Functional calculus on integer-valued measures and martingale representation formulas for jump processes

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Overview

Martingale representation theorem for random measures

Let $J(dtdy)$ be an integer-valued random measure on $[0,T] \times \mathbb{R}^d \backslash \{0\}$ with compensator $\mu(dtdy)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
The filtration $(\mathcal{F}_t)$ generated by J is said to have the predictable representation property if any $\mathcal{F}_t$-adapted square-integrable martingale is such that

$$Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d \backslash \{0\}} \psi(s,y)(J-\mu)(ds\, dy)$$

with $\psi : [0,T] \times \mathbb{R}^d \backslash \{0\} \times \Omega \to \mathbb{R}^d \backslash \{0\}$, $\mathcal{F}_t$-predictable.
Example: The predictable representation property for J holds if J is a Poisson random measure (Itô, Ikeda-Watanabe). Not true in general for measures with non-deterministic compensators (Cohen 2013).

Martingale representation formulas

- Problem of finding an explicit representation appears in many applications like hedging, control of jump processes or BSDEs with jumps.
- Has been approached through Malliavin calculus for jump processes (Bismut 73, Lokka 05, Solé-Utzet-Vives 05) and Markovian techniques (Jacod-Méléard-Protter 00).
- In these results, ψ is represented in the form: $\psi = {}^{p}E[D_{t,z}Y|\mathcal{F}_t]$, where D is an appropriate "Malliavin" derivative operator, for which many constructions have been proposed.

Outline

- We introduce a pathwise calculus for functionals of integer-valued measures.
- Use it to provide an explicit version of the martingale representation formula for functionals of integer-valued measures.
- These results extend the Functional Itô calculus to integer-valued random measures.

Canonical space of integer-valued random measures

$\mathcal{M} = \mathcal{M}([0,T] \times \mathbb{R}^d \backslash \{0\})$ space of σ-finite integer-valued measures on $[0,T] \times \mathbb{R}^d \backslash \{0\}$:

$$j : \mathcal{B}([0,T] \times \mathbb{R}^d \backslash \{0\}) \to \mathbb{N}$$

such that j is σ-finite and there exists a sequence of $(t_i, z_i) \in [0,T] \times \mathbb{R}^d \backslash \{0\}$ with elements neither necessarily distinct nor ordered such that

$$j(.) = \sum_{i=0}^{\infty} \delta_{(t_i, z_i)}(.)$$

Stopped measure

For any $t \in [0, T]$ and $j \in \mathcal{M}([0, T] \times \mathbb{R}^d \backslash \{0\})$, define

$$j_t(.) := (. \cap ([0, t] \times (\mathbb{R}^d \backslash \{0\}))).$$

Similarly

$$j_{t-}(.) := (. \cap ([0, t) \times (\mathbb{R}^d \backslash \{0\}))).$$

Non-anticipative functionals on measures

A map $F : [0, T] \times \mathcal{M}([0, T] \times (\mathbb{R}^d \backslash \{0\}) \to \mathbb{R}$ is called

- a non-anticipative functional if $F(t, j) = F(t, j_t)$.
- a predictable functional if $F(t, j) = F(t, j_{t-})$.

Definition

For $z \in \mathbb{R}^d$, define the pathwise finite difference operator $\nabla_{j,z}$

$$\nabla_{j,z} F(t,j) = F(t, j_{t-} + \delta_{(t,z)}) - F(t, j_{t-}) \qquad \text{(FD)}$$

Denote

$$\begin{aligned} \nabla_j F : [0,T] \times \mathcal{M} \times (\mathbb{R}^d - \{0\}) &\mapsto \mathbb{R} \\ (t,j,z) &\to \nabla_{j,z} F(t,j) \end{aligned}$$

Then the operator $\nabla_j : F \mapsto \nabla_j F$ maps non-anticipative functionals into predictable functionals.

Integral functionals of integere-valued measures

Proposition: integral functionals

Let $\psi : [0, T] \times \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ be a kernel with support bounded away from 0, and

$$F(t, j) = \int_0^t \int_{\mathbb{R}^d \backslash \{0\}} \psi(s, y)(j - \mu)(ds\ dy),$$

with $\mu : \mathcal{B}([0, T] \times \mathbb{R}_0^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d \backslash \{0\}) \to \mathbb{R}^+$ predictable in j and σ-finite. Then F is non-anticipative and $\nabla_j F = \psi$, i.e.

$$\forall (t, z) \in [0, T] \times \mathbb{R}^d, \quad \nabla_{j,z} F(t, j) = \psi(t, z).$$

Constructing the probability space

We now consider a integer-valued *random* measure with law $\mathbb{P}$ on $(\Omega, \mathcal{F})$, with compensator $\mu(dt\ dz)$, where

- $\Omega = \mathcal{M}([0, T] \times \mathbb{R}^d \backslash \{0\})$; for $\omega \in \Omega$,

$$\omega(t, .) = j_t(.)$$

- $\mathcal{F}$ a σ-algebra making the $J(A)$, $A \in \mathcal{B}([0, T] \times \mathbb{R}_0^d)$ measurable.
- The filtration $\mathbb{F}$ generated by J.

We will now show that the pathwise operator ∇_j admits a unique closure on the space of square-integrable $\mathbb{F}$-martingales, which is the adjoint of the stochastic integral with respect to the $J - \mu$.

Space of square-integrable integrands

Assume that the compensator μ of J satisfies

$$\mu(dsdy) = \nu(\{s\} \times dy)ds \text{ and } \int_0^T \int_{\mathbb{R}^d\setminus\{0\}} |z|^2 \vee 1\mu(dsdz) < \infty, \mathbb{P}\text{-a.s.}$$

define
$\mathcal{L}^2_{\mathbb{P}}(\mu)$: $\{$ space of predictable random fields $\psi : [0, T] \times R^d \to \mathbb{R}$ such that

$$\|\psi\|^2_{\mathcal{L}^2_{\mathbb{P}}(\mu)} := E[\int_{[0,T]\times\mathbb{R}^d\setminus\{0\}} |\psi(s, y)|^2 \mu(ds\ dy)] < \infty\}$$

and

$\mathcal{I}^2_{\mathbb{P}}(\mu) :=$

$$\{Y : [0, T] \times \Omega \to \mathbb{R} | Y(t) = \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} \psi(s, y)(J - \mu)(dsdy), \psi \in \mathcal{L}^2_{\mathbb{P}}(\mu)\}$$

$$\|Y\|^2_{\mathcal{I}^2_{\mathbb{P}}(\mu)} := E[|Y(T)|^2]$$

Set $\mathcal{S}$ of regular simple predictable fields

$\psi : [0, T] \times \mathbb{R}^d \times \mathcal{M}([0, T] \times \mathbb{R}^d \backslash \{0\}) \to \mathbb{R}^d$ belongs to $\mathcal{S}$ if

- ψ is predictable: $\psi(t, z, j) = \psi(t, z, j_{t-})$
- and

$$\psi(t, z, j_t) = \sum_{\substack{i=0\\k=1}}^{I,K} \psi_{ik}(j_{t_i}) \mathbb{1}_{(t_i, t_{i+1}]}(t) \mathbb{1}_{A_k}(z)$$

with $A_k \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$,$0 \notin \overline{A_k}$ and

$$\psi_{ik} = g_{ik}(S^{\frac{1}{n}}_{\tau_1}, \dots, S^{\frac{1}{n}}_{\tau_n}),$$

g_{ik} bounded and $0 \leq \tau_1 \leq \tau_n \leq t_i$,

$$S^{\epsilon}_t := j([0, t] \times (\epsilon, \infty)^d)$$

Stochastic operators

The operator of compensated stochastic integration w.r.t J is defined as

$$I : \mathcal{L}^2_{\mathbb{P}}(\mu) \to \mathcal{I}^2_{\mathbb{P}}(\mu)$$
$$\psi \mapsto \int_0^{\cdot} \int_{\mathbb{R}^d \setminus \{0\}} \psi(s,y)(J-\mu)(dsdy)$$

The operator ∇_J is defined on $I(\mathcal{S})$ as

$$\nabla_J : I(\mathcal{S}) \to \mathcal{L}^2_{\mathbb{P}}(\mu)$$
$$F(t, J_t) = \int_0^{\cdot} \int_{\mathbb{R}^d \setminus \{0\}} \psi(s,y)(J-\mu)(dsdy)$$
$$\mapsto \nabla_{j,z} F(t, J_t)$$
$$= F(t, J_{t-} + \delta_{t,z}) - F(t, J_{t-})$$
$$= \psi(t,z)$$

Density of regular integral functionals

Proposition

The set $I(\mathcal{S})$ of processes Y that have the following functional representation

$$Y(.) = F(., J) = \int_0^{.} \int_{\mathbb{R}^d \setminus \{0\}} \psi(s, y)(J - \mu)(dsdy) \quad (1)$$

with $\psi \in \mathcal{S}$, is dense in $\mathcal{I}^2_{\mathbb{P}}(\mu)$.

In other words, integral processes having a regular functional representation are dense in $\mathcal{I}^2_{\mathbb{P}}(\mu)$.

∇_J as the adjoint of the stochastic integral

Theorem

The operator $\nabla_J : I(\mathcal{S}) \to \mathcal{L}^2_{\mathbb{P}}(\mu)$ is closable in $\mathcal{I}^2_{\mathbb{P}}(\mu)$, and is the adjoint of the stochastic integral in the sense of the following integration by parts.

$$
\begin{aligned}
< Y, I(\phi) >_{\mathcal{I}^2_{\mathbb{P}}(\mu)} &:= E\left[Y(T)\int_0^T \int_{\mathbb{R}^d\setminus\{0\}} \phi(s,y)(J-\mu)(dsdy)\right] \\
&= E\left[\int_0^T \int_{\mathbb{R}^d\setminus\{0\}} \nabla_J Y(s,y)\phi(s,y)\mu(dsdy)\right] \\
&=: < \nabla_J Y, \phi >_{\mathcal{L}^2_{\mathbb{P}}(\mu)}
\end{aligned}
$$

Representation theorem for square-integrable martingales

Martingale representation formula

If the filtration $\mathbb{F}$ generated by J has the martingale representation property, and if a process $Y(.) = F(., J)$ –with F an non-anticipative functional– is a square integrable martingale, then

$$Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \nabla_J Y(s, y)(J - \mu)(dsdy)$$

Including the continuous component

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ constructed similarly as before, and $\mathbb{F}$ generated by an integer valued random measure J with compensator

$$\mu(dsdy) = \nu(\{s\} \times dy)ds \text{ and } \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 \mu(dsdz) < \infty, \mathbb{P}\text{-a.s.},$$

and a continuous martingale X, any square-integrable martingale writes, $\mathbb{P}$-a.s.

$$Y(T) = Y(0) + \int_0^T \nabla_X Y(s) dX(s) + \int_0^T \int_{\mathbb{R}_0^d} \nabla_J Y(s,z) \widetilde{J}(ds\ dz),$$

with $\nabla_X Y$ defined as follows.

Defining $\mathcal{S}_c$ as:

Set $\mathcal{S}_c$ of regular simple predictable processes

$\psi : [0,T] \times \times \mathcal{D}([0,T]) \times \mathcal{M}([0,T] \times \mathbb{R}^d \backslash \{0\}) \to \mathbb{R}^d$ belongs to $\mathcal{S}_c$ if

- ϕ is predictable: $\psi(t,z,x,j) = \psi(t,z,x_{t-},j_{t-})$
- and

$$\phi(t, x_t, j_t) = \sum_{\substack{i=0 \\ k=1}}^{I} \phi_i(x_{t_i}, j_{t_i}) \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

with

$$\phi_i = g_i(x(\tau_1), \cdots, x(\tau_n), S_{\tau_1}^{\frac{1}{n}}, \ldots, S_{\tau_n}^{\frac{1}{n}}),$$

$g_{ik} \in C_c^\infty(\mathbb{R}^{2n}, \mathbb{R}^n)$ and $0 \leq \tau_1 \leq \tau_n \leq t_i$,

$$S_t^\epsilon := j([0,t] \times (\epsilon, \infty)^d)$$

Define $\mathcal{L}^2_{\mathbb{P}}([X]) :=$ { space of predictable processes $\psi : [0, T] \times \Omega \to \mathbb{R}$ such that

$$\|\psi\|^2_{\mathcal{L}^2_{\mathbb{P}}([X])} := E[\int_{[0,T]\times\mathbb{R}^d\setminus\{0\}} |\psi(s,y)|^2[X](ds\ dy)] < \infty\}$$

and

$$\mathcal{I}^2_{\mathbb{P}}([X]) :=$$

$$\{Y : [0, T] \times \Omega \to \mathbb{R} | Y(t) = \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} \phi(s)dX(s), \psi \in \mathcal{L}^2_{\mathbb{P}}([X])\}$$

$$\|Y\|^2_{\mathcal{I}^2_{\mathbb{P}}([X])} := E[|Y(T)|^2]$$

Defining:

$$\begin{aligned} I_X : & \mathcal{L}^2_{\mathbb{P}}([X]) \to \mathcal{I}^2_{\mathbb{P}}([X]) \\ & \phi \mapsto \int_0^{\cdot} \phi(s) dX(s), \end{aligned}$$

The operator

$$\begin{aligned} \nabla_x : & I_X(\mathcal{S}_c) \to \mathcal{I}^2_{\mathbb{P}}([X]) \\ & F(t, x_t, j_t) \mapsto \lim_{h \to 0} \frac{F(t, x_t + h\mathbb{1}_{[t,\infty)}, j_t) - F(t, x_t, j_t)}{h} \\ & = \phi(t) \end{aligned}$$

can be closed in $\mathcal{I}^2_{\mathbb{P}}([X])$ in the same fashion as in the jump case, and the closure is $\mathcal{I}^2_{\mathbb{P}}([X])$ itself.

Continuous- and jump-part comparison

Creating the martingale-generating measure

$$M(ds\ dz) := \mathbb{1}_{\{z=0\}} dX(s) + z\widetilde{J}(ds\ dz),$$

the martingale representation formula rewrites as:

$$Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d} \nabla Y(s,z) M(ds\ dz) \quad \mathbb{P}\text{-a.s.}$$

where

$$\nabla Y(s,z) := \begin{cases} \nabla_X Y(s,y) \text{ if } z = 0 \\ \frac{\nabla_J Y(s,z)}{z} \text{ otherwise.} \end{cases}$$

The continuous component is the limit operator of the operator appearing in the jump case.

Example: representation of the supremum of a Lévy process

The representation formula for the supremum $\overline{X}$ of a Lévy process X

1. was proved by Shiryaev and Yor (2004) using Itô's formula.
2. was reproved more recently by Rémillard-Renaud(2011) using Malliavin calculus.

Main challenges in the functional Itô case:

1. Infinite variation: infinite variation, induced by a continuous component and/or an infinite jump activity destroys the pathwise characterisation of the quantities.
2. In case the Lévy process has a continuous component: the supremum is not a vertically differentiable functional.

$\hookrightarrow$ we need to truncate the jumps and smoothen the functional.

Functional approximation

Define the Lévy process

$$X(t) = X(0) + \mu t + \sigma W(t) + \int_0^t \int_{|z|<1} z\widetilde{J}(dsdz) + \int_0^t \int_{|z|\geq 1} zJ(dsdz)$$

and its approximation

$$X^n(t) = X(0)+\mu t+\sigma W(t)+\int_0^t \int_{(-1,-\frac{1}{n})\cup(\frac{1}{n},1)} z\widetilde{J}(dsdz)+\int_0^t \int_{|z|\geq 1} zJ(dsdz)$$

It can be shown that

$$E[\overline{X}(T)|\mathcal{F}_t] = \overline{X}(t) + \int_{\overline{X}(t)-X(t)}^{\infty} F_{T-t}(u)du,$$

with $F_{T-t}(u) = \mathbb{P}(\overline{X}(T-t) \leq u)$.

Furthermore, consider the approximation of the supremum functional,

$$L^a(f,t) = \frac{1}{a}\log(\int_0^t e^{af(s)}ds).$$

Define the approximation:

$$Y^{a,n}(t) = L^a(X^n,t) + \int_{L^a(X^n,t)-X^n(t)}^{\infty} F_{T-t}(u)du$$

Since $X^n \underset{n\to\infty}{\overset{L^2}{\longrightarrow}} X$ and $L^a(f,T) \underset{a\to 0}{\longrightarrow} \sup_{s\in[0,T]} f(s)$, one can show:

$$\lim_{n\to\infty}\lim_{a\to\infty} E[|Y^{a,n}(T) - \overline{X}(T)|^2] = 0$$

Operator computations

We can now compute

$$\nabla_J Y^{a,n}(t,z) = \int_{L^a(X^n,t)-X^n(t)-z}^{L^a(X^n,t)-X^n(t)} F_{T-t}(u)du$$
$$\underset{\substack{a\to\infty \\ n\to\infty}}{\longrightarrow} \int_{\overline{X}(t)-X(t)-z}^{\overline{X}(t)-X(t)} F_{T-t}(u)du = \nabla_J \overline{X}(t,z)$$

and

$$\nabla_W Y^{a,n}(t) = \lim_{h\to 0} \frac{1}{h} \int_{L^a(X^n,t)-X^n(t)-\sigma h}^{L^a(X^n,t)-X^n(t)} F_{T-t}(u)du$$
$$= F_{T-t}(L^a(X^n,t) - X^n(t))$$
$$\underset{\substack{a\to\infty \\ n\to\infty}}{\longrightarrow} \sigma F_{T-t}(\overline{X}(T) - X(t)) = \nabla_W \overline{X}(t)$$

References

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