On the support of extremal martingale measures with given marginals: the countable case

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Motivation: robust finance

- Financial context: $(S_t)_{t=0}^2$ is the price of a risky asset, set $S_0 = 1$, $S_1 = X$ and $S_2 = Y$. Assume that S is defined on the canonical space.
- European Call prices (with maturities 1 and 2, all strikes $K > 0$) give the marginals μ, ν of the underlying at time 1 and 2 (Breeden-Litzenberger formula).
- No-arbitrage $\Rightarrow$ (S_t) should be a martingale under some probability compatible with marginals.

The natural set of pricing measures is the convex set:

$$\mathcal{M}(\mu, \nu) := \{Q : \; X \sim \mu, Y \sim \nu, \mathbb{E}^Q [Y|X] = X, Q - a.s.\}.$$

Basic properties of $\mathcal{M}(\mu, \nu)$

- Strassen (1965): $\mathcal{M}(\mu, \nu) \neq \emptyset$ if and only if $\mu \preceq \nu$ for the convex order, i.e.

$$\int f d\mu \leq \int f d\nu, \quad \text{for all convex functions } f$$

- From now on we will always assume $\mu \preceq \nu$ and their means are normalized to one.
- $\mathcal{M}(\mu, \nu)$ is compact for the weak convergence. Hence, it has *extremal points*: $Q \in \mathcal{M}(\mu, \nu)$ s.t. $Q = \alpha Q_1 + (1 - \alpha) Q_2$ with $\alpha \in (0, 1)$ and $Q_i \in \mathcal{M}(\mu, \nu)$ implies $Q = Q_1 = Q_2$.
- Extremal measures are solutions to max/min problems over $\mathcal{M}(\mu, \nu)$ (cf. Bauer Max Theorem).

Connection with optimal transport: primal problem

Let μ, ν be two marginals on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $\mu \preceq \nu$, with means normalized to one.

Let $f(x, y)$ be a given pay-off.

- Sup-problem: $\overline{P}(\mu, \nu, f) := \sup_{Q \in \mathcal{M}(\mu,\nu)} \mathbb{E}^Q[f(X, Y)]$.
- Inf-problem: $\underline{P}(\mu, \nu, f) := \inf_{Q \in \mathcal{M}(\mu,\nu)} \mathbb{E}^Q[f(X, Y)]$.

They give the robust super-(resp. sub-)replication prices of f.

For a duality theory: see Beiglböck-Henry-Labordère-Penkner (2013) and, more recently, Beiglböck-Nutz-Touzi (2015).

Two explicit examples of extremal measures

- Hobson-Klimmek (2013) solve the inf-problem for an *ATM forward start straddle of type II*: $f(x,y) = |y - x|$ and find that the optimizer is basically a trinomial model:

$$Q^{HK}(Y \in \{p(X), X, q(X)\}) = 1,$$

  for decreasing functions p, q.
- Henry-Labordère-Touzi (2013) solve the sup-problem for payoff s.t. $f_{xyy} > 0$ and, building on Beiglböck-Juillet (2012), find that the optimizer is basically a binomial model.
- Both such measures are extremal in $\mathcal{M}(\mu, \nu)$.

The problem and the related iterature

Can we describe the support of extremal points of $\mathcal{M}(\mu, \nu)$?

- Martingale measures: the extremal measures gives the binomial models (e.g. Dellacherie (1968), Jacod & Shiryaev (1998)).
- Prob measures with given marginals. Three types of characterisation:
  - *Density in $L^1(Q)$ of the set of all additive functions $\varphi(x) + \psi(y)$ (Douglas (1964), Lindenstrauss (1965))*.
  - No cycles in the support (Birkhoff (1946), Denny (1980), Letac (1966) ...).
  - Support as union of graphs of two functions whose compounded iterates have no fixed points (Denny (1980)).

Douglas-Lindenstrauss and the weak completeness

Theorem (via Douglas, Lindenstrauss)

$Q \in \mathcal{M}(\mu,\nu)$ is extremal if and only if the set

$$\left\{\varphi(x)+h(x)(y-x)-\psi(y) : (\varphi,\psi) \in L^1(\mu)\otimes L^1(\nu), h : \mathbb{R} \to \mathbb{R}\right\}\cap L^1(Q)$$

is dense in $L^1(Q)$.

Definition (Weak a.s. completeness)

$Q \in \mathcal{M}(\mu,\nu)$ is weakly a.s. complete if $\forall f : \mathbb{R}^2 \to \mathbb{R}$, $\exists(\varphi,\psi,h)$ s.t.

$$f(x,y) = \varphi(x) + h(x)(y-x) - \psi(y),\ Q - a.s.$$

The countable case

Let $Q \in \mathcal{M}(\mu, \nu)$ satisfy weak a.s. completeness.

- Assume that $\mathcal{X} := supp(\mu)$, $\mathcal{Y} := supp(\nu)$ are countable.
- Notice that weak a.s. completeness implies extremality.
- Set $\mathcal{Y}_Q(x) = \{y \in \mathcal{Y} : Q(x, y) > 0\}$, $x \in \mathcal{X}$.
- Hence: $|\mathcal{Y}_Q(x_1) \cap \mathcal{Y}_Q(x_2)| \leq 2$, for all pairs $x_1 \neq x_2$.

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- Hence: $|\mathcal{Y}_Q(x_1) \cap \mathcal{Y}_Q(x_2)| \leq 2$, for all pairs $x_1 \neq x_2$.
- *Proof:* Assume $\mathcal{Y}_Q(x_1) \cap \mathcal{Y}_Q(x_2) \supset \{y_1, y_2, y_3\}$ for some $x_1 \neq x_2$. Take f be s.t. $f(x_1, \cdot)$ is "strictly convex" while $f(x_2, \cdot)$ is "strictly concave" over $\{y_1, y_2, y_3\}$. Such a function cannot satisfy

$$f(x_i, y) = \varphi(x_i) + h(x_i)(y - x_i) - \psi(y), \quad y \in \mathcal{Y}_Q(x_i).$$

A sufficient condition: 2-link property

The property of the previous slide suggests the following sufficient condition:

Proposition

Let $Q \in \mathcal{M}(\mu, \nu)$. Assume that there exists a numbering $\mathcal{X} = (x_n)$ such that

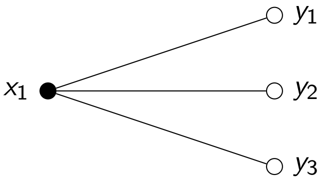
$$|\mathcal{Y}_Q(x_{n+1}) \cap \bigcup_{i=1}^{n} \mathcal{Y}_Q(x_i)| \leq 2, \quad \forall n \geq 1, \tag{0.1}$$

thus Q is extremal.

Remark: This is very much related to the notion of 2-degeneracy in graph theory.

Sketch of the proof: step 1

Let $f(x, y)$ be arbitrary. Assume there is a numbering $\mathcal{X} = (x_n)$ with 2-link property. Consider $\{x_1\} \times \mathcal{Y}_Q(x_1)$:

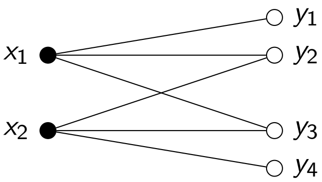


Fix $\varphi(x_1)$ and $h(x_1)$ arbitrarily, then $\psi(y_i)$ are determined by

$$f(x_1, y_i) + \psi(y_i) = \varphi(x_1) + h(x_1)(y_i - x_1), \quad y_i \in \mathcal{Y}_Q(x_1).$$

Sketch of the proof: step 2

Consider now $\{x_1, x_2\} \times (\mathcal{Y}_Q(x_1) \cup \mathcal{Y}_Q(x_2))$:



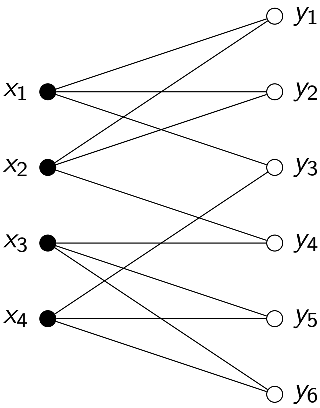
Now, step 1 gives $\psi(y_2)$ and $\psi(y_3)$, hence there is only one choice for intercept $\varphi(x_2)$ and slope $h(x_2)$ in the affine part of

$$f(x_2, y_i) + \psi(y_i) = \varphi(x_2) + h(x_2)(y_i - x_1), \quad y_i \in \mathcal{Y}_Q(x_2).$$

To conclude: iterate.

2-link property is not necessary

Let $Q \in \mathcal{M}(\mu, \nu)$ with support



Q doesn't satisfy 2-link ppty, nonetheless it's weak a.s. complete.

Hence extremal.

Conclusion

- Abstract sufficient and necessary condition via an application of Douglas-Lindenstrauss theorem.
- Some sufficient conditions in terms of the combinatorial properties of the conditional supports.
- Sufficient and necessary condition in terms of a generalized 2-link property: hopefully ready soon!
- Understanding the structure the support can give a way of generating models fulfilling weak completeness.
- Many questions remain open.