

Model-independent bounds for Asian options

A dynamic programming approach

Alexander M. G. Cox¹ Sigrid Källblad²

¹University of Bath

²CMAP, École Polytechnique

7th General AMaMeF and Swissquote Conference
EPFL, 9th August, 2015



Model-independent bounds for option prices

- Aim: make statements about the price of options given very mild modelling assumptions
- Incorporate market information by supposing the prices of vanilla call options are known
- Typically want to know the largest/smallest price of an exotic option (Lookback option, Barrier option, Variance option, Asian option, . . .) given observed call prices, but with (essentially) no other assumptions on behaviour of underlying

- Option priced on an asset $(S_t)_{t \in [0, T]}$, option payoff X_T
- Dynamics of S unspecified, but suppose paths are continuous, and we see prices of call options at all strikes K and at maturity time T
- Assume for simplicity that all prices are discounted — this won't affect our main results
- Under risk-neutral measure, S should be a (local-)martingale, and we can recover the law of S_T at time T , μ say, from call prices $C(K)$

Existing Literature

Rich literature on these problems:

- Starting with Hobson ('98) connection with Skorokhod Embedding problem $\rightarrow$ explicit optimal solutions for many different payoff functions (Brown, C., Dupire, Henry-Labordère, Hobson, Klimmek, Obłój, Rogers, Spoida, Touzi, Wang, . . .)
- More recently, model-independent duality has been proved by Dolinsky-Soner ('14):

$$\sup_{\mathbb{Q}: S_T \sim \mu} \mathbb{E}^{\mathbb{Q}}[X_T] = \text{price of cheapest super-replication strategy}$$

Here the super-replication strategy will use both calls and dynamic trading in underlying, and is **model-independent**. The sup is taken over measures $\mathbb{Q}$ for which S is a martingale. (See also Hou-Obłój and Beiglböck-C.-Huesmann-Perkowski-Prömel)

- The problem of finding the martingale S which maximises the expectation above is commonly called the **Martingale Optimal Transport** problem (MOT)

Explicit solutions and ‘convexity’

To date, explicit solutions to MOT have largely been constructed using the connection to the Skorokhod Embedding problem (SEP):

- Since S is a (continuous) martingale, it is the (continuous) time change of a Brownian motion, $S_t = B_{\tau_t}$. When the option payoff function X_T is independent of the time-scale (e.g. maximum), then choosing a model for S with $S_T \sim \mu$ is equivalent to finding a stopping time τ_T such that $B_{\tau_T} \sim \mu$ (the SEP).
- Finding a given model which maximises $\mathbb{E}^Q[X_T]$ corresponds to finding a solution to the SEP with a certain optimality property
- Needs payoff to be invariant under time changes
- Historically, an optimal solution was produced using ad-hoc methods. In Beiglböck-C.-Huesmann, this was formalised in a **monotonicity principle**
- The monotonicity principle captures a certain type of convexity — essentially all known optimal solutions to the SEP exploit this convexity

Model-independent bounds for Asian options

- In this talk, want to consider Asian options, $X_T = F\left(\int_0^T S_r dr\right)$ for **arbitrary** F
- The SEP methodology will not be effective: $A_T = \int_0^T S_r dr$ is very dependent on the choice of the time-change.
- In the case of convex F , Jensen gives an easy solution: essentially, jump immediately to final law (see also Stebegg ('14))
- Our methods are not specific to Asian options: should(!) generalise

Dynamic programming approach

- One of the difficulties inherent in constructing solutions to the Martingale Optimal Transport problem (MOT) is that 'local' optimal behaviour is driven by the 'global' requirement that $S_T \sim \mu$
- Key idea: 'localise' the condition on the terminal law
- At any given time the 'state' of our process should be enhanced to include also the conditional terminal law:

$$\xi_t(A) = \mathbb{P}(S_T \in A | \mathcal{F}_t), \quad \xi_0 = \mu$$

Note: implies $\xi_t(A)$ is a martingale for any A

- Model ξ_t rather than S_t — S_t can be recovered by $S_t = \int x \xi_t(dx)$

Measure valued martingales

- Introduce the set of integrable probability measures:

$$\mathcal{P}^1 := \{\mu \in \mathcal{M}(\mathbb{R}_+) : \mu(\mathbb{R}_+) = 1, \int |x| \mu(dx) < \infty\},$$

and the set of singular probability measures:

$$\mathcal{P}^s = \{\mu \in \mathcal{M}(\mathbb{R}_+) : \mu = \delta_y, y \in \mathbb{R}_+\}$$

- We say an adapted process $\xi_t \in \mathcal{P}^1$ is a **measure-valued martingale** (MVM) if for any $f \in C_b(\mathbb{R}_+)$, $\xi_r(f)$ is a martingale
- An MVM (on $[0, T]$) is terminating if $\xi_T \in \mathcal{P}^s$
- See Horowitz ('85); Walsh ('86); Dawson ('91); Eldan ('13).

Examples of measure valued martingale

The previous result tells us that we can construct an MVM from any (suitably) stopped Brownian motion:

- Exit from an interval: let $a < 0 < b$, $H := \inf\{t \geq 0 : B_t \notin (a, b)\}$, then

$$\xi_t := \frac{b - B_{t \wedge H}}{b - a} \delta_a + \frac{B_{t \wedge H} - a}{b - a} \delta_b$$

- ‘Bass’ solution: Let μ have distribution function F_μ , Φ the d.f. of an $N(0, 1)$, $h := F_\mu^{-1} \circ \Phi$, so $h(B_1) \sim \mu$. Then:

$$\xi_t(A) := \mathbb{P}(h(B_t) \in A | \mathcal{F}_t)$$

is an MVM with $\xi_0 = \mu$. Note that if $\mu = N(0, 1)$, we get the easy case where $\xi_t = N(B_t, 1 - t)$ for $t \in [0, 1]$

Measure valued martingales

Suppose we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_r)_{r \in [0-, T]}, \mathbb{P})$ satisfying the usual conditions, such that $\mathcal{F}_{0-}$ is trivial.

Lemma

- 1 If $(\xi_r)_{r \in [0, T]}$ is a terminating measure-valued martingale with $\xi_{0-} = \mu$, then $S_r := \int x \xi_r(dx)$ is a non-negative UI martingale with $S_T \sim \mu$
- 2 If $(S_r)_{r \in [0, T]}$ is a non-negative UI martingale with $S_T \sim \mu$, then $\xi_r(A) := \mathbb{P}(S_T \in A | \mathcal{F}_r)$ is a terminating MVM with $\xi_{0-} = \mu$

[MOT] and [MVM] problem formulation

Given an integrable probability measure μ on $\mathbb{R}_+$ and a sufficiently nice function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

[MOT] find a probability space $(\Omega, \mathcal{H}, (\mathcal{H}_t)_{t \in [0, T]}, \mathbb{P})$ and a càdlàg UI martingale $(S_t)_{t \in [0, T]}$ on this space with $S_T \sim \mu$ which maximises $\mathbb{E}[F(A_T)]$ over the class of such probability spaces and processes.

[MVM] find a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_r)_{r \in [0-, \infty)}, \mathbb{P})$, a progressively measurable process $\lambda_r \in [0, 1]$, and a terminating measure-valued $(\mathcal{G}_r)_{r \in [0-, \infty)}$ -martingale $(\xi_r)_{r \in [0-, \infty]}$ with $\xi_{0-} = \mu$ and $\int x \xi_r(dx)$ **continuous** a.s., which maximises $\mathbb{E}[F(A_T)]$ with A_T given by:

$$T_r = \int_0^r \lambda_s ds, \quad A_T = \int_0^T \left\{ \int x \xi_{T_s^{-1}}(dx) \right\} ds$$

Equivalence of [MOT] and [MVM]

Lemma

[MOT] and [MVM] are equivalent. Moreover, if F is bounded above, then the value remains the same in [MVM] if we restrict to MVMs in a Brownian filtration which are (pathwise) continuous and almost surely terminate in finite time.

Here, (pathwise) continuity of MVMs is in the topology derived from the 1-Wasserstein metric on $\mathcal{P}^1$:

$$d_{\mathcal{W}_1}(\lambda, \mu) := \sup \left\{ \int \varphi(x) (\lambda - \mu)(dx) : \varphi \text{ is 1-Lipschitz} \right\}.$$

Dynamic Programming Principle

Formulate dynamically: suppose that at time r , we have 'real' time $T_r = t$, current law $\xi_r = \xi \in \mathcal{P}^1$, running average $A_{T_r} = a$, and we wish to find:

$$U(r, t, \xi, a) = \sup \mathbb{E} [F(A_T) | T_r = t, \xi_r = \xi, A_{T_r} = a],$$

where the supremum is taken over all time-change determining processes $(\lambda_u)_{u \in [r, \infty)}$ and continuous, finitely-terminating models $(\xi_u)_{u \in [r, \infty)}$.

Lemma

Suppose F is a non-negative, Lipschitz function. The function $U : \mathbb{R}_+ \times [0, T] \times \mathcal{P}^1 \times \mathbb{R}_+$ is continuous (here the topology on $\mathcal{P}^1$ is the topology derived from the Wasserstein-1 metric), and independent of r .

$\Rightarrow$ Can approximate ξ by finite atomic measures

Approximation by atomic measures

When approximating by finitely supported measures, we have some nice structure to exploit:

- Suppose initially ξ supported on $0 \leq x_0 \leq x_1 \leq \dots \leq x_N$, then at any later time ξ supported on some subset $x_{\alpha_0} \leq x_{\alpha_1} \leq \dots \leq x_{\alpha_m}$, where $\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \subset \{0, 1, \dots, N\}$
- Consider $\xi_r^i = \xi_r(\{x_i\})$ — each ξ^i is a martingale with values on $[0, 1]$, constrained by $\sum_i \xi^i = 1$. I.e. ξ takes values on a simplex
- Consider a sequence of problems, where we run until the first time one of the ξ_i 's hits zero: problem reduces to a smaller simplex
- Recalling that we can assume a Brownian filtration, $d\xi_r^i = w_r^i dW_r$, w_r^i part of control to be chosen
- Control also incorporates 'speed' how fast ξ_t evolves relative to 'real' time

Value function

Fix α, ξ with $|\alpha| = k + 1$ then we write

$$\xi^\alpha = (\xi^{\alpha_0}, \xi^{\alpha_1}, \dots, \xi^{\alpha_k}) \in \Delta^{k+1} := \{\mathbf{z} \in \mathbb{R}_+^{k+1} : \sum z_i = 1\}$$

$$\mathbf{x}^\alpha = (x_{\alpha_0}, x_{\alpha_1}, \dots, x_{\alpha_k})$$

$$\mathbb{S}^{k+1} = \{\mathbf{z} \in \mathbb{R}^{k+1} : \|\mathbf{z}\| = 1\}$$

Then the value function of interest is

$$V_\alpha(u, t, \xi^{\alpha_1}, \dots, \xi^{\alpha_k}, a) = U(u, t, \sum \xi^{\alpha_i} \delta_{x_{\alpha_i}}, a)$$

Main Result

Theorem

Suppose $F(a)$ is continuous, non-negative. Then V_α is independent of u and the smallest non-negative solution (in the viscosity sense) to

$$\max \left\{ \frac{\partial V_\alpha}{\partial t} + \mathbf{x}^\alpha \cdot \boldsymbol{\xi}^\alpha \frac{\partial V_\alpha}{\partial a}, \sup_{\mathbf{w} \in \mathbb{S}^k} \left[\text{tr}(\mathbf{w}\mathbf{w}^T D_{\boldsymbol{\xi}}^2 V_\alpha) \right] \right\} = 0$$

for $\boldsymbol{\xi} \in (\Delta^{k+1})^\circ$, and $t < T$, with the boundary conditions

$$\begin{aligned} V_\alpha(u, T, \boldsymbol{\xi}^\alpha, a) &= F(a) \\ V_\alpha(u, t, \boldsymbol{\xi}^\alpha, a) &= V_{\alpha'}(u, t, \boldsymbol{\xi}^{\alpha'}, a), \quad \text{when } \boldsymbol{\xi}^\alpha \in \partial \Delta^{k+1} \end{aligned}$$

Here α' is the subset of α corresponding to non-zero entries of $\boldsymbol{\xi}^\alpha$, and $\boldsymbol{\xi}^{\alpha'}$ is the vector of non-zero values of $\boldsymbol{\xi}^\alpha$.

If F is bounded, there is a unique bounded solution.

Example: Convex F

Lemma

Suppose the function F is convex and Lipschitz. Then for all $\xi \in \mathcal{P}^1(\mathbb{R}_+)$:

$$U(t, \xi, a) = \int F(a + (T - t)x) \xi(dx).$$

Moreover, an optimal model is given by:

$$\begin{aligned} S_{0-} &= \int x \xi(dx) \\ S_t &= S_T, \quad t \geq 0, \end{aligned}$$

where $S_T \sim \xi$.

- Proof: check that U verifies our PDE
- Result due to Stebegg ('14): also provides a model-independent super-hedging strategy

Example: Non-convex F

Consider

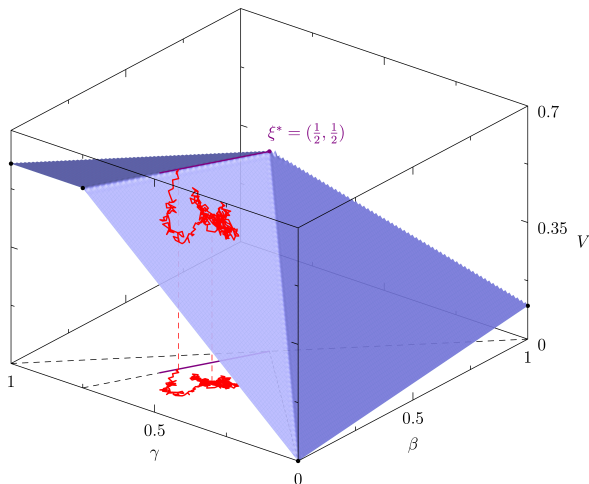
$$F(A_T) = (A_T - K_1)_+ - (A_T - K_2)_+, \quad K_1 < K_2$$

- Conjecture: at time 0, run to final distribution for 'small' final values, or to K_2 for 'large' final values
- Simplify to three point case:

$$\xi_0 = (1 - \beta - \gamma)\delta_{-1} + \beta\delta_0 + \gamma\delta_1$$

- Explicit value function corresponding to conjectured solution can be computed, PDE can be verified for this solution $\implies$ optimality
- Note that general duality results (Dolinsky-Soner, ...) give existence of a model-independent super-hedging strategy

Example: non-convex F



Value function at $t = 0$ when $a = 0$, $T = 1$, $K_1 = -0.1$, $K_2 = 0.5$.

Conclusions

- Formulated the model-independent pricing problem for Asian options in terms of a measure-valued martingale
- In this formulation, we can apply standard dynamic programming arguments, no convexity assumption required
- By discretising, can formulate as a PDE $\mapsto$ characterisation of value function
- Solve simple problems explicitly via verification