_i}(t, T)$. For instance, in the case of a FRA we have:

Proposition (C., Fontana, Gnoatto (2015))

The value at time t of a FRA with reset date T and settlement date $T + \delta_i$, fixed rate K and notional 1 is given by

$$
\begin{aligned}
\Pi_t^{FRA} &= \big(B(t, T)S^{\delta_i}(t, T) - B(t, T + \delta)(1 + \delta_i K)\big) \\
&= \big(\exp(c_i + \phi(T - t, u_i, 1) + \langle\psi(T - t, u_i, 1), X_t\rangle) \\
&\qquad - \exp(\phi(T + \delta - t, 0, 1) + \langle\psi(T + \delta - t, 0, 1), X_t\rangle)(1 + \delta_i K)\big).
\end{aligned}
$$

Pricing of caplets

Proposition (C., Fontana, Gnoatto (2015))

Consider an affine spot multi-curve model. Let $\bar{K} := 1 + \delta_i K$ *where* K *denotes the strike rate. Then the price at time* t *of a caplet with notional* 1, *reset date* T, *and settlement date* $T + \delta_i$, *is given by*

$$\Pi_t^{CPLT} = B_t \left(\frac{1}{2} \varphi_{t,\mathcal{Y}_T}(-\mathrm{i}) + \frac{1}{\pi} \int_0^\infty \Re \left(e^{-\mathrm{i}\zeta \log(\bar{K})} \frac{\varphi_{t,\mathcal{Y}_T}(\zeta - \mathrm{i})}{-\zeta(\zeta - \mathrm{i})} \right) d\zeta \right),$$

where $\mathcal{Y}_T := c_i - \phi(\delta_i, 0, 1) + \langle u_i - \psi(\delta_i, 0, 1), X_T \rangle$ *and*

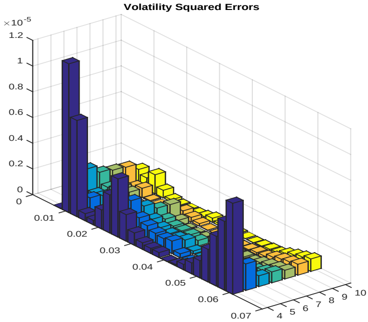
$$\begin{aligned}
\varphi_{t,\mathcal{Y}_T}(\zeta) &:= \mathbb{E}\left[\frac{B(T, T+\delta)}{B_T} e^{\mathrm{i}\zeta \mathcal{Y}_T} \middle| \mathcal{F}_t \right] \\
&= \exp\left((1 - \mathrm{i}\zeta)\phi(\delta_i, 0, 1) + \mathrm{i}\zeta c_i + Z_t \right) \\
&\quad \times \exp\left(\phi\left(T - t, \mathrm{i}\zeta u_i + (1 - \mathrm{i}\zeta)\psi(\delta_i, 0, 1), 1 \right) \right) \\
&\quad \times \exp\left(\left\langle \psi\left(T - t, \mathrm{i}\zeta u_i + (1 - \mathrm{i}\zeta)\psi(\delta_i, 0, 1), 1 \right), X_t \right\rangle \right).
\end{aligned}$$

A (preliminary) calibration example

- Goal: Model calibration to the initial term structure of OIS bonds and $L_0^M(T, T+\delta)$ (or equivalently $S^{M,\delta}(0,T)$) for $\delta = 6M$ as well as a surface of (Bachelier) implied caplet volatilities
- Model: deterministic shift extension of the following basic model

$$r_t = l + \lambda X_t^1, \quad \log S^\delta(t,t) = u_1^1 X_t^1 + u_1^2 X_t^2,$$

where X^1 is a CIR process and X^2 a Gamma process.



A tractable affine spot multi-curve model based on Wishart processes

- Let X be a Wishart process in S_2^+ and of the form

$$dX_t = \left(\kappa Q^\top Q + MX_t + X_t M^\top\right) dt + \sqrt{X_t} dW_t Q + Q^\top dW_t^\top \sqrt{X_t},$$

  where W is 2×2 matrix of Brownian motions, $\kappa \geq 1$ and Q, M are 2×2 matrices and with $M_{21} = 0$.
- Let the OIS short rate be given by $r = \lambda_2 X_{22}$ with $\lambda_2 > 0$ and the log multiplicative spreads by $\log S^\delta(t, t) = u_1 X_{11,t}$ with $u_1 > 0$.
- Since the diagonal elements of Wishart processes are stochastically correlated CIR processes this is a natural extension of the classical CIR model to the multiple curve setting.

Closed form expression for caplets in the Wishart model

Theorem (C., Fontana, Gnoatto (2015))

Let X be a Wishart process as above. Consider an affine spot multi-curve model with

- *OIS short rate $r_t = \lambda_2 X_{22,t}$ with $\lambda_2 > 0$,*
- *log multiplicative spread $\log S^{\delta}(t,t) = u_1 X_{11,t}$ with $u_1 > 0$.*

The of a caplet with strike K, reset date T and settlement date $T+\delta$ is given by

$$\Pi_0^{CPLT} = S^{\delta}(0,T)B(0,T)(1-\widetilde{F}_T(C)) - (1+\delta K)B(0,T+\delta)(1-F_T(C)),$$

- *$C = \log(1+\delta K) + \phi(\delta,0,1)$ where ϕ is the constant part of the characteristic exponent of $(X,Z) = (X, -\int_0^{\cdot} \lambda_2 X_{22,s} ds)$;*
- *F_T and $\widetilde{F}_T$ denote the cumulative distribution function of a weighted sum of two independent non-centrally χ^2-distributed random variables corresponding to $\mu_{1,T}Y_{1,T} + \mu_{2,T}Y_{2,T}$ and $\widetilde{\mu}_{1,T}\widetilde{Y}_{1,T} + \widetilde{\mu}_{2,T}\widetilde{Y}_{2,T}$, where*

Closed form expression for caplets in the Wishart model

Theorem (cont.)

- *the weights $\mu_{i,T}$ and $\widetilde{\mu}_{i,T}$ are the (positive) eigenvalues of*

$$\sqrt{V(T)}\,\mathrm{diag}(u_1, -\psi_{22}(\delta, 0, 1))\sqrt{V(T)}$$
$$\sqrt{\widetilde{V}(T)}\,\mathrm{diag}(u_1, -\psi_{22}(\delta, 0, 1))\sqrt{\widetilde{V}(T)}$$

with

$$V(T) := \int_0^T \exp\left(\int_t^T A_s ds\right) Q^\top Q \exp\left(\int_t^T A_s^\top ds\right) dt,$$
$$\widetilde{V}(T) := \int_0^T \exp\left(\int_t^T \widetilde{A}_s ds\right) Q^\top Q \exp\left(\int_t^T \widetilde{A}_s^\top ds\right) dt,$$
$$A_s := M + 2\psi\,(T - s, 0, 1)\, Q^\top Q,$$
$$\widetilde{A}_s := M + 2\psi\left(T - s, \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix}, 1\right) Q^\top Q.$$

Closed form expression for caplets in the Wishart model

Theorem (cont.)

- $Y_{i,T} \sim \chi^2(\kappa, y_{ii,T})$ and $\widetilde{Y}_{i,T} \sim \chi^2(\kappa, \widetilde{y}_{ii,T})$ *where*

$$
\begin{aligned}
y_T &= (O_T^\top V(T)^{-\frac{1}{2}} \Psi^\top(T) x \Psi(T) V(T)^{-\frac{1}{2}} O_T),\\
\widetilde{y}_T &= (\widetilde{O}_T^\top \widetilde{V}(T)^{-\frac{1}{2}} \widetilde{\Psi}^\top(T) x \widetilde{\Psi}(T) \widetilde{V}(T)^{-\frac{1}{2}} \widetilde{O}_T),\\
\Psi(T) &= \exp\left(\int_0^T \left(M^\top + 2Q^\top Q \psi\left(T-t, 0, 1\right) dt\right)\right),\\
\widetilde{\Psi}(T) &= \exp\left(\int_0^T \left(M^\top + 2Q^\top Q \psi\left(T-t, \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix}\right), 1\right) dt\right),
\end{aligned}
$$

and O_T *and* $\widetilde{O}_T$ *are the orthogonal matrices of*

$$
\begin{aligned}
\sqrt{V(T)}\,\mathrm{diag}(u_1, -\psi_{22}(\delta, 0, 1))\sqrt{V(T)} &= O_T\,\mathrm{diag}(\mu_1, \mu_2) O_T^\top,\\
\sqrt{\widetilde{V}(T)}\,\mathrm{diag}(u_1, -\psi_{22}(\delta, 0, 1))\sqrt{\widetilde{V}(T)} &= \widetilde{O}_T\,\mathrm{diag}(\mu_1, \mu_2) \widetilde{O}_T^\top.
\end{aligned}
$$

Computational aspects

The pricing of a caplet in the above model thus requires

- the knowledge of ψ and integration (on the $[0, T]$) to obtain $V(T)$, $\widetilde{V}(T)$, $\Psi(T)$, $\widetilde{\Psi}(T)$,
- eigenvalue and eigenvector computations of two symmetric matrices
- computation of the distribution function of a positive weighted sum of two independent non-centrally χ^2-distributed random variables, e.g. via an Laguerre series expansions (e.g. Martinez and Blazques (2005)) of the form

$$F_T(C) = F_T(C, \kappa, y_T, \mu_T) = g(\kappa, C) \sum_{j=0}^{n} \alpha_j(\kappa, y_T, \mu_T) L_j^{(\kappa)}(kC),$$

where g is a function depending on κ, C and α_j are the coefficients (depending on κ, y_T, μ_T) of the Laguerre polynomials $L_j^{(\kappa)}$ (evaluated at kC where k denotes a constant).

Relation to other multi-curve models based on affine models

- Affine short rate models for both OIS bond prices and risky (artificial) bond prices e.g., by
  - Kenyon,
  - Kijima et al.,
  - Morino and Runggaldier,
  - Grasselli and Miglietta

  can be embedded in the present affine multi-curve spot model approach.
- Multi-curve affine Libor models recently proposed by Grbac et al. can be embedded in a general HJM framework via a natural extension to a continuum of maturities and are also related to the present affine multi-curve spot model approach.

Conclusion

- We propose a spot multi-curve model approach based on affine processes, where
  - the OIS short rate and
  - (logarithmic) multiplicative spreads between LIBOR rates and simply compounded OIS rates

  are modeled as an affine functions of a common affine process.

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Thank you for your attention

For more information:

C. C., C. Fontana and A. Gnoatto (2015), Affine multiple yield curve models, in preparation.