

# Rational Multi-Curve Models with Counterparty-Risk Valuation Adjustments

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*Joint work with A. Macrina, T. M. Nguyen and D. Skovmand*

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# Outline

- 1 Post-Crisis Interest Rate Markets and Models
- 2 Rational Multi-Curve Models
- 3 Rational Bilateral Counterparty Risk Model
- 4 Conclusion

# Libor

Most interest-rate derivatives have Libor-indexed cash-flows (Libor fixings)

## What is Libor?

- Libor stands for London InterBank Offered Rate. It is produced for 10 currencies with 15 maturities quoted for each, ranging from overnight to 12 Months producing 150 rates each business day. Libor is computed as a trimmed average of the interbank borrowing rates assembled from the Libor contributing banks.
- More precisely, every contributing bank has to submit an answer to the following question: "At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am?"

# OIS

In most currencies there is also an interbank market of overnight loans, at a rate dubbed OIS (spot) rate in reference to the related swap market

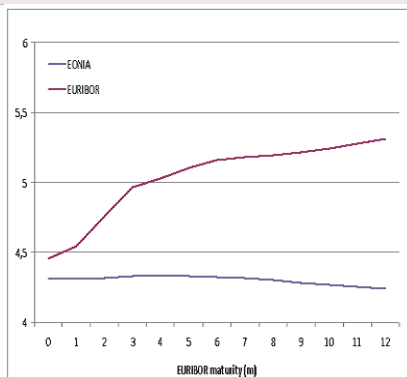
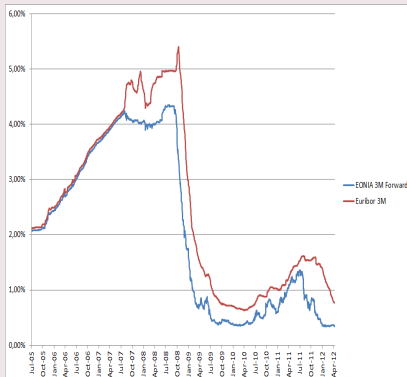
- In some currencies the OIS rate (like the Eonia rate for the euro) can be viewed as a short-tenor limit of Libor
- In others (like US dollar) this view is simplistic since the panel of the Libor and of the OIS rate is not the same, and the OIS rate reflects actual transaction rates (as opposed to a purely collected Libor)

# LOIS

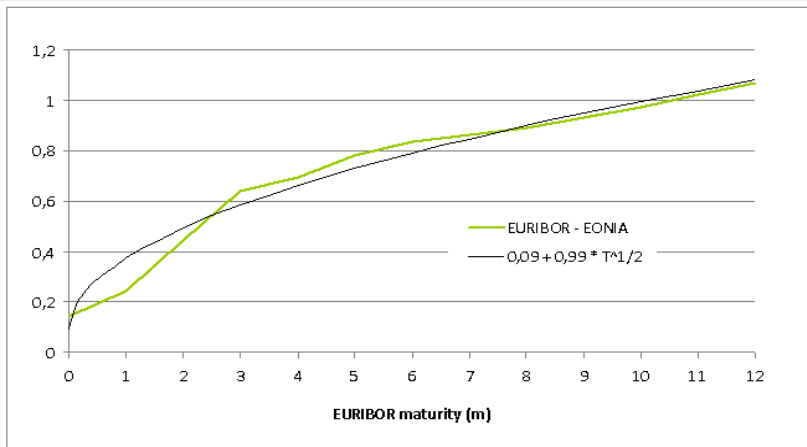
Divergence Euribor (“*L*”) / Eonia-swap (“*R*”) rates

*Left:* Sudden divergence between the 3m Euribor and the 3m Eonia-swap rate that occurred on Aug 6 2007

*Right:* Term structure of Euribor vs Eonia-swap rates, Aug 14 2008

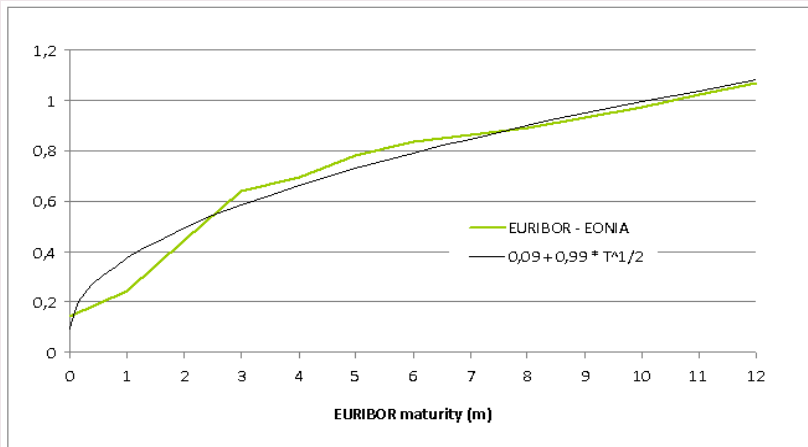


## Square root fit of the LOIS corresponding to the data of Aug 14 2008



Crépey, S. and Douady, R.: LOIS: Credit and Liquidity. *Risk Magazine* June 2013.

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# Clean Valuation vs Adjustments

## Interaction between the multiple-curve and the counterparty risk/funding issues

- **Clean valuation** = derivation of a “fully collateralized” price  $P_t$  at an OIS collateral rate
  - Fully collateralized at an OIS collateral rate → no CVA/DVA/FVA
  - OIS discounting versus Libor fixings → **Multiple-curve**
- Computation of a **CVA+DVA+FVA=TVA** correction  $\Theta_t$  to account for counterparty risk and excess-funding costs
  - $\Theta_0$  = price of a dividend-paying option on  $P_\tau$ 
    - $\tau$  (first) default time of a party
    - Dividends Excess-funding benefit/cost



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# Multiple-Curve Clean Valuation of Interest Rate Derivatives

“Classical” clean valuation formula  $\beta_t P_t = \mathbb{E} \left( \int_t^T \beta_s dD_s \mid \mathbb{G}_t \right)$  with

$$\beta_t = e^{-\int_0^t r_s ds}$$

- Libor fixings  $dD_t$
- Appropriate choice of the OIS rate as the clean discount rate  $r_t$ 
  - Perverse incentives for traders otherwise
  - Calibration constraints to market data = clean prices discounted at OIS

## OIS discounting versus Libor fixings

- In a multiple curve environment one loses the usual consistency between discounting and fixing of classical one-curve interest rates models
- Increased complexity of clean valuation of Libor derivatives
- Also more degrees of freedom for the calibration...

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# Multiple Curve (Clean Valuation) Models

- Short-rate model of Kenyon (2010)
- Affine (short-rate) interbank risk model of Filipović and Trolle (2011)
- Market models of Mercurio (2009, 2010)
  - Setting a new market standard in terms of the **FRA rates**  $L_t^{T,S}$
- Market model of Bianchetti (2009)
  - Cross-currency mathematical analogy
- HJM multi-currency model of Fujii et al. (2010)
  - Choice of collateral currency and cheapest-to-deliver option
- Hybrid HJM-Market “parsimonious” models of Moreni and Pallavicini (2010 and 2012)
  - Best of both worlds?
- Defaultable HJM model of Crépey, Grbac and Nguyen (2011)
- HJM Lévy driven model of Crépey, Grbac, Ngor and Skovmand (2013)
- General HJM framework of Cuchiero, Fontana and Gnoatto (2014)

## Markovian perspective



- With TVA in mind, “static” **calibrability** is not the only clean valuation tractability issue
- **TVA  $\sim$  option on  $P_\tau$**   $\rightarrow$  Tractability should also be considered at the dynamic level of plugging a clean price process  $P_t$  into a TVA Monte-Carlo engine
  - American Monte Carlo valuation of the TVA  $\Theta_t$  and sometimes even of its “underlying”  $P_t$ 
    -  Cesari, G. et al.: *Modelling, Pricing, and Hedging Counterparty Credit Exposure*. Springer Finance, 2010.
  - Marked branching particles
    -  Henry-Labordère, P.: Cutting CVA's complexity, *Risk* 2012.

$\rightarrow$  Markovian perspective on a clean price process  $P_t$  also key

## Affine diffusions, Lévy drivers,...???

- Tractable calibration possible either with affine diffusions or by means of (possibly time-inhomogenous) Lévy drivers
- Less factors should be the focus with TVA in mind

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## Rational (often Pricing Kernel based) Models

Discount bond price processes and associated short interest rate models with **rational form**.

- Flesaker & Hughston (1996) introduced the so-called rational log-normal model for discount bond prices. Their approach to interest-rate modelling can be used to develop rational models based on a generic martingale.
- For further early contributions and studies in this context we refer to Rogers (1997), Rutkowski (1997), Musiela & Rutkowski (1997), Hunt & Kennedy (2000), Jin & Glasserman (2001), etc.
- More recent work on rational pricing models include Brody & Hughston (2004), Hughston & Rafailidis (2005), Brody, Hughston & Mackie (2012), Akahori, Hishida, Teichmann & Tsuchiya (2014), Filipović, Larsson & Trolle (2014), Macrina (2014) and Macrina & Parbhoo (2014).

- So far, most of the work on (linear-)rational interest rate and pricing (kernel) models has focused on the **relevance of these models from the viewpoint of economics**.
  - The **transparent relation between model specifications under the risk-neutral and the real-world probability measures** provided by the underlying pricing kernel structure has been appreciated for some time.
- This work emphasizes the appeal of these models also from a **financial engineering perspective**
  - Especially in connection with the **post-crisis multi-curve and counterparty risk issues**.



S. Crépey, A. Macrina, T. M. Nguyen and D. Skovmand: A Lévy Rational Multi-Curve Models with Counterparty-Risk Valuation Adjustments. Forthcoming in *Quantitative Finance*.



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 $\{A_t^{(i)}\}$ : zero-initialised  $(\{\mathcal{F}_t\}, \mathbb{M})$ -martingales of the form  $A(t, X_t^{(i)})$ .  
 $\{X_t^{(i)}\}$ :  $(\{\mathcal{F}_t\}, \mathbb{M})$ -Markov processes.
- **Rational pricing measure**  $\mathbb{M}$  such that prices discounted at the **rational discount factor**  $h_t = c_1(t) + b_1(t)A_t^{(1)}$  follow  $\mathbb{M}$  martingales  
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 $c_1(t) = P_{0t}$
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# Swaption Clean Valuation

- Interest rate swap: exchange of two streams of future interest payment (a fixed rate against a LIBOR rate) based on a nominal  $N$ .

$$Sw_t = \sum_{i=1}^n N\delta[KP_{tT_i} - L(t; T_{i-1}, T_i)], \quad t \leq T_0.$$

- Swaption**: option between two parties to enter the above swap at the expiry  $T_k$  (maturity date of the option).

$$Sw_{ntT_k} = \frac{1}{h_t} \mathbb{E}^M[h_{T_k}(Sw_{T_k})^+ | \mathcal{F}_t].$$

- For  $A_t^{(i)} = S_t^{(i)} - 1$ , where  $S_t^{(i)}$  is a positive  $\mathbb{M}$ -mart. with  $S_0^{(i)} = 1$ 
  - unit-initialised exponential Lévy martingale  $S_t^{(i)}$

$$Sw n_{0T_k} = N\delta \mathbb{E}^{\mathbb{M}} \left[ \left( c_2 S_{T_k}^{(2)} + c_3 S_{T_k}^{(3)} - c_1 S_{T_k}^{(1)} + \tilde{c}_0 \right)^+ \right],$$

where

$$c_2 = \sum_{i=k+1}^m b_2(T_{i-1}, T_i), \quad c_3 = \sum_{i=k+1}^m b_3(T_{i-1}, T_i), \quad c_1 = K \sum_{i=k+1}^m b_1(T_i),$$

$$c_0 = \sum_{i=k+1}^m [L(0; T_{i-1}, T_i) - KP_{0T_i}], \quad \tilde{c}_0 = c_0 + c_1 - c_2 - c_3.$$

- **One-factor lognormal** model:

$$A_t^{(1)} = A_t^{(3)} = 0, A_t^{(2)} = \exp \left( a_2 X_t^{(2)} - \frac{1}{2} a_2^2 t \right) - 1$$

→ **BS** swaption pricing formula

- **Univariate NIG** model:

$$A_t^{(1)} = A_t^{(3)} = 0, S_t^{(2)} = e^{X_t^{(2)} - t \psi_2(1)}, \{X_t^{(2)}\}: \text{Lévy process with cumulant } \psi_2$$

→ **univariate Fourier** transform swaption pricing formula

- **Two-factor lognormal** model:

$$A_t^{(i)} = \exp \left( a_i X_t^{(i)} - \frac{1}{2} a_i^2 t \right) - 1, i = 1, 2, 3 \text{ for real constants } a_i \text{ and standard Brownian motions } \{X_t^{(1)}\} = \{X_t^{(3)}\} \text{ and } \{X_t^{(2)}\} \text{ with correlation } \rho.$$

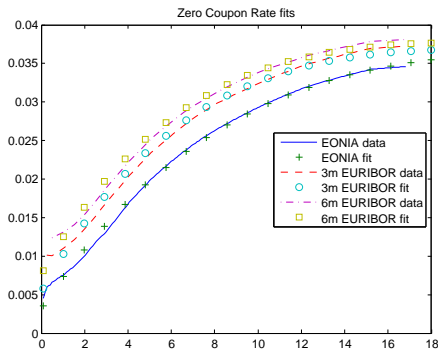
→ **bivariate log-normal** swaption pricing formula with root-finding

# Model Calibration

- **Market data:** EUR market Bloomberg data of January 4, 2011
  - Eonia, 3m Euribor and 6m Euribor initial term structures
  - 3m and 6m tenor Libor swaptions
- **Initial term structures fitted by construction**
- First phase (**swaption smile calibration**): calibrate the nonmaturity/tenor dependent parameters (parameters of the driving martingales  $A^{(2)}$ ) to the smile of the  $9y \times 1y$  swaption with (most liquid) tenor  $\delta = 3m$ . This phase also gives us the values of  $b_2(9, 9.25)$ ,  $b_2(9.25, 9.5)$ ,  $b_2(9.5, 9.75)$  and  $b_2(9.75, 10)$ , which we assume to be equal.
- Second phase (**ATM swaption term structure calibration**): use at-the-money swaptions data with tenor  $\delta = 3m$  and  $6m$ , termination  $T_n = 10$  years and expiries  $T_k$  ranging from 1 to 9 years.

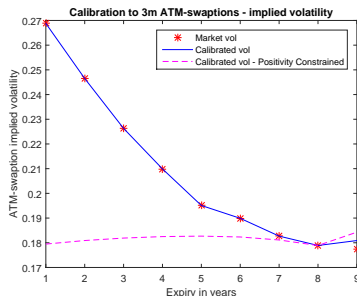
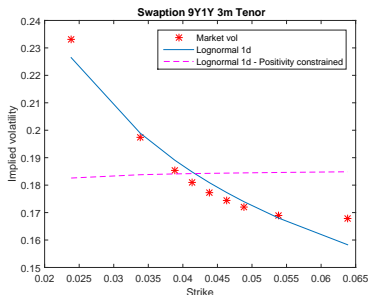
# Initial term structures

- Bloomberg EUR market data from the 4th of January 2011
- Regularized initial Eonia, 3m-Euribor and 6m-Euribor term structures fitted to Nelson-Siegel-Svensson parameterizations



# Swaption calibration of the one-factor lognormal model

3m tenor: 9Y1Y smile and  $\Delta Y(10 - \Delta)Y$  ATM

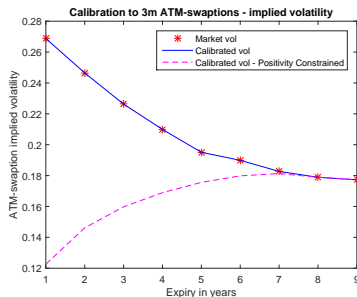
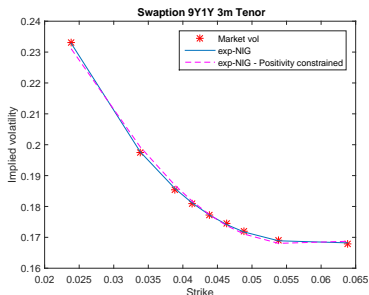


- unconstrained model achieves a reasonably good calibration, but not satisfactory.
- enforcing positivity leads to a poor fit to the data.
- while positivity of rates and spreads are not achieved, the model assigns only small probabilities to negative values.



# Swaption calibration of exp-NIG model

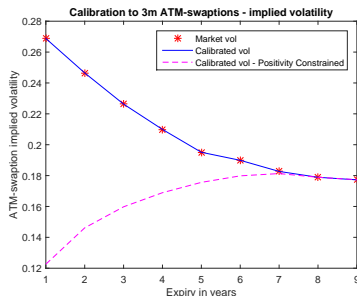
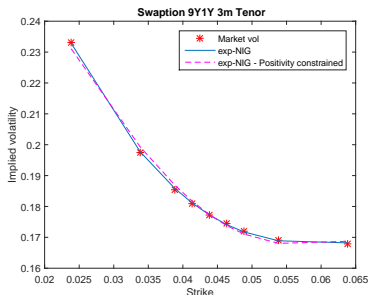
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- unconstrained model achieves a good calibration.
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- volatility structure cannot be made to match swaptions with maturity smaller than 7 years.
- without enforcing positivity, the model assigns an unrealistically high probability mass to negative values.

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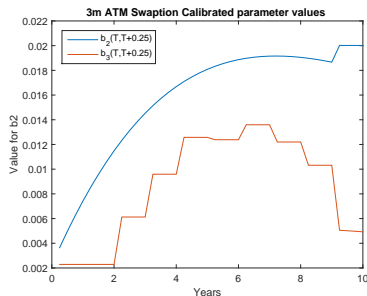
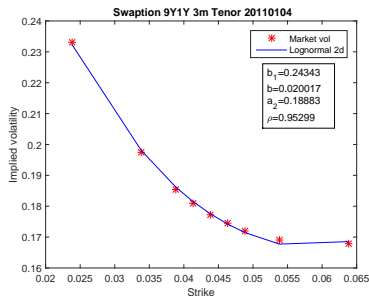
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# Swaption calibration of a two-factor lognormal model

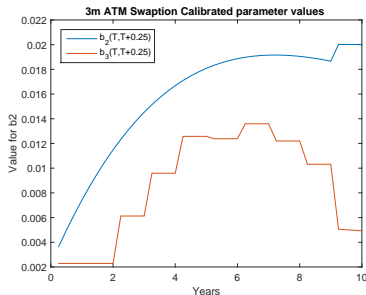
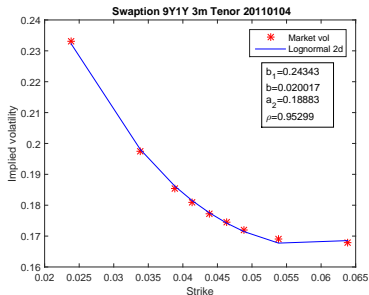
3m tenor: 9Y1Y smile (left) and  $b_2$  and  $b_3$  functions (right)



- The quality of the fit appears quite satisfactory and comparable to the unconstrained exponential-NIG calibration.
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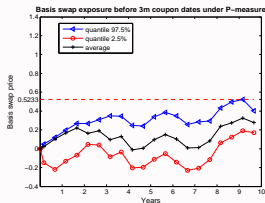
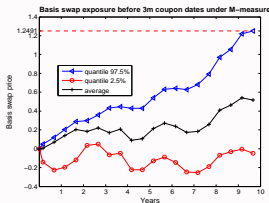


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# Potential Future Exposure

- For regulatory purposes, an important risk measure is the potential future exposure (PFE), i.e. the maximum of the 97.5% quantile curve of the exposure (positive part of the price process, if there is no collateral involved)
- In principle, this should be computed with respect to the statistical measure  $\mathbb{P}$
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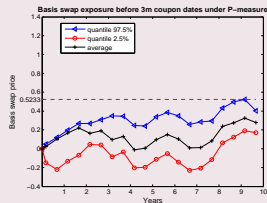
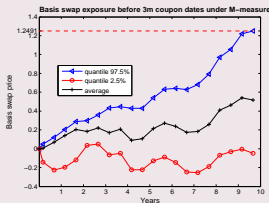
## PFE of a Basis Swap



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## PFE of a Basis Swap



# Outline

- 1 Post-Crisis Interest Rate Markets and Models
- 2 Rational Multi-Curve Models
- 3 Rational Bilateral Counterparty Risk Model**
- 4 Conclusion

- $\{X_t^{(i)}\}$ ,  $i = 4, 5, 6$ : Markov processes assumed to be  $\mathbb{M}$ -independent between them and of the Markov processes  $i = 1, 2, 3$ .
- $h_t = c_1(t) + b_1(t)A_t^{(1)}$  as before,  $k_t = \prod_{i \geq 4} k_t^{(i)}$  where  $k_t^{(i)} := c_i(t) + b_i(t)A_t^{(i)}$
- **Rational pricing measure**  $\mathbb{M}$  such that (pre-default) prices discounted at the **rational discount factor**  $h_t k_t$  follow  $\mathbb{M}$  martingales
  - $\tau_c = \tau_4 \wedge \tau_6$ ,  $\tau_b = \tau_5 \wedge \tau_6$ , hence  $\tau = \tau_b \wedge \tau_c = \tau_4 \wedge \tau_5 \wedge \tau_6$ , where  $\tau_4, \tau_5$  and  $\tau_6$  are independent exponential times  
 $\rightarrow \gamma_t^c = \gamma_t^{(4)} + \gamma_t^{(6)}$ ,  $\gamma_t^b = \gamma_t^{(5)} + \gamma_t^{(6)}$ ,  $\gamma_t = \gamma_t^{(4)} + \gamma_t^{(5)} + \gamma_t^{(6)}$
- OIS discount bond price  $P_{tT}$  and LIBOR processes  $L(t; T_{i-1}, T_i)$  as before



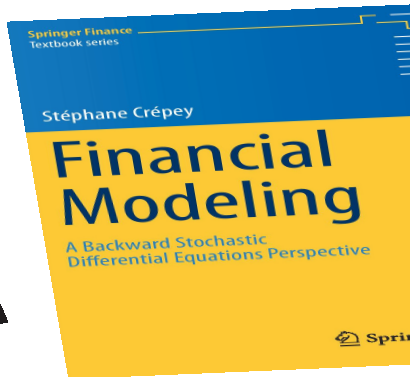
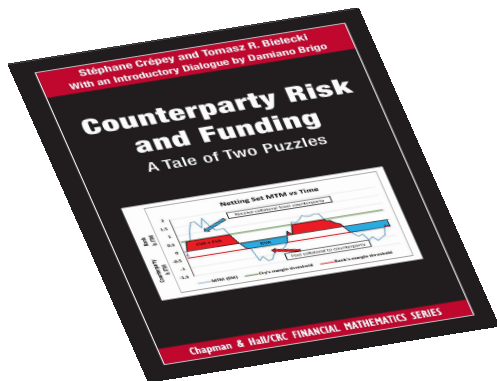
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$$h_t k_t \Theta_t = \mathbb{E}_t^{\mathbb{M}} \left[ \int_t^T h_s k_s f_s(\Theta_s) ds \right], \quad t \in [0, T], \text{ where}$$

$$f_t(\vartheta) = cva_t - dva_t + fva_t(\vartheta)$$

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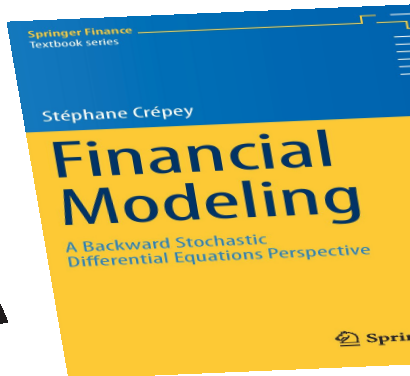


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# TVA Computations

$$\begin{aligned}
 f_t(\vartheta) = & \underbrace{\gamma_t^c(1 - R_c)(P_t - \Gamma_t)^+}_{\text{costly Credit Valuation Adjustment (CVA)}} \\
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- $\tilde{\lambda}_t := \bar{\lambda}_t - \gamma_t^b \Lambda$  External borrowing basis net of the credit spread  
 $\Lambda$  Loss given default of the funder of the bank  
 $\tilde{\lambda}$  Liquidity borrowing funding basis
- The positive (negative) TVA terms can be considered as “deal adverse” (“deal friendly”) as they increase the TVA and therefore decrease the (buying) price for the bank

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## Basis Swap Example

- $\gamma^b = 5\%$ ,  $\gamma^c = 7\%$ ,  $\gamma = 10\%$ ,  $R_b = R_c = 40\%$ ,  $b = \bar{b} = \lambda = \tilde{\lambda} = 1.5\%$ . Both legs of the basis swap are equal to EUR 27.96.

m	Regr TVA	CVA	DVA	FVA	Sum	MC TVA
$10^4$	0.0447	0.0614	-0.0243	0.0067	0.0438	0.0438
$10^5$	0.0443	0.0602	-0.0234	0.0067	0.0435	0.0435

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- Models with rational form provide particularly tractable interest rate models, which can be readily extended to **multi-curve rational interest rate models** while retaining tractability. These models
  - can be **efficiently calibrated** to swaption data
  - are **particularly easy to simulate** since their market factors are deterministic functions of basic processes such as Brownian motions,
  - **require no jumps** to be introduced in their dynamics in order to achieve acceptable calibration accuracy (traders dislike models with jumps from a hedging perspective).
- The same class of rational models allows for the development of manageable **rational credit-intensity models** necessary for the analysis of counterparty-risk valuation adjustments.
  - The **transparent relation between the measures  $\mathbb{P}$  and  $\mathbb{Q}$**  can be employed to derive sound risk-measure computations (under  $\mathbb{P}$ ) that are consistent with counterparty-risk pricing adjustments (under  $\mathbb{Q}$ ).