Functional vs Banach space stochastic calculus & strong-viscosity solutions to semilinear parabolic path-dependent PDEs

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Outline

1. Functional vs Banach space stochastic calculus
   - Functional Itô calculus via regularization
   - Comparing the two approaches
2. Path-dependent PDE
   - Path-dependent SDE
   - Strict solutions
3. Strong-viscosity solutions
   - Towards a weaker notion of solution
   - Definition, existence and uniqueness

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Functional Itô & Banach space stochastic calculus

Functional Itô calculus is an extension of classical Itô calculus designed *ad hoc* for functionals $F(t, X_{\cdot+t}, X_t)$ depending on time t, past and present values of the process X.

- B. Dupire (2009) **Functional Itô calculus**. *Portfolio Research Paper*, Bloomberg.
- R. Cont & D.-A. Fournié (2013) **Functional Itô calculus and stochastic integral representation of martingales**. *Annals of Probability*, 41 (1), 109-133.

Banach space stochastic calculus also gives an expansion of $F(t, X_{\cdot+t}, X_t)$, but considering the path $X_{\cdot+t} = (X_{s+t})_{s\in[-T,0]}$ as an element of the Banach space $B = C([-T, 0])$ (B can be a *generic* separable Banach space).

- C. Di Girolami & F. Russo (2010) **Infinite dimensional stochastic calculus via regularization and applications**. *Ph.D. Thesis*, preprint inria-00473947.
- C. Di Girolami & F. Russo (2014) **Generalized covariation for Banach space valued processes, Itô formula and applications**. *Osaka J. Math.*, 51 (3), 729-783.

First step: functional Itô calculus via regularization

Using *regularization techniques*, instead of discretization techniques of Föllmer type, is not the only issue.

We also investigate other possible improvements of functional Itô calculus:

- To define functional derivatives, we do not need to extend a functional from $C([-T,0])$ to $\mathbb{D}([-T,0])$, but to a space $\mathscr{C}([-T,0])$ which gets stuck as much as possible to the "natural" space $C([-T,0])$.
- Time and path plays two distinct roles in our setting.
  $\Longrightarrow$ we define the horizontal derivative independently of the time derivative.

The space $\mathscr{C}([-T,0])$: motivation

In the classical literature on path-dependent SDEs, it is usual to consider the state space

$$\underbrace{L^2([-T,0])}_{\text{past}} \times \underbrace{\mathbb{R}}_{\text{present}}$$

Here we take

$$\underbrace{C_b([-T,0[)}_{\text{past}} \times \underbrace{\mathbb{R}}_{\text{present}}$$

where

$$C_b([-T,0[) \;=\; \{f\colon [-T,0[\to \mathbb{R}\colon\; f \text{ is bounded and continuous}\}.$$

Definition

Definition

$\mathscr{C}([-T,0])$: set of bounded functions $\eta\colon [-T,0] \to \mathbb{R}$ continuous on $[-T,0[$, equipped with an inductive topology which induces the following **convergence**

$$\eta_n \underset{\text{in } \mathscr{C}([-T,0])}{\overset{n\to\infty}{\longrightarrow}} \eta$$

if:

(i) $\|\eta_n\|_\infty \leq C$.

(ii) $\sup_{t\in K} |\eta_n(t) - \eta(t)| \to 0, \quad \forall$ compact set $K \subset [-T,0[$.

(iii) $\eta_n(0) \to \eta(0)$.

Remarks

- $C([-T,0])$ is dense in $\mathscr{C}([-T,0])$, when endowed with the topology of $\mathscr{C}([-T,0])$.

- **Examples of continuous functionals:**

  (a) $\mathcal{U}(\eta) = g(\eta(t_1), \dots, \eta(t_n))$, with $-T \le t_1 < \cdots < t_n \le 0$ and $g \colon \mathbb{R}^n \to \mathbb{R}$ continuous.

  (b) $\mathcal{U}(\eta) = \int_{[-T,0]} \varphi(t) d^- \eta(t)$, with $\varphi \colon [-T,0] \to \mathbb{R}$ a càdlàg bounded variation function.

- The functional $\mathcal{U}(\eta) = \sup_{t \in [-T,0]} \eta(t)$ is *not* continuous.

Functional derivatives

Definition

Let $u \colon \mathscr{C}([-T,0]) \to \mathbb{R}$ and $\eta \in \mathscr{C}([-T,0])$.
(i) Horizontal derivative at η:

$$D^H u(\eta) := \lim_{\varepsilon \to 0^+} \frac{u(\eta(\cdot)1_{[-T,0[} + \eta(0)1_{\{0\}}) - u(\eta(\cdot - \varepsilon)1_{[-T,0[} + \eta(0)1_{\{0\}})}{\varepsilon}.$$

(ii) First-order vertical derivative at η:

$$D^V u(\eta) := \partial_a \tilde{u}(\eta_{|[-T,0[}, \eta(0)).$$

(iii) Second-order vertical derivative at η:

$$D^{VV} u(\eta) := \partial^2_{aa} \tilde{u}(\eta_{|[-T,0[}, \eta(0)).$$

- $\tilde{u}(\gamma, a) := u(\gamma 1_{[-T,0[} + a 1_{\{0\}}), \qquad \forall (\gamma, a) \in C_b([-T,0[) \times \mathbb{R}.$

The space $C^{1,2}(([0,T] \times \text{past}) \times \text{present})$

Definition

$\mathcal{U} \colon [0,T] \times C([-T,0]) \to \mathbb{R}$ is in $C^{1,2}(([0,T] \times \text{past}) \times \text{present})$ if:

- $\mathcal{U}$ admits a (necessarily unique) continuous extension $u \colon [0,T] \times \mathscr{C}([-T,0]) \to \mathbb{R}$.
- $\partial_t u$, $D^H u$, $D^V u$, $D^{VV} u$ exist and are continuous.

Then we define on $[0,T] \times C([-T,0])$:

$$D^H \mathcal{U} := D^H u, \qquad D^V \mathcal{U} := D^V u, \qquad D^{VV} \mathcal{U} := D^{VV} u.$$

Functional Itô's formula

Theorem

Let $\mathcal{U}$ *be in* $C^{1,2}(([0,T] \times \text{past}) \times \text{present})$ *and* $X = (X_t)_{t\in[0,T]}$ *be a real continuous finite quadratic variation process.*

$$\begin{aligned}\mathcal{U}(t,\mathbb{X}_t) &= \mathcal{U}(0,\mathbb{X}_0) + \int_0^t \big(\partial_t \mathcal{U}(s,\mathbb{X}_s) + D^H \mathcal{U}(s,\mathbb{X}_s)\big)ds \\ &\quad + \int_0^t D^V \mathcal{U}(s,\mathbb{X}_s) d^- X_s + \frac{1}{2}\int_0^t D^{VV}\mathcal{U}(s,\mathbb{X}_s) d[X]_s\end{aligned}$$

for all $0 \le t \le T$*, where* $\mathbb{X} = (\mathbb{X}_t)_t$ *denotes the* **window process** *associated with* X*, defined by*

$$\mathbb{X}_t := \{X_{t+s},\ s \in [-T,0]\}.$$

Comparing the two approaches

Identification of the functional derivatives

Our aim is to prove formulae which allow to express functional derivatives in terms of differential operators arising in the Banach space stochastic calculus.

▶ Notation: we denote by $D\mathcal{U}$ the Fréchet derivative of $\mathcal{U}$, which can be written as

$$D\mathcal{U}(\eta)\varphi = \int_{[-T,0]} \varphi(x) D_{dx}\mathcal{U}(\eta) = \int_{[-T,0]} \varphi(x)\big(D_{dx}^{\perp}\mathcal{U}(\eta) + D^{\delta_0}\mathcal{U}(\eta)\delta_0(dx)\big)$$

for some uniquely determined finite signed Borel measure $D_{dx}\mathcal{U}(\eta)$ on $[-T,0]$.

- Vertical derivative

$$D^V\mathcal{U}(\eta) \;=\; D^{\delta_0}\mathcal{U}(\eta)$$

- Horizontal derivative

$$D^H\mathcal{U}(\eta) \;=\; ?$$

Identification of $D^H\mathcal{U}$: definition of χ_0

- χ_0 **subspace of** $\mathcal{M}([-T,0]^2)$: $\mu \in \mathcal{M}([-T,0]^2)$ belongs to χ_0 if

$$\mu(dx,dy) = g_1(x,y)dxdy + \lambda_1\delta_0(dx)\otimes\delta_0(dy) + g_2(x)dx\otimes\lambda_2\delta_0(dy) + \lambda_3\delta_0(dx)\otimes g_3(y)dy + g_4(x)\delta_y(dx)\otimes dy,$$

with $g_1 \in L^2([-T,0]^2)$, $g_2, g_3 \in L^2([-T,0])$, $g_4 \in L^\infty([-T,0])$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

Identification of $D^H\mathcal{U}$

Theorem

Let $\eta \in C([-T,0])$ be such that the quadratic variation $[\eta]$ on $[-T,0]$ exists. Let $\mathcal{U}\colon C([-T,0]) \to \mathbb{R}$ be C^2-Fréchet such that:

(i) $D^2\mathcal{U}\colon C([-T,0]) \longrightarrow \chi_0$.

(ii) $D_x^{2,Diag}\mathcal{U}(\eta)$ ($\longleftrightarrow g_4$) *has a $[\eta]$-zero set of discontinuity (e.g., if it is countable).*

(iii) *There exist continuous extensions of $\mathcal{U}$ and $D^2_{dx\,dy}\mathcal{U}$*

$$u\colon \mathscr{C}([-T,0]) \to \mathbb{R}, \qquad D^2_{dx\,dy}u\colon \mathscr{C}([-T,0]) \to \chi_0.$$

(iv) *The horizontal derivative $D^H\mathcal{U}(\eta)$ exists at η.*

Then

$$D^H\mathcal{U}(\eta) = \int_{[-T,0]} D_{dx}^{\perp}\mathcal{U}(\eta)d^+\eta(x) - \frac{1}{2}\int_{[-T,0]} D_x^{2,Diag}\mathcal{U}(\eta)d[\eta](x)$$

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Semilinear parabolic path-dependent PDE

Consider the semilinear parabolic path-dependent PDE on $[0,T] \times C([-T,0])$:

$$\partial_t \mathcal{U} + D^H \mathcal{U} + b(t,\eta) D^V \mathcal{U} + \frac{1}{2}\sigma(t,\eta)^2 D^{VV}\mathcal{U} + F(t,\eta,\mathcal{U},\sigma(t,\eta)D^V\mathcal{U}) = 0,$$

$$\mathcal{U}(T,\eta) = H(\eta).$$

Standing Assumption (A). b, σ, F, H are Borel measurable functions satisfying, for some positive constants C and m,

$$\begin{aligned}
|b(t,\eta) - b(t,\eta')| + |\sigma(t,\eta) - \sigma(t,\eta')| &\le C\|\eta - \eta'\|, \\
|F(t,\eta,y,z) - F(t,\eta,y',z')| &\le C\big(|y-y'| + |z-z'|\big), \\
|b(t,0)| + |\sigma(t,0)| &\le C, \\
|F(t,\eta,0,0)| + |H(\eta)| &\le C\big(1 + \|\eta\|^m\big),
\end{aligned}$$

for all $t \in [0,T]$, $\eta,\eta' \in C([-T,0])$, $y,y',z,z' \in \mathbb{R}$.

Path-dependent SDE

For every $(t,\eta) \in [0,T] \times C([-T,0])$, consider the path-dependent SDE:

$$\begin{cases} dX_s = \; b(s,\mathbb{X}_s)dt + \sigma(s,\mathbb{X}_s)dW_s, & s \in [t,T], \\ X_s \; = \; \eta(s-t), & s \in [-T+t,t]. \end{cases}$$

W is a real Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$. $\mathbb{F}$ is the completion of the natural filtration generated by W.

Proposition

$\forall\ (t,\eta) \in [0,T] \times C([-T,0])$, $\exists!$ *(up to indistinguishability)* $\mathbb{F}$*-adapted continuous process* $X^{t,\eta} = (X_s^{t,\eta})_{s\in[-T+t,T]}$ *solution to the path-dependent SDE. Moreover, for any* $p \geq 1$ *there exists a positive constant* C_p *such that*

$$\mathbb{E}\Big[\sup_{s\in[-T+t,T]} \big|X_s^{t,\eta}\big|^p\Big] \;\leq\; C_p\big(1 + \|\eta\|^p\big).$$

Strict solutions

Definition

A map $\mathcal{U}$ in $C^{1,2}(([0,T[\times\text{past})\times\text{present})$ and $C([0,T]\times C([-T,0]))$, satisfying the path-dependent PDE, is called a **strict solution**.

- **Notation**
  - $\mathbb{S}^2(t,T)$, $t\leq T$, the set of real càdlàg adapted processes $Y=(Y_s)_{t\leq s\leq T}$ such that

$$\|Y\|^2_{\mathbb{S}^2(t,T)} := \mathbb{E}\Big[\sup_{t\leq s\leq T}|Y_s|^2\Big] < \infty.$$

  - $\mathbb{H}^2(t,T)$, $t\leq T$, the set of real predictable processes $Z=(Z_s)_{t\leq s\leq T}$ such that

$$\|Z\|^2_{\mathbb{H}^2(t,T)} := \mathbb{E}\Big[\int_t^T |Z_s|^2 ds\Big] < \infty.$$

Strict solutions: Feynman-Kac formula & uniqueness

Theorem

Let $\mathcal{U}\colon [0,T] \times C([-T,0]) \to \mathbb{R}$ *be a strict solution to the path-dependent PDE, satisfying the polynomial growth condition*

$$|\mathcal{U}(t,\eta)| \;\leq\; C\big(1 + \|\eta\|_\infty^m\big), \qquad \forall\,(t,\eta) \in [0,T] \times C([-T,0]),$$

for some positive constant m*. Then, we have*

$$\mathcal{U}(t,\eta) \;=\; Y_t^{t,\eta}, \qquad \forall\,(t,\eta) \in [0,T] \times C([-T,0]),$$

where

$$(Y_s^{t,\eta}, Z_s^{t,\eta})_s \;=\; (\mathcal{U}(s,\mathbb{X}_s^{t,\eta}), \sigma(s,\mathbb{X}_s^{t,\eta}) D^V \mathcal{U}(s,\mathbb{X}_s^{t,\eta}) 1_{[t,T[}(s))_s$$

with $(Y^{t,\eta}, Z^{t,\eta}) \in \mathbb{S}^2(t,T) \times \mathbb{H}^2(t,T)$*, is the solution to the Backward Stochastic Differential Equation (BSDE)*

$$Y_s^{t,\eta} \;=\; H(\mathbb{X}_T^{t,\eta}) + \int_s^T F(r,\mathbb{X}_r^{t,\eta}, Y_r^{t,\eta}, Z_r^{t,\eta})dr - \int_s^T Z_r^{t,\eta} dW_r.$$

Strict solutions: existence (I)

Theorem

Suppose that b, σ, F, H are cylindrical and smooth, i.e.

$$b(t,\eta) = \bar{b}\Big(\int_{[-t,0]} \varphi_1(x+t)d^-\eta(x), \dots, \int_{[-t,0]} \varphi_N(x+t)d^-\eta(x)\Big)$$

$$\sigma(t,\eta) = \bar{\sigma}\Big(\int_{[-t,0]} \varphi_1(x+t)d^-\eta(x), \dots, \int_{[-t,0]} \varphi_N(x+t)d^-\eta(x)\Big)$$

$$F(t,\eta,y,z) = \bar{F}\Big(t, \int_{[-t,0]} \varphi_1(x+t)d^-\eta(x), \dots, \int_{[-t,0]} \varphi_N(x+t)d^-\eta(x), y, z\Big)$$

$$H(\eta) = \bar{H}\Big(\int_{[-T,0]} \varphi_1(x+T)d^-\eta(x), \dots, \int_{[-T,0]} \varphi_N(x+T)d^-\eta(x)\Big)$$

where

(i) $\bar{b}$, $\bar{\sigma}$, $\bar{F}$, $\bar{H}$ *are continuous and satisfy Assumption* **(A)** *with* $x \in \mathbb{R}^N$ *in place of* η.

(ii) $\bar{b}$ *and* $\bar{\sigma}$ *are of class* C^3 *with partial derivatives from order* 1 *up to order* 3 *bounded.*

Strict solutions: existence (II)

Theorem (cont'd)

(iii) *For all* $t \in [0,T]$, $\bar{F}(t,\cdot,\cdot,\cdot) \in C^3(\mathbb{R}^N)$ *and moreover we assume the validity of the properties below.*

(a) $\bar{F}(t,\cdot,0,0)$ *belongs to* C^3 *and its third order partial derivatives satisfy a polynomial growth condition uniformly in* t.

(b) $D_y\bar{F}$, $D_z\bar{F}$ *are bounded on* $[0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, *as well as their derivatives of order one and second with respect to* $x_1,\ldots,x_N,y,z$.

(iv) $\bar{H} \in C^3(\mathbb{R}^N)$ *and its third order partial derivatives satisfy a polynomial growth condition.*

(v) $\varphi_1,\ldots,\varphi_N \in C^2([0,T])$.

Then, the map $\mathcal{U}$ *given by*

$$\mathcal{U}(t,\eta) \;=\; Y_t^{t,\eta}, \qquad \forall\,(t,\eta) \in [0,T] \times C([-T,0]),$$

is the unique strict solution to the path-dependent PDE.

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Towards a weaker notion of solution

Consider the lookback-type payoff:

$$H(\eta) \; = \; \sup_{x \in [-T,0]} \eta(x), \qquad \forall\, \eta \in C([-T,0]).$$

In this case, we expect that the map

$$\mathcal{U}(t,\eta) \; = \; \mathbb{E}\big[H(\mathbb{W}_T^{t,\eta})\big], \qquad \forall\,(t,\eta) \in [0,T] \times C([-T,0])$$

is "virtually" a solution to the path-dependent PDE:

$$\begin{aligned} \partial_t \mathcal{U} + D^H \mathcal{U} + \frac{1}{2} D^{VV} \mathcal{U} &= 0, \\ \mathcal{U}(T,\eta) &= H(\eta). \end{aligned}$$

Unfortunately, $\mathcal{U}$ is not continuous with respect to the topology of $\mathscr{C}([-T,0])$, therefore it can not be a *strict solution*.

- $\mathcal{U}$ is a strong-viscosity solution.

Strong-viscosity solutions: introduction

Various definitions of **viscosity-type solutions** for path-dependent PDEs have been given.

We recall in particular:

- I. Ekren, C. Keller, N. Touzi, and J. Zhang (2014) **On viscosity solutions of path dependent PDEs**. *Annals of Probability*, 42 (1), 204-236.

We propose a notion of viscosity-type solution with the following peculiarities:

- it is a purely analytic object;
- it can be easily adapted to more general equations than classical partial differential equations.

▶ We call it strong-viscosity solution to distinguish it from the classical notion of viscosity solution and from the definition introduced by Ekren, Keller, Touzi, Zhang.

Strong-viscosity solutions: path-dependent case

Path-dependent PDE on $[0,T] \times C([-T,0])$:

$$\partial_t\mathcal{U} + D^H\mathcal{U} + b(t,\eta)D^V\mathcal{U} + \frac{1}{2}\sigma(t,\eta)^2 D^{VV}\mathcal{U} + F(t,\eta,\mathcal{U},\sigma(t,\eta)D^V\mathcal{U}) = 0,$$

$$\mathcal{U}(T,\eta) = H(\eta).$$

Standing Assumption (A). b, σ, F, H are Borel measurable functions satisfying, for some positive constants C and m,

$$\begin{aligned}
|b(t,\eta) - b(t,\eta')| + |\sigma(t,\eta) - \sigma(t,\eta')| &\leq C\|\eta - \eta'\|, \\
|F(t,\eta,y,z) - F(t,\eta,y',z')| &\leq C\big(|y-y'| + |z-z'|\big), \\
|b(t,0)| + |\sigma(t,0)| &\leq C, \\
|F(t,\eta,0,0)| + |H(\eta)| &\leq C\big(1 + \|\eta\|^m\big),
\end{aligned}$$

for all $t \in [0,T]$, $\eta, \eta' \in C([-T,0])$, $y, y', z, z' \in \mathbb{R}$.

Definition (I)

Definition

A function $\mathcal{U}\colon [0,T]\times C([-T,0])\to\mathbb{R}$ is called a **strong-viscosity solution** to the path-dependent PDE if there exists a sequence $(\mathcal{U}_n, H_n, F_n, b_n, \sigma_n)_n$ of Borel measurable functions satisfying:

(i) For some positive constants C and m,

$$
\begin{aligned}
|b_n(t,\eta)-b_n(t,\eta')|+|\sigma_n(t,\eta)-\sigma_n(t,\eta')| &\leq C\|\eta-\eta'\| \\
|F_n(t,\eta,y,z)-F_n(t,\eta,y',z')| &\leq C(|y-y'|+|z-z'|) \\
|b_n(t,0)|+|\sigma_n(t,0)| &\leq C \\
|\mathcal{U}_n(t,\eta)|+|H_n(\eta)|+|F_n(t,\eta,0,0)| &\leq C\big(1+\|\eta\|_\infty^m\big)
\end{aligned}
$$

for all $t\in[0,T]$, $\eta,\eta'\in C([-T,0])$, $y,y'\in\mathbb{R}$, $z,z'\in\mathbb{R}$. Moreover, the functions $\mathcal{U}_n(t,\cdot)$, $H_n(\cdot)$, $F_n(t,\cdot,\cdot,\cdot)$, $n\in\mathbb{N}$, are equicontinuous on compact sets, uniformly with respect to $t\in[0,T]$.

Definition (II)

Definition (cont'd)

(ii) $\mathcal{U}_n$ is a strict solution to

$$\begin{cases} \partial_t \mathcal{U}_n + D^H \mathcal{U}_n + b_n(t,\eta) D^V \mathcal{U}_n + \frac{1}{2}\sigma_n(t,\eta)^2 D^{VV} \mathcal{U}_n \\ + F_n(t,\eta,\mathcal{U}_n,\sigma_n(t,\eta) D^V \mathcal{U}_n) \ = \ 0, \quad \forall\, (t,\eta) \in [0,T[\times C([-T,0]), \\ \mathcal{U}_n(T,\eta) \ = \ H_n(\eta), \quad \forall\, \eta \in C([-T,0]). \end{cases}$$

(iii) $(\mathcal{U}_n, H_n, F_n, b_n, \sigma_n)_n$ converges pointwise to $(\mathcal{U}, H, F, b, \sigma)$ as $n \to \infty$.

Feynman-Kac formula & uniqueness

Theorem

Let $\mathcal{U}\colon [0,T]\times C([-T,0])\to\mathbb{R}$ be a strong-viscosity solution to the path-dependent PDE. Then, we have

$$\mathcal{U}(t,\eta) \;=\; Y_t^{t,\eta}, \qquad \forall\,(t,\eta)\in[0,T]\times C([-T,0]),$$

where $(Y_s^{t,\eta},Z_s^{t,\eta})_{s\in[t,T]}\in\mathbb{S}^2(t,T)\times\mathbb{H}^2(t,T)$, with $Y_s^{t,\eta}=\mathcal{U}(s,\mathbb{X}_s^{t,\eta})$, is the unique solution in $\mathbb{S}^2(t,T)\times\mathbb{H}^2(t,T)$ to the BSDE

$$Y_s^{t,\eta} \;=\; H(\mathbb{X}_T^{t,\eta})+\int_s^T F(r,\mathbb{X}_r^{t,\eta},Y_r^{t,\eta},Z_r^{t,\eta})dr-\int_s^T Z_r^{t,\eta}dW_r,$$

for all $t\leq s\leq T$. In particular, there exists at most one strong-viscosity solution to the path-dependent PDE.

Existence

Consider the path-dependent heat equation

$$\begin{cases} \partial_t \mathcal{U} + D^H \mathcal{U} + \frac{1}{2} D^{VV} \mathcal{U} \;=\; 0, & \forall\, (t,\eta) \in [0,T[\times C([-T,0]), \\ \mathcal{U}(T,\eta) \;=\; H(\eta), & \forall\, \eta \in C([-T,0]). \end{cases}$$

Theorem

Suppose that H is continuous. Then, the map

$$\mathcal{U}(t,\eta) \;=\; \mathbb{E}\big[H(\mathbb{W}_T^{t,\eta})\big],$$

for all $(t,\eta) \in [0,T] \times C([-T,0])$, is the unique strong-viscosity solution to the path-dependent heat equation.

- H can be in particular the lookback-type payoff

$$H(\eta) \;=\; \sup_{x\in[-T,0]} \eta(x), \qquad \forall\, \eta \in C([-T,0]).$$

THANK YOU!