Models for Bank Runs and Mean Field Games of Timing

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Motivation:

lectures / talks at Systemic Risk Summer School / Conference (Vancouver July 2014)

Rochet-Vives, **Fong-Gossner-Hörner-Sannikov**

As **economists**, use a **continuum of players** (atomless measure space) **Morris-Shin**,**He-Xiong**,

Try to understand models for **finitely many players** in the Mean Field framework

Work in progress (no numerics)

Joint work with **Dan Lacker** and **Geoffrey Zhu**

Games of Timing

N players with individual states $X_t^{N,i}$ at time t

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\nu}_t^N)dt + \sigma(t, X_t^{N,i})dW_t^i, \qquad i = 1, \cdots, N$$

interacting through the empirical distribution

$$\overline{\nu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}.$$

Each player chooses a $\mathcal{F}^{X^i}$ stopping time τ^i and try to maximize

$$J^i(\tau^1, \cdots, \tau^N) = \mathbb{E}\Big[g(\tau^i, X_{\tau^i}, \overline{\mu}^N([0, \tau^i])\Big]$$

where

- $\overline{\mu} = \frac{1}{N}\sum_{i=1}^N \delta_{\tau^i}$ empirical distribution of the τ^i's
- $g(t, x, y, p)$ is the reward to a player for
  - exercising his timing decision at time t when
  - his private signal is $X_t^I = x$,
  - the proportion of players who already exercised their right is p.

MFG Formulation

$N \to \infty$ and follow a **representative player**
e.g. $b(t, x, \nu) = b(t, x)$ states do not interact (for simplicity)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

- $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \le t \le T}$ information available to agent at time t
- $\mathcal{S}^X$ set of $\mathbb{F}^X$-stopping times

MFG of Timing

1. **Best Response Optimization**: for each fixed environment $\boldsymbol{\mu} \in \mathcal{P}([0, T])$ solve

$$\hat{\theta} \in \arg \sup_{\theta \in \mathcal{S}^X, \theta \le T} \mathbb{E}[g(\theta, X_\theta, \mu([0, \theta]))]$$

2. **Fixed Point Step**: find $\boldsymbol{\mu}$ so that

$$\mu([0, t]) = \mathbb{P}[\hat{\theta} \le t]$$

Solution with Randomized Stopping Times

Recall that even if

$$\begin{cases} \lim_{n\to\infty}(X, Y_n) = (X, Y) & \text{in law} \\ Y_n \ \text{is a function of} \ X \end{cases}$$

Y is not necessarily a function of X, i.e. $Y \in \sigma\{X\}$ may not hold.

Assume the reward $g : [0, T] \times \mathbb{R} \times \mathcal{P}([0, T]) \ni ((t, x, \mu) \mapsto g(t, x, \mu)\mathbb{R}$ is

- bounded
- continuous in (t, x) for μ fixed
- Lipschitz continuous in μ for (t, x) fixed

Then

$$\Pi : \mathcal{P}([0, T]) \times \mathcal{P}(C([0, T] \times [0, T])) \mapsto \mathbb{R}$$

$$(\mu, \xi) \hookrightarrow \Pi(\nu, \xi) = \int_{C([0,T])\times[0,T]} g(t, x_t, \mu)\xi(dx, dt)$$

is continuous

NB: 3rd assumption **NOT SATISFIED** for functions $t \hookrightarrow \mu([0, t])$,
unless $t \in \mathbb{T} \subset [0, T]$ with $\mathbb{T}$ finite!

Existence Proof

- Space $\tilde{\mathcal{S}}$ of randomized stopping times is compact (Baxter-Chacon)
- **Berge's maximum theorem** implies

$$\mathcal{P}([0,T]) \ni \nu \hookrightarrow \arg\sup_{\xi \in \tilde{\mathcal{S}}} \Pi(\nu, \xi)$$

  is upper hemicontinuous and compact-valued
- Followed by the projection on the first marginal, it is still upper hemicontinuous and compact-valued
- **Kakutani's fixed point theorem** implies existence

Existence with usual stopping times

Work directly on the **space of stopping times** (**complete lattice**) and use **Tarski's fixed point theorem**

Assume (for example):

- **Time increments of g are monotone in ν**

so we can check

$$\tau \hookrightarrow \arg\sup_{\tau' \in \mathcal{S}} \mathbb{E}[g(\tau', X_{\tau'}, F_\tau(\tau'))]$$

is monotone. Here $F_\tau(t) = \mathbb{P}[\tau \leq t]$ is the c.d.f. of τ.

Unfortunately, NOT THE CASE for "Bank Run" models !!!

A Continuous Time Model for Bank Runs

Gossner's talk

- N depositors
- Amount of each individual (initial & final) deposit $D_0^i = 1/N$
- Current interest rate r
- Depositors promised return $\bar{r} > r$
- Y_t = value of the assets of the bank at time t,
- Y_t Itô process, $Y_0 \geq 1$
- $L(y)$ liquidation value of bank assets if $Y = y$
- Bank has a credit line of size $L(Y_t)$ at time t at rate $\bar{r}$
- Bank uses credit line each time a depositor runs (withdraws his deposit)

Bank Run Model (cont.)

- Assets mature at time T, no transaction after that
- If $Y_T \geq 1$ every one is paid in full
- If $Y_T < 1$ **exogenous default**
- **Endogenous default** at time t if depositors try to withdraw **more** than $L(Y_t)$

Bank Run Model (cont.)

Each depositor $i \in \{1, \cdots, N\}$

- has access to a **private signal** X_t^i at time t

$$dX_t^i = dY_t + \sigma dW_t^i, \qquad i = 1, \cdots, N$$

- **chooses a time** τ^i at which to **ATTEMPT** to withdraw his/her deposit
- collects **return** $\bar{r}$ until time τ^i
- tries to **maximize**

$$J^i(\tau^1, \cdots, \tau^N) = \mathbb{E}\Big[g(\tau^i, Y_{\tau^i}, \tau^{-i})\Big]$$

where for example:

- $g(t, Y_t, \tau^1, \cdots, \tau^N) = e^{(\bar{r}-r)t\wedge\tau} + e^{-rt\wedge\tau}(L(Y_t) - N_t/N)^+ \wedge \frac{1}{N}$
- $N_t = \#\{i;\ \tau^i \le t\}$ number of withdrawals before t
- $\tau = \inf\{t;\ L(Y_t) < N_t/N\}$

BANK RUN MODEL: CASE OF FULL INFORMATION

Again **Gossner**'s talk

Assume

- $\sigma = 0$, i.e. Y_t is **public knowledge** !
- the function $y \hookrightarrow L(y)$ is also public knowledge

In **ANY** equilibrium

$$\tau^i = \inf\{t;\ L(Y_t) \leq 1\}$$

- Depositors withdraw at the **same time** (**run on the bank**)
- Each depositor gets his deposit back (**no one gets hurt!**)

Highly Unrealistic

Depositors should **wait longer** because of **noisy private signals**

GAMES OF TIMING

N players, states (observations / private signals) X_t^i at time t

$$dX_t^i = dY_t + \sigma dW_t^i, \qquad i = 1, \cdots, N$$

Y_t common unobserved signal (Itô process)

$$dY_t = \mu_t dt + \sigma_t dW_t^0$$

Each player maximizes

$$J^i(\tau^1, \cdots, \tau^N) = \mathbb{E}\Big[g(\tau^i, X_{\tau^i}, Y_{\tau^i}, \overline{\mu}^N([0, \tau^i])\Big]$$

where

- each τ^i is a $\mathcal{F}^{X^i}$ stopping time
- $\overline{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\tau^i}$ empirical distribution of the τ^i's
- $g(t, x, y, p)$ is the reward to a player for
  - exercising his timing decision at time t when
  - his private signal is x,
  - the unobserved signal is y,
  - the proportion of players who already exercised their right is p.

MFG FORMULATION

$$\begin{cases} dY_t = \mu_t dt + \sigma_t dW_t^0 \\ dX_t = dY_t + \sigma dW_t, \end{cases}$$

- $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \le t \le T}$ information available to depositor at time t
- $\mathcal{S}^X$ set of $\mathbb{F}^X$-stopping times
- $\boldsymbol{\mu}$ random environment $C([0,T]) \ni \omega^0 \mapsto \mu(\omega^0, \cdot) \in \mathcal{P}([0,T])$ proxy for the distribution of stopping time given $W^0_{[0,T]} = \omega^0$.

MFG of Timing

1. **Best Response Optimization**: for each random environment $\boldsymbol{\mu}$ solve

$$\hat{\theta} \in \arg \sup_{\theta \in \mathcal{S}^X, \theta \le T} \mathbb{E}[g(\theta, X_\theta, Y_\theta, \mu(W^0_{[0,T]}, [0,\theta]))]$$

2. **Fixed Point Step**: find $\boldsymbol{\mu}$ so that

$$\mu(W^0_{[0,T]}, [0,t]) = \mathbb{P}[\hat{\theta} \le t | W^0_{[0,t]}]$$

Existence in the Presence of the Common Noise

Strategy from solution of **MFGs with common noise** (**R.C.-Delarue-Lacker**,**R.C.-Delarue**)

- Discretize **Time** and **Time Increments of** W^0 on finite meshes
- Define approximate MFGs of timing problems with **projected common signal** $\mathbf{W}^{0,\ell,n}$
- Extend above existence result for no-common noise case to **approximate problems**
- Use (cheap) tightness arguments for joint laws of approximate solutions
- Check that **limit points are solutions** (with randomized stopping times)

Discretization Procedure

Fix $L = 2^\ell$ and $N = 2^n$, consider the dyadic time grid

$$t_i = \frac{iT}{N}, \qquad i = 0, 1, \cdots, N,$$

and the state space:

$$\mathbb{J} = \{-L, -L + \frac{1}{L}, -L + \frac{2}{L}, \cdots, L - \frac{2}{L}, L - \frac{1}{L}, L\}^d$$

Use projection $\Pi_L^{(d)}(x_1, \ldots, x_d) = (\Pi_L^{(1)} x_1, \ldots, \Pi_L^{(1)} x_d)$ onto $\mathbb{J}$ where

$$\Pi_L^{(1)} x = \begin{cases} L^{-1}\lfloor Lx \rfloor & \text{if } |x| \leq L, \\ L\text{sign}(x) & \text{if } |x| > L, \end{cases}$$

Now, for $k \geq 1$ define (recursively):

$$\Pi_{L,k+1}^{(d)}(x^1, \ldots, x^k, x^{k+1}) = (y^1, \ldots, y^k, y^{k+1})$$

provided that

$$(y^1, \ldots, y^k) = \Pi_{L,k}^{(d)}(x^1, \ldots, x^k), \qquad \text{and} \qquad y^{k+1} = \Pi_L^{(d)}\big(y^k + x^{k+1} - x^k\big) \in \mathbb{R}^d.$$

and finally

$$\big(V_1, \ldots, V_{N-1}\big) = \Pi_{L,N-1}^{(d)}\big(W_{t_1}^0, \ldots, W_{t_{N-1}}^0\big).$$

Discretized Random Environment

$$\boldsymbol{\nu} = (\nu^0, \ldots, \nu^{N-1}) \in \prod_{i=0}^{N-1} \mathcal{P}([0, t_{i+1}])^{\mathbb{J}^i}$$

(compact space) where for each $i \in \{0, 1, \ldots, N-1\}$,

$$\nu^i : \mathbb{J}^i \mapsto \mathcal{P}([0, t_{i+1}]).$$

Think of

- $\mathbb{J}^i$ as the set of all the possible values $(v_1, \ldots, v_i)$ of the discretizations of the path ω^0 up to time t_i,
- $\nu^i(v_1, \ldots, v_i)$ as the conditional distribution of the stopping time given the discretization of the path ω^0 up to time t_i.

Given $\boldsymbol{\nu} = (\nu^0, \ldots, \nu^{N-1})$, we construct its continuous time extension

$$\nu(v_1, \cdots, v_{N-1})[0, t] = \begin{cases} \nu^i(v_1, \cdots, v_i)[0, t] & \text{whenever } t \in [t_i, t_{i+1}) \\ & \qquad \text{for some } 0 \le i \le N-1; \\ \nu^{N-1}(v_1, \cdots, v_{N-1})[0, T]. & \end{cases}$$

and the (continuous time) random environment

$$\nu(\omega^0, [0, t]) = \nu(V_1(\omega^0), \cdots, V_{N-1}(\omega^0))[0, t]$$

If we can solve the optimal stopping problem, the fixed point step requires that

$$\mathbb{P}[\hat{\tau} \le t | V_1 = v_1, \ldots, V_i = v_i] = \nu(v_1, \cdots, v_{N-1})[0, t]$$

Solution of the Approximate MFG Problem

We start with a finite discrete input $\boldsymbol{\nu} = (\nu^0, \nu^1, \cdots, \nu^{N-1})$, **as before** we define the real valued function Π on $\prod_{i=0}^{N-1} \mathcal{P}([0, t_{i+1}])^{\mathbb{J}^i} \times \mathcal{M}$ by:

$$(\nu, \xi) \hookrightarrow \Pi(\nu, \xi) = \int_{\Omega \times [0,T]} g(t, X_t, Y_t, \nu(V^1, \cdots, V^N, [0, t]))\xi(d\omega, dt)$$

with $\Omega = \Omega^0 \times \Omega^1$, and $\omega = (\omega^0, \omega^1)$

As before

The function Π is jointly continuous,

as long as $\nu \hookrightarrow \nu(V^1, \cdots, V^N, [0, t])$ is continuous, so

as long as set $\mathbb{T}$ of times is finite !

Depositors can only withdraw their funds at a finite set of specific times

Solution of the Approximate MFG Problem

If $\mathbb{T}$ is finite Since the space $\mathcal{M}$ of randomized stopping times is compact for the Baxter-Chacon topology, Berge's maximum theorem implies that the correspondence

$$\prod_{i=0}^{N-1} \mathcal{P}([0, t_{i+1}])^{\mathbb{J}^i} \ni \nu \hookrightarrow \arg \sup_{\xi \in \mathcal{M}} \Pi(\nu, \xi)$$

is compact valued and upper hemicontinuous. Composed with the map $\Xi : \ \xi \hookrightarrow \mathbb{P}[\tau \leq t | W^0_{[0,T]} = \omega^0]$ if $\xi \leftrightarrow \tau$, it is still compact valued and upper hemicontinuous, Kakutani's fixed point theorem does the trick.

Using the Full Common Noise

- For each n and ℓ we have a solution with randomized stopping time

$$\hat{\tau}^{n,\ell} \longleftrightarrow \hat{\xi}^{n,\ell} \longleftrightarrow \hat{\nu}^{n,\ell}$$

- Let $\mathbb{P}^{n,\ell} \in \mathcal{P}(\Omega^0 \times \Omega^1 \times [0,T] \times \mathcal{P}([0,T]))$ be defined by

$$\mathbb{P}^{n,\ell}(d\omega^0, d\omega^1, d\mu, dt) = \hat{\xi}^{n,\ell}(d\omega^0, d\omega^1, dt)\delta_{\hat{\nu}^{n,\ell}(\omega^0,dt)}(d\mu)$$

- We treat $\mathbb{P}^{n,\ell}$ as a weak solution of the problem of MFG of timing.
- The $\mathbb{P}^{n,\ell}$ have $\mathbb{P}^0$ and $\mathbb{P}^1$ as first marginals and $\mathcal{P}([0,T])$ is compact
- The set of $\mathbb{P}^{n,\ell}$ is **tight**, so we can extract limit points who are also weak solution.

OK when g is Lip - 1 in its measure argument or (**in the bank run model**) when $\mathbb{T}$ is finite.

Back to the Finite Number of Players

Goal:h Have each player choose a (distributed) stopping time (i.e. from his own private information) at the possible cost of producing an **approximate Nash equilibria**

The MFG optimal (randomized) stopping time $\hat{\tau}$ can be viewed as stopping time on $\Omega^0 \times \Omega^1 \times [0,1]$ for $\mathbb{P}^0 \otimes \mathbb{P}^1 \otimes$ Leb.

Set

$$\Omega = \Omega^0 \times \Omega^{1,1} \times [0,1] \times \Omega^{1,2} \times [0,1] \cdots\cdots \times \Omega^{1,N} \times [0,1] \cdots\cdots$$

and for $\omega = (\omega^0, \omega^1, \lambda^1, \omega^2, \lambda^2, \cdots, \cdots)$ set

$$\theta^i(\omega) = \hat{\tau}(\omega^0, \omega^{1,i}, \lambda^i)$$

then we control the rate of convergence for the LLN

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \delta_{\theta^i} = \mathbb{P}[\hat{\tau} \le t | W^0_{[0,T]} = \omega^0]$$

and prove that $(\theta^1, \cdots, \theta^N)$ is an ϵ^N-Nash equilibrium with $\lim_{N\to\infty} \epsilon^N = 0$.

Still, the θ^i depend upon the knowledge of the common noise. Can be remedied for Gaussian models of **Y** by Kalman filtering.

Remaining Challenges

For the Mean Field Game (even without the common noise)

- Approximate solution with hitting / threshold rules
- Develop numerical procedures
- Understand role of the size of the observation noise
- Understand role of the volatility of the common signal **Y**

For the finite player games

- Find approximate Nash equilibria with distributed (randomized) stopping times