Arbitrage-Free Pricing of XVA

Agostino Capponi

Columbia University

joint work with Maxim Bichuch (WPI) and Stephan Sturm (WPI)

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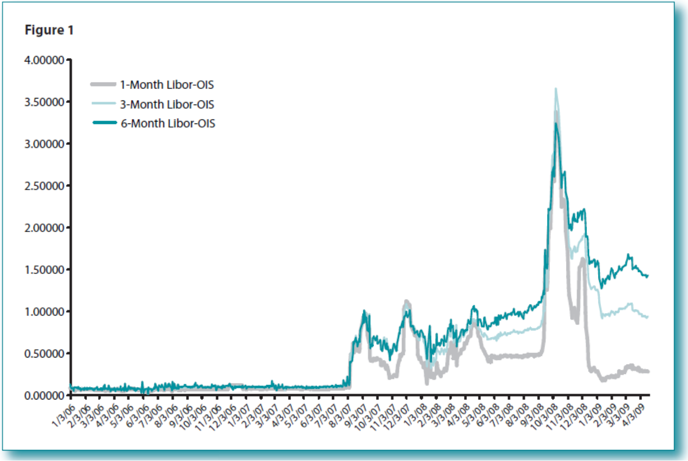
The LIBOR-OIS Spread



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- Widening of spreads is due to counterparty credit risk
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- 2013: Many banks (Barclays, JPM, BoA,...) introduce XVA desks

Literature

- Practitioner literature: Piterbarg (2010, 2012), Burgard & Kjaer (2010, 2011), Mercurio (2013)
- (Corporate) Finance literature: Hull & White (2012, 2013)
- Financial Mathematics literature: Bielecki & Rutkowski (2013), Brigo (2014), Crépey (2011, 2013), Crépey, Bielecki and Brigo (2014)

Main Contributions

- Develop a framework to characterize the total valuation adjustment (XVA) of a European style claim on a stock in presence of
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- Develop a framework to characterize the total valuation adjustment (XVA) of a European style claim on a stock in presence of
  - counterparty credit risk
  - funding spread
- Derive a nonlinear backward stochastic differential equation (BSDE) associated with the replicating portfolios of long and short positions in the claim.
- Develop an explicit representation of XVA in case of symmetric rates, but in presence of counterparty risk

The Market Model

The market model (I)

- **Treasury desk**: borrowing and lending at rates r_f^-, r_f^+, respectively
- **Stock** (S_t): used to the hedge market risk of transaction. Trading happens through repo market at rates r_r^-, r_r^+ (Duffie (1996))
- **Risky bonds** (P_t^I, P_t^C): underwritten by investor/counterparty and used to hedge default risk. Trading does not happen in the repo market

Stock Short-Selling

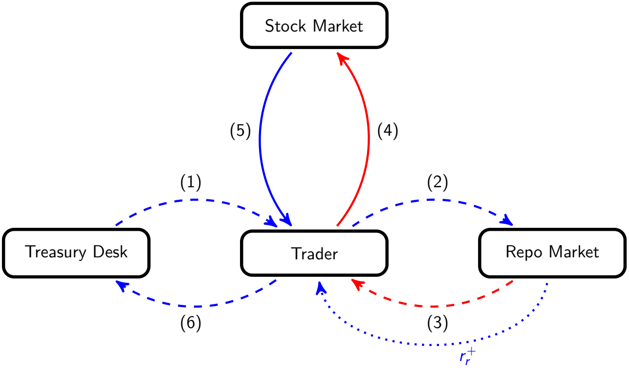

FIGURE: Security driven repo activity: Solid lines are purchases/sales, dashed lines borrowing/lending, dotted lines interest due; blue lines are cash, red lines are stock.

Stock Purchasing

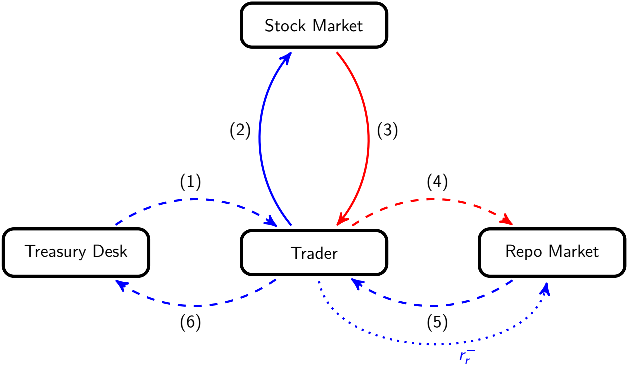

Figure: Cash driven repo activity: Solid lines are purchases/sales, dashed lines borrowing/lending, dotted lines interest due; blue lines are cash, red lines are stock.

The Market Model

The market model (II)

- We consider the dynamics

$$
\begin{aligned}
dS_t &= \mu S_t \, dt + \sigma S_t \, dW_t \\
dP_t^I &= \mu_I P_t^I \, dt - P_{t-}^I \, d\mathbb{1}_{\{\tau_I \leqslant t\}} \\
&= (\mu_I - h_I) P_t^I \, dt - P_{t-}^I \, d\varpi_t^I \\
dP_t^C &= \mu_C P_t^C \, dt - P_{t-}^C \, d\mathbb{1}_{\{\tau_C \leqslant t\}} \\
&= (\mu_C - h_C) P_t^C \, dt - P_{t-}^C \, d\varpi_t^C
\end{aligned}
$$

for independent default times τ_I, τ_C with constant default intensities h_I, h_C and martingales ϖ^I, ϖ^C

The market model (III)

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Proposition

No-arbitrage conditions:
Necessary: $r_r^+ \leqslant r_f^-$, $r_f^+ \leqslant r_f^-$, $r_f^+ < \mu_I$, $r_f^+ < \mu_C$.
Sufficient: *Necessary plus* $r_r^+ \leqslant r_f^+ \leqslant r_r^-$

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- Collateral is provided in form of cash (80%)
- Collateral can be reinvested (rehypothecated) (96%)
- The collateral provider receives interests at rate r_c^+. The collateral taker pays interests at rate r_c^-.

The Market Model

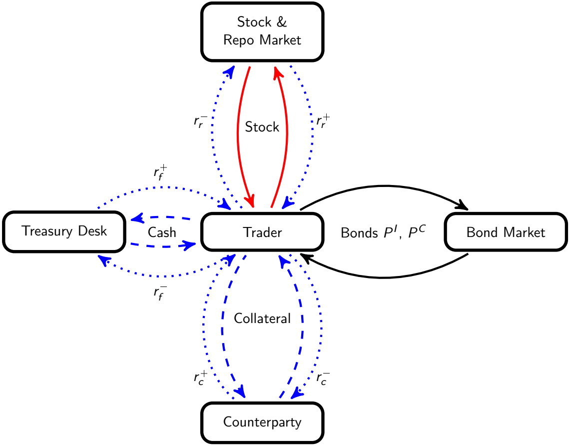

Figure: Solid lines are purchases/sales, dashed lines borrowing/lending, dotted lines interest due; blue lines are cash, red lines stock purchases for cash and black lines bond purchases for cash.

Closeout Payments and Valuation

- The closeout value of the claim is decided by a valuation agent (either party or third party) in accordance with market practices (ISDA)

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- We can then introduce a valuation measure $\mathbb{Q}$ under which r_D-discounted prices are $\mathbb{Q}$ martingales.
- The XVA will be computed under $\mathbb{Q}$

Collateral and Close-Out Valuation

- Collateral is a percentage α of the price of the contract

$$C_t = \alpha \mathbb{1}_{\{\tau_I \wedge \tau_C > t\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-r_D(T-t)} \Phi(S_T) \,\middle|\, \mathcal{F}_t \right]$$
$$:= \alpha \mathbb{1}_{\{\tau_I \wedge \tau_C > t\}} \hat{V}(t, S_t)$$

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&:= \alpha \mathbb{1}_{\{\tau_I \wedge \tau_C > t\}} \hat{V}(t, S_t)
\end{aligned}
$$

- Set $\tau = \tau_I \wedge \tau_C \wedge T$. The close-out payment is

$$
\begin{aligned}
\theta_\tau(\hat{V}) &= \theta_\tau(C, \hat{V}) \\
&:= \hat{V}(\tau, S_\tau) + \mathbb{1}_{\{\tau_C < \tau_I\}} L_C Y^- - \mathbb{1}_{\{\tau_I < \tau_C\}} L_I Y^+,
\end{aligned}
$$

where $Y := \hat{V}_\tau - C_\tau$ is the residual value of the claim at default

Wealth Dynamics

The dynamics of the wealth is given by

$$
\begin{aligned}
dV_t = \Big(& r_f^+ \left(\xi_t^f B_t^{r_f}\right)^+ - r_f^- \left(\xi_t^f B_t^{r_f}\right)^- + (r_D - r_r^-)\left(\xi_t S_t\right)^+ \\
& - (r_D - r_r^+)\left(\xi_t S_t\right)^- + r_D \xi_t^I P_t^I + r_D \xi_t^C P_t^C \Big)\, dt \\
& - r_c^- \left(\psi_t^c B_t^{r_c}\right)^+ dt + r_c^+ \left(\psi_t^c B_t^{r_c}\right)^- dt \\
& + \underbrace{(\cdots)}_{\text{martingales}}
\end{aligned}
$$

with $B_t^{r_f}$ funding account, $B_t^{r_c}$ collateral account, ξ_t, and ψ_t number of shares in the securities and various accounts

Arbitrage Pricing

Definition

A price $P \in \mathbb{R}$, of a derivative security with terminal payoff $\xi \in \sigma(S_t; t \leqslant T)$ is called *hedger's arbitrage-free*, if for all $\gamma \in \mathbb{R}$ buying γ securities for the price γP and hedging in the market with an admissible strategy does not create hedger's arbitrage.

Replicating Wealth

- $V_t^+(\gamma)$: wealth process when replicating the claim $\gamma\Phi(S_T)$, $\gamma > 0$. This means hedging the position after selling γ securities with terminal payoff $\Phi(S_T)$.
- $\left(-V_t^-(\gamma)\right)$: wealth process when replicating the claim $-\gamma\Phi(S_T)$, $\gamma > 0$. This means hedging the position after buying γ securities with terminal payoff $\Phi(S_T)$.

BSDE formulation

Set

$$
\begin{aligned}
f^+(t, v, z, z^I, z^C; \hat{V}) = -\Big(& r_f^+ \big(v + z^I + z^C - \alpha \hat{V}_t\big)^+ \\
& - r_f^- \big(v + z^I + z^C - \alpha \hat{V}_t\big)^- \\
& + (r_D - r_r^-) \frac{1}{\sigma} z^+ - (r_D - r_r^+) \frac{1}{\sigma} z^- \\
& - r_D z^I - r_D z^C \\
& + r_c^+ \alpha \hat{V}_t - (r_c^- - r_c^+) \big(\alpha \hat{V}_t\big)^- \Big) \\
f^-(t, v, z, z^I, z^C; \hat{V}) = & -f^+(t, -v, -z, -z^I, -z^C; -\hat{V}_t)
\end{aligned}
$$

BSDE formulation

The BSDEs

$$\begin{cases} -dV_t^+(\gamma) & = f^+\left(t, V_t^+, Z_t^+, Z_t^{I,+}, Z_t^{C,+}; \hat{V}\right) dt \\ & \quad - Z_t^+ \, dW_t^{\mathbb{Q}} - Z_t^{I,+} \, d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,+} \, d\varpi_t^{C,\mathbb{Q}} \\ V_\tau^+(\gamma) & = \gamma\Big(\theta_\tau(\hat{V})\mathbb{1}_{\{\tau<T\}} + \Phi(S_T)\mathbb{1}_{\{\tau=T\}}\Big) \end{cases}$$

$$\begin{cases} -dV_t^-(\gamma) & = f^-\left(t, V_t^-, Z_t^-, Z_t^{I,-}, Z_t^{C,-}; \hat{V}\right) dt \\ & \quad - Z_t^- \, dW_t^{\mathbb{Q}} - Z_t^{I,-} \, d\varpi_t^{I,\mathbb{Q}} - Z_t^{C,-} \, d\varpi_t^{C,\mathbb{Q}} \\ V_\tau^-(\gamma) & = \gamma\Big(\theta_\tau(\hat{V})\mathbb{1}_{\{\tau<T\}} + \Phi(S_T)\mathbb{1}_{\{\tau=T\}}\Big) \end{cases}$$

describe the wealth dynamics for buying/selling γ options

- Existence and uniqueness of solution can be guaranteed

No arbitrage

Theorem

Let Φ be a function of polynomial growth. If $V_0^- \leqslant V_0^+$, then all prices in the closed interval $[\pi^{inf} = V_0^-, V_0^+ = \pi^{sup}]$ are free of hedger's arbitrage.

XVA

- We define the total value adjustment XVA_t as

Definition

The seller's XVA is given as

$$\mathrm{XVA}_t^{sell} = V_t^+ - \hat{V}(t, S_t)$$

and the buyer's XVA as

$$\mathrm{XVA}_t^{buy} = V_t^- - \hat{V}(t, S_t).$$

The Extended Piterbarg Model

Extension of Piterbarg's model

- Allow for default of investor and counterparty
- Default risk is hedged by risky bonds
- Maintain Piterbarg's assumption of symmetric rates: $r_f = r_f^+ = r_f^-$, $r_r = r_r^+ = r_r^-$, $r_c = r_c^+ = r_c^-$
- BSDE becomes linear and $\mathrm{XVA}_t^{sell} = \mathrm{XVA}_t^{buy}$

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- **Note**: If $r_f = r_r = r_c = r_D$ we have no funding costs and recover the classical CVA/DVA setting
- In particular

$$\mathrm{XVA}_t = \mathrm{DVA}_t - \mathrm{CVA}_t$$

Price decomposition

$$
\begin{aligned}
e^{r_f t} V_t \mathbb{1}_{\{\tau \geqslant t\}} &= \mathbb{E}^{\mathbb{Q}}\Big[\big(B_T^{r_f}\big)^{-1} \Phi(S_T) \Gamma_t^T \mathbb{1}_{\{\tau = T\}} \,\Big|\, \mathcal{G}_t \Big] \\
&+ \mathbb{E}^{\mathbb{Q}}\Big[\big(B_{\tau_I}^{r_f}\big)^{-1} l_I \hat{V}(\tau_I, S_{\tau_I}) \Gamma_t^{\tau_I} \mathbb{1}_{\{t<\tau_I<\tau_C \wedge T; \hat{V}(\tau_I, S_{\tau_I}) \geqslant 0\}} \\
&\quad + \big(B_{\tau_I}^{r_f}\big)^{-1} \hat{V}(\tau_I, S_{\tau_I}) \Gamma_t^{\tau_I} \mathbb{1}_{\{t<\tau_I<\tau_C \wedge T; \hat{V}(\tau_I, S_{\tau_I}) < 0\}} \,\Big|\, \mathcal{G}_t \Big] \\
&+ \mathbb{E}^{\mathbb{Q}}\Big[\big(B_{\tau_C}^{r_f}\big)^{-1} l_C) \hat{V}(\tau_C, S_{\tau_C}) \Gamma_t^{\tau_C} \mathbb{1}_{\{t<\tau_C<\tau_I \wedge T; \hat{V}(\tau_C, S_{\tau_C}) < 0\}} \\
&\quad + \big(B_{\tau_C}^{r_f}\big)^{-1} \hat{V}(\tau_C, S_{\tau_C}) \Gamma_t^{\tau_C} \mathbb{1}_{\{t<\tau_C<\tau_I \wedge T; \hat{V}(\tau_C, S_{\tau_C}) \geqslant 0\}} \,\Big|\, \mathcal{G}_t \Big] \\
&+ \mathbb{E}^{\mathbb{Q}}\Big[\alpha(r_f - r_c) \int_{t \wedge \tau}^{\tau} \big(B_s^{r_f}\big)^{-1} \hat{V}(s, S_s) \Gamma_t^s \, ds \,\Big|\, \mathcal{G}_t \Big].
\end{aligned}
$$

with $l_I = 1 - (1-\alpha) L_I$ and $l_c = 1 - (1-\alpha) L_C$.

The Extended Piterbarg Model

- Direct computation leads to

$$\mathrm{XVA}_t = (A-1)\hat{V}(t, S_t),$$

where $A = \frac{V_t}{\hat{V}_t}$ is explicit

- Hedging strategies are explicit and given by

$$\xi_t = A \times \hat{V}_S(t, S_t),$$
$$\xi_t^i = \frac{A \times \hat{V}(t, S_t) - \theta_i(\hat{V}(t, S_t))}{P_t^i}, \quad i \in \{I, C\}.$$

and

$$\theta_C(\hat{v}) := \hat{v} + L_C((1-\alpha)\hat{v})^-,$$
$$\theta_I(\hat{v}) := \hat{v} - L_I((1-\alpha)\hat{v})^+.$$

The Extended Piterbarg Model

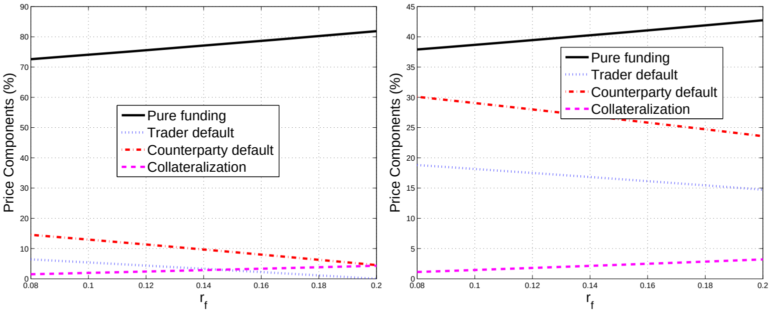
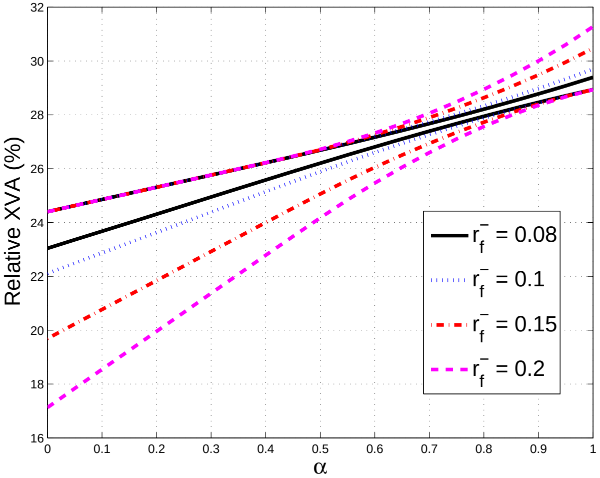


FIGURE: Left graph: $h_I^{\mathbb{Q}} = 0.15$, $h_C^{\mathbb{Q}} = 0.2$. Right graph: $h_I^{\mathbb{Q}} = 0.5$, $h_C^{\mathbb{Q}} = 0.5$.

XVA with Differential Rates

- What if borrowing and lending rates differ?: $r_f^- \neq r_f^+$, $r_r^- \neq r_r^+$, $r_c^- \neq r_c^+$
- BSDE becomes nonlinear: $V_t^+ \neq V_t^-$. We have a no-arbitrage interval for prices
- But, we can show that the semilinear PDE v corresponding to the BSDE V admits a unique classical solution

Band and funding spreads



Conclusion

- Developed an arbitrage-free valuation framework for XVA of an European style claim
- Seller's and buyer's XVA characterized as the solution of a nonlinear BSDEs with random terminal condition
- Funding component of XVA is predominant unless trader and his counterparty are very risky
- The no-arbitrage band widens as funding spreads and collateral levels increase

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