A long-range dependence model in fixed income markets

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Ioannidis, C. and Monoyios, M. (2001). Long-Range Dependence in Daily Interest Rate. http://bura.brunel.ac.uk/handle/2438/3778.

They employ a number of parametric and non-parametric techniques to establish the existence of long-range dependence in daily interbank offer rates for four countries. They test for long memory using classical R/S analysis, variance-time plots and Lo's (1991) modified R/S statistic, they shuffle the data to destroy long and short memory in turn, and they repeat their non-parametric tests. From these non-parametric tests they find strong evidence of the presence of long memory in all four series independently of the chosen statistic. The data consists of four time series of overnight interest rates, for the US, UK, France and Germany, from January 1981 until December 1998.

- Backus, D. K., Zin, S. E. (1993). Long-Memory Inflation Uncertainty: Evidence from the Term Structure of Interest Rates. Journal of Money, Credit and Banking, 25(3), 681-700.

They show that a fractional difference model can explain why the data (1947-1986 period, based on monthly data for U.S. Treasury bonds) about long forward rates show a variance that decays at a hyperbolic rate. They conjecture that the long memory in short rates is inherited from inflation, money growth and monetary policy.

This talk is based in

- Corcuera, J. M., Farkas, G., Schoutens, W., and Valkeila, E. (2013). A short rate model using ambit processes. In Malliavin Calculus and Stochastic Analysis (pp. 525-553). Springer US.

The model

Assume that

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \mu_t$$

Notice that the process r is not a semimartingale if $g' \notin L^2((0,\infty))$.

Bond prices

Denote $P(t, T)$ the price at t of the zero-coupon bond with maturity time T. Then

$$P(t, T) = E_{P^*}\left(\exp\left(-\int_t^T r_s \mathrm{d}s \right) \middle| \mathcal{F}_t \right).$$

Bond prices

$$\int_t^T r_s \mathrm{d}s = \int_{-\infty}^t \sigma_u c(u; t, T) W(\mathrm{d}u) + \int_t^T \sigma_u c(u; u, T) W(\mathrm{d}u) + \int_t^T \mu_s \mathrm{d}s,$$

where

$$c(u; t, T) := \int_t^T g(s-u)\mathrm{d}s, \ \ t \geq u$$

Bond prices

Then

$$P(t,T) = \exp\left(A(t,T) - \int_{-\infty}^{t} \sigma_u c(u;t,T) W(\mathrm{d}u)\right),$$

where

$$\begin{aligned} A(t,T) &= \log E_{P^*}(\exp(-\int_t^T \sigma_u c(u;u,T) W(\mathrm{d}u) - \int_t^T \mu_s \mathrm{d}s)|\mathcal{F}_t) \\ &= \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)\mathrm{d}u - \int_t^T \mu_s \mathrm{d}s. \end{aligned}$$

and the variance of the yield $-\frac{1}{T-t}\log P(t,T)$ is given by

$$\mathrm{var}\left(-\frac{1}{T-t}\log P(t,T)\right) = \frac{1}{(T-t)^2}\int_{-\infty}^{t} \sigma_u^2 c^2(u;t,T)\mathrm{d}u.$$

Forward rates

$$
\begin{aligned}
& f(t,T) := -\partial_T \log P(t,T) \\
= \quad & -\int_t^T \sigma_u^2 g(T-u)c(u;u,T)\mathrm{d}u + \int_{-\infty}^t \sigma_u g(T-u)W(\mathrm{d}u) + \mu_T
\end{aligned}
$$

and

$$\mathrm{var}\,(f(t,T)) = \int_{-\infty}^t \sigma_u^2 g^2(T-u)\mathrm{d}u.$$

Forward rates

Note that

$$\begin{aligned} \mathrm{d}_t f(t, T) &= \sigma_t^2 g(T-t) c(t; t, T)\mathrm{d}t + \sigma_t g(T-t) W(\mathrm{d}t) \\ &= \alpha(t, T)\mathrm{d}t + \sigma(t, T) W(\mathrm{d}t), \end{aligned}$$

with

$$\begin{aligned} \sigma(t, T) &= \sigma_t g(T-t), \\ \alpha(t, T) &= \sigma_t^2 g(T-t) c(t; t, T). \end{aligned}$$

Completeness of the market

It is easy to see that

$$\begin{aligned}&\tilde{P}(t, T)\\&= P(0, T)\exp(-\int_0^t \sigma_u c(u; u, T)W(\mathrm{d}u) - \frac{1}{2}\int_0^t \sigma_u^2 c(u; u, T)^2 \mathrm{d}u),\end{aligned}$$

Then,

$$\mathrm{d}\tilde{P}(t, T) = -\tilde{P}(t, T)\sigma_t c(t; t, T)W(\mathrm{d}t), \ \ t \geq 0,$$

Completeness of the market

Let X be a P^*-square integrable payoff. Consider the $(\mathcal{F}_t)$-martingale

$$M_t := E_{P^*}\left(X|\mathcal{F}_t\right), t \geq 0,$$

then by (an extension) of Brownian martingale representation theorem we can write

$$\mathrm{d}M_t = H_t W(\mathrm{d}t),$$

where H is an adapted square integrable process.

Completeness of the market

Let $\left(\phi_t^0, \phi_t^1\right)$ be a self-financing portfolio built with a bank account and a T-bond, its value process is given by

$$V_t = \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 P(t, T),$$

and, by the self-financing condition,

$$\mathrm{d}\tilde{V}_t = \phi_t^1 \mathrm{d}\tilde{P}(t, T),$$

so if we take

Completeness of the market

$$\phi_t^1 = -\frac{H_t}{\tilde{P}(t,T)\sigma_t c(t;t,T)}$$

we can replicate X. In particular the bond with maturity T^* can be replicated by taking

$$\frac{P(t,T^*)c(t;t,T^*)}{P(t,T)c(t;t,T)}$$

bonds with maturity time $T \geq T^*$.

Option prices

Consider a bond with maturity $\bar{T} > T$, where T is the maturity time of a call option for this bond with strike K. Its price is given by

$$\Pi(t;T) = P(t,\bar{T})P^{\bar{T}}(P(T,\bar{T}) \geq K|\mathcal{F}_t) - KP(t,T)P^{T}(P(T,\bar{T}) \geq K|\mathcal{F}_t)$$

$$= P(t,\bar{T})P^{\bar{T}}(\frac{P(T,T)}{P(T,\bar{T})} \leq \frac{1}{K}|\mathcal{F}_t) - KP(t,T)P^{T}(\frac{P(T,\bar{T})}{P(T,T)} \geq K|\mathcal{F}_t).$$

Where P^T, is the T-forward measure and analogously for $P^{\bar{T}}$.

Option prices

Define

$$U(t, T, \bar{T}) := \frac{P(t, T)}{P(t, \bar{T})}.$$

Then

$$U(t; T, \bar{T}) \\ = \exp\{-A(t, \bar{T}) + A(t, T) - \int_{-\infty}^{t} \sigma_u (c(u; t, T) - c(u; t, \bar{T})) W(\mathrm{d}u)\}.$$

If we take the $\bar{T}$-forward measure $P^{\bar{T}}$, we will have that

$$W(\mathrm{d}u) = W^{\bar{T}}(\mathrm{d}u) - a(u)\mathrm{d}u,$$

where $W^{\bar{T}}(\mathrm{d}u)$ is a Brownian measure in $\mathbb{R}$ again.

Then, since $U(t, T, \bar{T})$ has to be a martingale with respect to $P^{\bar{T}}$, $a(u)$ is deterministic and we will have that

$$
\begin{aligned}
U(t; T, \bar{T}) &= \exp\Big\{-\int_{-\infty}^{t} \sigma_u (c(u; t, T) - c(u; t, \bar{T})) W^{\bar{T}}(\mathrm{d}u) \\
&\quad -\frac{1}{2}\int_{-\infty}^{t} \sigma_u^2 (c(u; t, T) - c(u; t, \bar{T}))^2 \mathrm{d}u\Big\}
\end{aligned}
$$

so

Option prices

$$U(T) := U(T; T, \bar{T}) = U(t; T, \bar{T})$$
$$\times \exp\left\{ \int_t^T \sigma_u c(u; T, \bar{T}) W^{\bar{T}}(\mathrm{d}u) - \frac{1}{2} \int_t^T \sigma_u^2 c(u; T, \bar{T})^2 \mathrm{d}u \right\}$$

and analogously

$$U(T)^{-1} = U(T; \bar{T}, T) = U^{-1}(t; T, \bar{T})$$
$$\times \exp\left\{ - \int_t^T \sigma_u c(u; T, \bar{T}) W^{T}(\mathrm{d}u) - \frac{1}{2} \int_t^T \sigma_u^2 c(u; T, \bar{T})^2 \mathrm{d}u \right\}.$$

Option prices

Therefore

$$\begin{aligned}\Pi(t;T) &= P(t,\bar{T})P^{\bar{T}}(U(T) \leq \frac{1}{K}|\mathcal{F}_t) - KP(t,T)P^{T}(U^{-1}(T) \geq K|\mathcal{F}_t) \\ &= P(t,\bar{T})P^{\bar{T}}(\log U(T) \leq -\log K|\mathcal{F}_t) \\ &\quad -KP(t,T)P^{T}(\log U^{-1}(T) \geq \log K|\mathcal{F}_t) \\ &= P(t,\bar{T})\Phi(d_+) - KP(t,T)\Phi(d_-),\end{aligned}$$

where

$$d\pm = \frac{\log \frac{P(t,\bar{T})}{KP(t,T)} \pm \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}}.$$

and

$$\Sigma^2_{t,T,\bar{T}} := \int_t^T \sigma_u^2 c(u;T,\bar{T})^2 \mathrm{d}u$$

Example

Assume that $\sigma_t = \sigma \mathbf{1}_{\{t \geq 0\}}$ and

$$g(t) = e^{-bt} \int_0^t e^{bs} \beta s^{\beta-1} \mathrm{d}s.$$

for $\beta \in (0, 1/2)$. Then

$$\mathrm{var}\left(f(t,T)\right) = \int_{-\infty}^t \sigma_u^2 g^2(T-u) \mathrm{d}u \sim T^{2\beta-2}.$$

and the volatility of the forward rates

$$\sigma(t,T) = \sigma^2 g(T-t) \sim T^{\beta-1}$$

when $T \to \infty$. That is more according to the data than the exponential decay in the Vasicek model.

The dynamics

We have postulated that

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \mu_t$$

and the question is if this process $(r_t)_{t\in\mathbb{R}}$ can be seen as the solution of certain stochastic differential equation. For instance, assume that

$$\mathrm{d}r_t = b(a - r_t)\mathrm{d}t + \sigma W(\mathrm{d}t)$$

then we have

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + e^{-bt}\int_0^t e^{bs}\sigma W(\mathrm{d}s),$$

The dynamics

and, if we take

$$r_0 = \int_{-\infty}^{0} e^{bs}\sigma W(\mathrm{d}s) + a,$$

we obtain that

$$r_t = a + \int_{-\infty}^{t} e^{-b(t-s)}\sigma W(\mathrm{d}s).$$

So, it corresponds to $g(t) = e^{-bt}$, $\sigma_s = \sigma$ and $\mu_t = a$.

The dynamics

Suppose that

$$W_t^g := \int_{-\infty}^t g(s,t)\,W(\mathrm{d}s).$$

Then formally

$$W_t^g(\mathrm{d}t) = g(t,t)\,W(\mathrm{d}t) + \left(\int_{-\infty}^t \partial_t g(s,t)\,W(\mathrm{d}s)\right)\mathrm{d}t,$$

The dynamics

and, for a deterministic function $f(\cdot,\cdot)$ we can define

$$\begin{aligned}
&\int_{-\infty}^{t} f(u,t) W_t^g(\mathrm{d}u) \\
= \;& \int_{-\infty}^{t} f(u,t) \left(g(u,u) W(\mathrm{d}u) + \left(\int_{-\infty}^{u} \partial_u g(s,u) W(\mathrm{d}s) \right) \mathrm{d}u \right) \\
= \;& \int_{-\infty}^{t} \left(\int_{-\infty}^{u} \left(f(u,t) - f(s,t) \right) \partial_u g(s,u) W(\mathrm{d}s) \right) \mathrm{d}u \\
& + \int_{-\infty}^{t} \left(\int_{s}^{t} f(s,t) \partial_u g(s,u) \mathrm{d}u \right) W(\mathrm{d}s) \\
& + \int_{-\infty}^{t} f(u,t) g(u,u) W(\mathrm{d}u)
\end{aligned}$$

The dynamics

$$
\begin{aligned}
& \int_{-\infty}^{t} f(u,t) W_t^g(\mathrm{d}u) \\
= \; & \int_{-\infty}^{t} \left(\int_{s}^{t} \left(f(u,t) - f(s,t) \right) \partial_u g(s,u) \mathrm{d}u \right) W(\mathrm{d}s) \\
& + \int_{-\infty}^{t} f(s,t) g(s,t) W(\mathrm{d}s) \\
= \; & \int_{-\infty}^{t} \left(\int_{s}^{t} \left(f(u,t) - f(s,t) \right) \partial_u g(s,u) \mathrm{d}u + f(s,t) g(s,t) \right) W(\mathrm{d}s).
\end{aligned}
$$

The dynamics

Then, the latest integral is well defined in an L^2 sense, provided

$$\int_{-\infty}^{t}\left(\int_{s}^{t}\left(f(u,t)-f(s,t)\right)\partial_u g(s,u)\mathrm{d}u+f(s,t)g(s,t)\right)^2\mathrm{d}s<\infty.$$

Then if we set the operator

$$K_t^g(f)(t,s):=\int_{s}^{t}\left(f(u,t)-f(s,t)\right)\partial_u g(s,u)\mathrm{d}u+f(s,t)g(s,t),$$

it is natural to define

$$\int_{-\infty}^{t}f(s,t)W_t^g(\mathrm{d}s):=\int_{-\infty}^{t}K_t^g(f)(s,t)W(\mathrm{d}s),$$

provided that $f(\cdot,t)\in\left(K_t^g\right)^{-1}\left(L^2(-\infty,t]\right)$.

The dynamics

Note that if $g(s,s)=0$, then we can write

$$K_t^g(f)(t,s) := \int_s^t f(u,t)\partial_u g(s,u)\mathrm{d}u,$$

and

$$\begin{aligned}
\int_{-\infty}^t f(s,t)W_t^g(\mathrm{d}s) &= \int_{-\infty}^t \left(\partial_t \int_s^t f(u,t)g(s,u)\mathrm{d}u\right) W(\mathrm{d}s) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t \int_s^t f(u,t)g(s,u)\mathrm{d}u W(\mathrm{d}s) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t f(u,t)\left(\int_{-\infty}^u g(s,u)W(\mathrm{d}s)\right)\mathrm{d}u \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t f(u,t)W_u^g \mathrm{d}u
\end{aligned}$$

The dynamics

Consider now

$$r_t = b\int_0^t (a - r_s)\mathrm{d}s + \sigma \int_0^t (t-s)^\beta W(\mathrm{d}s)$$

with $\beta \in (0, 1/2)$, then if we define

$$W_t^\beta := \int_0^t (t-s)^\beta W(\mathrm{d}s),$$

$$r_t = b\int_0^t (a - r_s)\mathrm{d}s + \sigma W^\beta(t).$$

In such a way that (r_t) is an Orstein-Uhlenbeck process driven by W^β.

The dynamics

We obtain

$$\begin{aligned} r_t &= r_0 e^{-bt} + a(1 - e^{-bt}) + e^{-bt} \int_0^t e^{bs} \sigma W^\beta(\mathrm{d}s) \\ &= r_0 e^{-bt} + a(1 - e^{-bt}) + \int_0^t \sigma g(t-s) W(\mathrm{d}u). \end{aligned}$$

Then, if $\beta \in (0, 1/2)$, we have

$$\begin{aligned} \int_0^t e^{-b(t-s)} W^\beta(\mathrm{d}s) &= \int_0^t \left(\int_u^t e^{-b(t-s)} \beta (s-u)^{\beta-1} \mathrm{d}s \right) W(\mathrm{d}u) \\ &= \int_0^t \left(\int_0^{t-u} e^{-b(t-s-u)} \beta s^{\beta-1} \mathrm{d}s \right) W(\mathrm{d}u). \\ &= \int_0^t e^{-b(t-u)} \left(\int_0^{t-u} e^{bs} \beta s^{\beta-1} \mathrm{d}s \right) W(\mathrm{d}u) \end{aligned}$$

The dynamics

In such a way that

$$g(t-s) = e^{-b(t-s)} \left(\int_0^{t-s} e^{bu} \beta u^{\beta-1} \mathrm{d}u \right)$$

Defaultable bonds

The purpose is to price a zero coupon bond with possibility of default. The payoff of this contract at the maturity time is $1_{\{\tau>T\}}$, where τ is the default time. Then, an arbitrage free price at time t is given by

$$D(t,T) = 1_{\{\tau>t\}} E\left(1_{\{\tau>T\}} e^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{G}_t\right), \ 0 \le t \le T,$$

where the expectation is taken with respect to a risk neutral probability and where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\ge0}$ represents the information available at the market.

Here we follow the Hazard process approach. In this approach we consider two filtrations, one is the default free filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that typically incorporates the history of the short rates. The default time is modeled by a random variable τ that is not necessarily an $\mathbb{F}$-stopping time, then the other filtration is $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \leq s\}, \quad 0 \leq s \leq t),$$

in such a way that τ is a $\mathbb{G}$-stopping time.

Defaultable bonds

Now, we assume that there exists an $\mathbb{F}$-adapted process $(\lambda_t)_{t\geq 0}$, such that

$$P^*(\tau > t | \mathcal{F}_t) = e^{-\int_0^t \lambda_s \mathrm{d}s},$$

then, it can be shown that

$$\begin{aligned} D(t,T) &= 1_{\{\tau > t\}} E_{P^*}\left(1_{\{\tau > T\}} e^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{G}_t \right) \\ &= 1_{\{\tau > t\}} E_{P^*}\left(e^{-\int_t^T (r_s + \lambda_s) \mathrm{d}s} \middle| \mathcal{F}_t \right). \end{aligned}$$

Then we need a model for $(r_t)_{t\geq 0}$ and $(\lambda_t)_{t\geq 0}$. A simple possible model is a Vasicek model for both processes

$$\begin{aligned} \mathrm{d}r_t &= b(a - r_t)\mathrm{d}t + \sigma \mathrm{d}W(t), \\ \mathrm{d}\lambda_t &= \breve{b}(\breve{a} - \lambda_t)\mathrm{d}t + \breve{\sigma}\mathrm{d}\breve{W}(t), \end{aligned}$$

where W and $\breve{W}$ are correlated Brownian motions, here $\mathcal{F}_t = \sigma($ $W_s, \breve{W}_s, 0 \leq s \leq t)$.

Defaultable bonds

The idea is to extend this model by considering ambit processes as noises in the stochastic differential equations. For instance we can have

$$\begin{aligned} r_t &= \int_{-\infty}^{t} \sigma_s g(t-s) W(\mathrm{d}s) + \mu_t, \\ \lambda_t &= \int_{-\infty}^{t} \breve{\sigma}_s \breve{g}(t-s) \breve{W}(\mathrm{d}s) + \breve{\mu}_t. \end{aligned}$$

Defaultable bonds

Then, the price of a defaultable zero coupon bond at time t will be given by

$$
\begin{aligned}
D(t, T) &= 1_{\{\tau > t\}} \exp \Big(A(t, T) \\
&\quad - \int_{-\infty}^{t} \left(\sigma_u c(u; t, T) W(\mathrm{d}u) + \breve{\sigma}_u \breve{c}(u; t, T) \right) \breve{W}(\mathrm{d}u) \Big),
\end{aligned}
$$

where

$$
\begin{aligned}
A(t, T) &= \frac{1}{2} \int_t^T (\sigma_u^2 c^2(u; t, T) + \breve{\sigma}_u^2 \breve{c}^2(u; t, T) \\
&\quad + 2\rho \sigma_u \breve{\sigma}_u c(u; t, T) \breve{c}(u; t, T)) \mathrm{d}u - \int_t^T (\mu_u + \breve{\mu}_u) \mathrm{d}u.
\end{aligned}
$$

and ρ is the correlation coefficient between W and $\breve{W}$.

Defaultable bonds

Interesting cases are $\sigma_u = \sigma 1_{\{u \geq 0\}}$, $\sigma_u = \breve{\sigma} 1_{\{u \geq 0\}}$, $\mu_u = \mu$, $\breve{\mu}_u = \breve{\mu}$,

$$
\begin{aligned}
g(t-s) &= e^{-b(t-s)} \int_0^{t-s} e^{bu} \beta u^{\beta-1} \mathrm{d}u, \\
\breve{g}(t-s) &= e^{-\breve{b}(t-s)} \int_0^{t-s} e^{\breve{b}u} \breve{\beta} u^{\breve{\beta}-1} \mathrm{d}u,
\end{aligned}
$$

$\beta, \breve{\beta} \in (0, 1/2)$. Then, we have that

$$
\mathrm{var}\left(-\frac{1}{T-t} \log D(t,T) \right) \sim T^{2(\beta \vee \breve{\beta})-2}.
$$

A CIR model

One of the drawbacks of the previous model is that it allows for negative short rates. An obvious way of avoiding this is to take

$$r_t = \sum_{i=1}^{d} \left(\int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s) \right)^2, \quad t \geq 0.$$

where $(W_i))_{1 \leq i \leq d}$ is a Brownian motion in $\mathbb{R}_+^d$.

A CIR model

$$r_t = \sum_{i=1}^{d} \int_0^t \int_0^t g(t-u)g(t-v)\sigma_s\sigma_u \mathrm{d}W_i(u)\mathrm{d}W_i(v),$$

then

$$\begin{aligned}\int_t^T r_s \mathrm{d}s &= \sum_{i=1}^{d} \int_0^t \int_0^t \sigma_u\sigma_v c_2(u,v;t,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\ &+2\sum_{i=1}^{d} \int_0^t \int_t^T \sigma_u\sigma_v c_2(u,v;u,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\ &+\sum_{i=1}^{d} \int_t^T \int_t^T \sigma_u\sigma_v c_2(u,v;u\vee v,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v),\end{aligned}$$

with $c_2(u,v;t,T) := \int_t^T g(s-u)g(s-v)\mathrm{d}s$.

A CIR model

$$
\begin{aligned}
P(0,T) &= E_{P^*}\left(\exp\left\{-\int_0^T r_s \mathrm{d}s\right\}\right)\\
&= E_{P^*}\left(\exp\left\{-\sum_{i=1}^{d}\int_0^T\int_0^T \sigma_u\sigma_v c_2(u,v;u\vee v,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v)\right\}\right)\\
&= \prod_{i=1}^{d} E\left(\exp\left\{-T\int_0^1\int_0^1 \sigma_{Tu}\sigma_{Tv}c_2(Tu,Tv;T(u\vee v),T)\mathrm{d}W_i(u)\mathrm{d}W_i(v)\right\}\right)\\
&= \left(1+\sum_{n=1}^{\infty}\frac{(2T)^n}{n!}\int_0^1\cdots\int_0^1 \begin{vmatrix} R(s_1,s_1) & \cdots & R(s_1,s_n)\\ \vdots & & \vdots \\ R(s_n,s_1) & \cdots & R(s_n,s_n)\end{vmatrix} \mathrm{d}s_1\cdots\mathrm{d}s_n\right)^{-d/2},
\end{aligned}
$$

with $R(u,v) = \sigma_{Tu}\sigma_{Tv}c_2(Tu,Tv;T(u\vee v),T)$.

Theorem

Assume $\sigma_t = \sigma \mathbf{1}_{\{t \geq 0\}}$, $g(s) = s^{\alpha}$, *for* $\alpha \in (0, 1/2)$, *let*

$$\tilde{R}(u,v) = \left[\frac{2(1-u)(1-v)}{2-u-v}\right]^{2\alpha+1} - \frac{1}{2}\left(\frac{|u-v|}{2}\right)^{2\alpha+1}\left[B\left(\frac{1}{2}-\alpha, \alpha+1\right) - B^{\gamma}\left(\frac{1}{2}-\alpha, \alpha+1\right)\right]$$

for $\gamma = \left(\frac{u-v}{2-(u+v)}\right)^2$, *and where* B *and* B^{γ} *are the beta and the incomplete beta functions respectively. Then, the price of a zero coupon bond, for the corresponding CIR model, is given by*

$$P(0,T) = \left[d^{R}\left(2\sigma^2 T\right)\right]^{-d/2} = \left[d^{\tilde{R}}\left(\frac{2\sigma^2 T^{2\alpha+2}}{1+2\alpha}\right)\right]^{-d/2} \approx \left[d^{\tilde{R}}_{Q_m}\left(\frac{2\sigma^2 T^{2\alpha+2}}{1+2\alpha}\right)\right]^{-d/2},$$

where

$$d^{R}(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} R(s_1,s_1) & \cdots & R(s_1,s_n) \\ \vdots & & \vdots \\ R(s_n,s_1) & \cdots & R(s_n,s_n) \end{vmatrix} \mathrm{d}s_1 \cdots \mathrm{d}s_n,$$

and we consider the Nyström-type approximation of $d(\lambda)$:

$$d^{R}_{Q_m}(\lambda) = \det\left[\delta_{ij} + \lambda w_i R(x_i, x_j)\right]_{i,j=1}^{m}.$$

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Bond prices

A CIR mode. The dynamics

A natural question, as we did in the previous section, is if the process

$$r_t = \sum_{i=1}^{d} \left(\int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s) \right)^2$$

can be seen as the solution of certain SDE. Write

$$Y_i(t) := \int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s),$$

then

$$r_t = \sum_{i=1}^{d} Y_i^2(t).$$

Assume that $g \in C^1$and it is square integrable, then Y is a semimartingale with

$$\mathrm{d}Y_i(t) = g(0)\sigma_t \mathrm{d}W_i(t) + \left(\int_0^t g'(t-s)\sigma_s \mathrm{d}W_i(s) \right) \mathrm{d}t,$$

suppose $g(0) \neq 0$ as well.

A CIR model. The dynamics

If we apply the Itô formula for continuous semimartingales we have

$$
\begin{aligned}
\mathrm{d}r_t &= \sum_{i=1}^{d} 2Y_i(t)\mathrm{d}Y_i(t) + \sum_{i=1}^{d} \mathrm{d}[Y_i, Y_i]_t \\
&= \sum_{i=1}^{d} 2g(0)\sigma_t Y_i(t)\mathrm{d}W_i(t) + \sum_{i=1}^{d} 2Y_i(t)\left(\int_0^t g'(t-s)\sigma_s \mathrm{d}W_i(s)\right)\mathrm{d}t \\
&\quad + \sum_{i=1}^{d} g^2(0)\sigma_t^2 \mathrm{d}t \\
&= 2g(0)\sigma_t\sqrt{r_t}\sum_{i=1}^{d}\frac{Y_i(t)}{\sqrt{r_t}}\mathrm{d}W_i(t) + \left(dg^2(0)\sigma_t^2 + \sum_{i=1}^{d} 2Y_i(t)\left(\int_0^t g'(t-s)\sigma_s \mathrm{d}W_i(s)\right)\right)\mathrm{d}t.
\end{aligned}
$$

A CIR mode. The dynamics

Then it is easy to see, by using the Lévy characterization of the Brownian motion, that

$$\sum_{i=1}^{d} \frac{Y_i(t)}{\sqrt{r_t}} \mathrm{d}W_i(t) = \mathrm{d}B(t),$$

where B is a Browian motion. Finally if $g'(t) = -bg(t)$, $g(0) = 1$, $\sigma_t = \sigma$, we have

$$\mathrm{d}r_t = (d\sigma^2 - 2br_t)\mathrm{d}t + 2\sigma\sqrt{r_t}\mathrm{d}B(t),$$

that is the dynamics of a CIR process.

A CIR mode. The dynamics

If g' is not square integrable then the process

$$Y_i(t) := \int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s),$$

is not a semimartingale and we cannot apply the usual Itô formula. In the particular case that

$$g(t-s) = e^{-b(t-s)} \int_0^{t-s} e^{bu} \beta u^{\beta-1} \mathrm{d}u, \beta \in (0, 1/2),$$

and $\sigma_u = \sigma$

$$Y_i(t) = \int_0^t \sigma e^{-b(t-s)} W_i^\beta(\mathrm{d}s)$$

$$W_i^\beta(t) := \int_0^t (t-s)^\beta W(\mathrm{d}s),$$

so

$$Y_i(t) = -b \int_0^t Y_i(s)\mathrm{d}s + \sigma W_i^\beta(t)$$

and the Itô formula for these processes is given by

$$\begin{aligned} \mathrm{d}r_t &= \sum_{i=1}^{d} 2\sigma Y_i(t)\delta W_i^{\beta}(t) - 2br(t)\mathrm{d}t + \sum_{i=1}^{d} \sigma^2 \left(\int_0^t (t-u)^{\beta} \mathrm{d}u \right) \mathrm{d}t \\ &= \left(d\sigma^2 t^{2\beta} - 2br(t) \right) \mathrm{d}t + 2\sigma\sqrt{r_t} \sum_{i=1}^{d} \frac{Y_i(t)}{\sqrt{r_t}} \delta W_i^{\beta}(t). \end{aligned}$$

But we do not have a characterization of the process

$$Z_t := \sum_{i=1}^{d} \int_0^t \frac{Y_i(s)}{\sqrt{r_s}} \delta W_i^{\beta}(s),\, t \geq 0.$$

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