Simple Arbitrage

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The (mixed) fractional Black-Scholes model

- Recall: Fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a centered Gaussian process Z with covariance structure

$$Cov(Z_t, Z_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- It exhibits long range dependence for $H > 1/2$.
- Consider a model with a constant bond and a stock given by

$$X_t = X_0 \exp\{\sigma W_t + \eta Z_t + at + bt^{2H}\},$$

where W is A Brownian motion and Z is a fractional Brownian motion independent of W. This model is called fractional Black-Scholes model, if $\sigma = 0$, and otherwise mixed fractional Black-Scholes model.

The (mixed) fractional Black-Scholes model

- The fractional Black-Scholes model fails the semimartingale property (except in the Brownian motion case), and so does the mixed fractional Black-Scholes model for Hurst parameter $H \in (0, 3/4] \setminus \{1/2\}$.
- Hence, by the fundamental theorem of asset pricing these models exhibit a free lunch with vanishing risk.
- Question: Are these models free of arbitrage, when the strategies are suitably restricted?

  We here consider 'simple strategies'?

Problem Setting

- Financial market with two assets (for simplicity) on $[0, T]$: constant bond $B_t = 1$, a stock X_t (adapted process with right-continuous paths).
- Simple strategy (number of risky shares held by the investor):

$$\Phi_t = \phi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}$$

where the τ_j's are a finite number of ordered stopping times with values in $[0, T]$ and the ϕ_j's are $\mathcal{F}_{\tau_j}$-measurable random variables.

Corresponding wealth process with zero initial endowment:

$$V_t(\Phi) = \sum_{j=0}^{n-1} \Phi_{\tau_{j+1}} (X_{t\wedge\tau_{j+1}} - X_{t\wedge\tau_j})$$

Problem Setting

- A simple strategy is said to be of Cheridito class, if there is an $h > 0$ such that for every j, $\tau_{j+1} \geq \tau_j + h$ on $\{\tau_{j+1} > \tau_j\}$.
- A simple arbitrage is a simple strategy which is an arbitrage:

$$V_T(\Phi) \geq 0 \ P\text{-a.s.}, \quad P(\{V_T(\Phi) > 0\}) > 0$$

- Questions:

  How to characterize absence of simple arbitrage?

  What kind of easy sufficient conditions are available for absence of arbitrage (for large subclasses of) simple strategies?

  How about simple arbitrage for models driven by fractional Brownian motion?

A first example of simple arbitrage

Example

Suppose W_t a Brownian motion and for some fixed $T > 0$

$$X_t = \begin{cases} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{W_t^2}{2(T-t)}}, & 0 \le t < T \\ 0, & t \ge T. \end{cases}$$

Then, X_t has continuous paths P-almost surely and is a local martingale (Itô formula).

As $X_0 = \frac{1}{\sqrt{2\pi T}} > 0$ and $X_T = 0$, the simple strategy $\Phi_t = -\mathbf{1}_{(0,T]}(t)$ is an arbitrage.

Here the arbitrage is obtained in the 'long run' by waiting up to time T.

It clearly is of Cheridito class.

No obvious arbitrage

- As in Guasoni, Rasonyi, Schachermayer (2010):

  $(X_t, \mathcal{F}_t)$ has no obvious arbitrage (NOA), if for all stopping times σ with $P(\{\sigma < T\}) > 0$ and every $\epsilon > 0$

  $$P(\{\sigma < T\} \cap \{\sup_{t\in[\sigma,T]} X_t < X_\sigma + \epsilon\}) > 0$$

  and

  $$P(\{\sigma < T\} \cap \{\inf_{t\in[\sigma,T]} X_t > X_\sigma - \epsilon\}) > 0$$

- If (NOA) is violated, e.g. there are σ, ϵ such that

  $$P(\{\sup_{t\in[\sigma,T]} X_t \geq X_\sigma + \epsilon\} | \{\sigma < T\}) = 1,$$

  then a simple arbitrage can be obtained by buying one share at time σ and selling it, once the stock price has increased by ϵ.

Conditional full support

- A continuous process S has conditional full support (CFS) if, for every time $0 \leq t \leq T$

$$\text{supp } P(S \in \cdot | \mathcal{F}_t^S) = C_{S_t}([t, T]) \quad \text{a.s.},$$

where $\mathcal{F}^S$ denotes the augmented filtration generated by S.

- If X has (CFS), then $(X_t, \mathcal{F}_t^X)$ satisfies (NOA), because the (CFS)-property extends automatically to stopping times, see Guasoni, Rasonyi, Schachermayer (2008).
- Examples: Fractional Brownian motion Z satisfies conditional full support for every choice of the Hurst parameter $0 < H < 1$ and so does mixed fractional Brownian motion $Z + W$, where W is a Brownian motion independent of W. Cf. Cherny (2008) or Gasbarra, Sottinen, van Zanten (2011).

Conditional full support and Cheridito class

Theorem (B./Sottinen/Valkeila, 2011)

Suppose $\log(X_t)$ *or* X_t *satisfies (CFS). If the simple strategy* Φ *belongs to the Cheridito class, i.e.* $\tau_{j+1} \geq \tau_j + h$ *for some constant* $h > 0$ *on* $\{\tau_{j+1} > \tau_j\}$*, then* Φ *is not a simple arbitrage for* $(X_t, \mathcal{F}_t^X)$*.*

Remark: The delay between two stopping times can be localized in a way that e.g. stopping times of the form

$$\tau_{j+1} = \inf\{t > \tau_j;\ X_t - X_{\tau_j} \geq b_t^j\} \wedge T$$

for positive continuous functions b^j are covered, and the above theorem still holds.

A second example of simple arbitrage

Question: If X satisfies (NOA), what kind of simple arbitrage is still possible?

Example

Suppose $X_t = \exp\{W_t + t^\alpha\}$ for some $0 < \alpha < 1/2$. Then clearly, $\log(X_t)$ satisfies (NOA). Define the stopping time

$$\tau := \inf\{t > 0;\ \log(X_t) < 0\} > 0$$

by the law of the iterated logarithm. Then, $\Phi_t = \mathbf{1}_{(0,\tau\wedge 1/N]}$ is a simple arbitrage for sufficiently large N.

Note that this arbitrage is 0-admissible, i.e.

$$V_t(\Phi) \geq 0 \quad \text{a.s.}$$

Characterizing absence of simple arbitrage

Theorem

Suppose X has right-continuous paths. Then the following assertions are equivalent:

(i) X is free of arbitrage with simple strategies.
(ii) X satisfies (NOA) and X has no 0-admissible arbitrage of the form $\pm\mathbf{1}_{(\sigma,\tau]}$ with stopping times $\sigma \le \tau \le T$.

Idea: Suppose a simple arbitrage is not 0-admissible. Then its wealth process drops with positive probability below some level $-\delta$. Buying this strategy at such a time, the wealth process must increase by δ (because it is nonnegative at time T). This gives rise to an obvious arbitrage.

Two-way crossing

- Question: How to exclude 0-admissible simple arbitrage?
- Assume that X has continuous paths.
- Suppose $\sigma \le T$ is a stopping time and

$$\sigma_{\pm} \;=\; \inf\{t \ge \sigma, \quad \pm(X_t - X_\sigma) > 0\} \wedge T.$$

  $(X_t, \mathcal{F}_t)$ satifies two-way crossing (TWC), if $\sigma_+ = \sigma_-$ a.s. for any stopping time $\sigma \le T$.
- (TWC) is a condition on the fine structure of the paths. Whenever the stock price moves from X_σ, it will cross the level X_σ infinitely often in time intervals of length ϵ for every $\epsilon > 0$.

A characterization of simple arbitrage

Lemma

Suppose X is continuous. Then:

$(X_t, \mathcal{F}_t)$ *satifies (TWC)* $\Leftrightarrow$ $(X_t, \mathcal{F}_t)$ *is free of 0-admissible simple arbitrage.*

Theorem

Suppose X is continuous. Then the following assertions are equivalent:

(i) $(X_t, \mathcal{F}_t)$ *has no simple arbitrage.*

(ii) $(X_t, \mathcal{F}_t)$ *satisfies (NOA) and (TWC).*

How to check (TWC)?

- Given a stopping time write $X^\tau_t = X_{\tau+t} - X_\tau,\ t \geq 0$.
- Try to decompose
$$X^\tau_t = Y^\tau_t + A^\tau_t,$$
where Y^τ_t satisfies a law of the iterated logarithm at $t = 0$.
- Try to control the path regularity of A^τ in a way that it is 'dominated' by the law of the iterated logarithm for Y^τ

Mixed fractional Black-Scholes model with $H > 1/2$

- Recall $\log(X_t) = \sigma W_t + \eta Z_t + at + bt^{2H}$
- Given a stopping time τ write

$$\begin{aligned} & \log(X_{\tau+t}) - \log(X_\tau) \\ = \; & \sigma W_t^\tau + \left(\eta(Z_{\tau+t} - Z_\tau) + at + b(\tau+t)^{2H} - b\tau^{2H}\right) \end{aligned}$$

- The process in brackets is $1/2$-Hölder continuous, because $H > 1/2$, and W^τ is a Brownian motion.
- By the law of the iterated logarithm for Brownian motion, $\log(X_{\tau+t}) - \log(X_\tau)$ crosses the level 0 infinitely often in arbitrary small time intervals.
- Hence, $\sigma_+ = \sigma_- = \sigma$.
- As mixed fBm satisfies (CFS), $\log(X_t)$ is free of simple arbitrage and so is X_t.

A sufficient condition for (TWC)

Combining the above argument with the Dambis/Dubins/Schwarz theorem one gets:

Theorem

Suppose $X_t = M_t + Y_t$, where M is a continuous $(\mathcal{F}_t)$-local martingale and Y_t is an $(\mathcal{F}_t)$-adapted process. We assume:

1) There is a strictly positive random variable ϵ such that for every $0 \leq s \leq t \leq T$

$$\langle M \rangle_t - \langle M \rangle_s \geq \epsilon(t-s).$$

2) Y is $1/2$-Hölder continuous, i.e. there is a positive random variable C such that for every $0 \leq s \leq t \leq T$

$$|Y_t - Y_s| \leq C|t-s|^{1/2}.$$

Then, $(X_t, \mathcal{F}_t)$ satisfies (TWC).

More mixed examples...

Similarly to the mixed fractional Black-Scholess model, one can also treat:

- 1-dimensional mixed stochastic volatility models (e.g. mixed Heston models)
- 1-dimensional mixed local volatility models.

Then, (CFS) for the log-prices can be derived from a result by Pakkanen (2010) under suitable conditions and (TWC) follows from the above sufficient condition under suitable conditions.

How about fractional Brownian motion?

- Suppose Z is a fractional Brownian motion with arbitrary Hurst parameter and σ is a stopping time with respect to its natural filtration.
- In a recent preprint (available on arXiv) R. Peyre shows: Z has no local extremum from the right at σ, i.e. $\sigma_+ = \sigma_- = \sigma$.
- As fractional Brownian motion also has conditional full support, it follows that the fractional Black-Scholes model is free of simple arbitrage.
- Recall that there is, however, an arbitrage with 'almost simple strategies', cp. Rogers (1997) and Cheridito (2003).

A rough sketch of Remi Peyre's argument

- Taking the Mandelbrot/van Ness respresentation into account, one can decompose

$$Z_{\sigma+t} - Z_\sigma = \mathbf{D}((Z_{\sigma+u} - Z_\sigma)_{-\infty<u\leq 0})(t) + Y_t^\sigma,$$

  where $Y_t^\sigma = C_H \int_0^t (t-s)^{H-1/2} dW_s^\sigma$ is a Lévy fractional Brownian motion independent of the past up to time σ and $\mathbf{D}$ is a suitable operator acting on an appropriate set of paths $(x_u)_{-\infty<u\leq 0}$.
- Denote by $\mathcal{A}$ the set of deterministic paths x such that

$$t \mapsto \mathbf{D}((x_u)_{-\infty<u\leq 0})(t) + Y_t^\sigma$$

  has a local extremum at $t=0$ with positive probability.

A rough sketch of Remi Peyre's argument

- It then suffices to show that the event

$$\{\exists\, r \in [0,T] :\ \mathbf{D}((Z_{r+u} - Z_r)_{-\infty<u\leq 0}) \in \mathcal{A}\}$$

has zero probability.

- Apply a law of the iterated logarithm for Lévy fractional Brownian motion to further reduce this to:

$$P\left(\#\{i \leq n;\ \sup_{0\leq r\leq T} \mathbf{D}((Z_{r+u} - Z_r)_{u\leq 0})(r^i) \geq \alpha H^{-1/2} r^{Hi} \log(i)^{1/2}\} \geq pn\right)$$

converges to zero (as n goes to infinity) for some $r, \alpha, p \in (0,1)$.

- Remove the supremum over r by sub-Gaussianity estimates on

$$\sup_{0\leq r\leq T} \left|\mathbf{D}((Z_{r+u} - Z_r)_{u\leq 0})(r^i) - \mathbf{D}((Z_u)_{u\leq 0})(r^i)\right|$$

References

- Cheridito, P. (2003) Arbitrage in fractional Brownian motion models. *Finance Stoch.* **7**, 533–553.
- Cherny, A. (2008) Brownian moving averages have conditional full support. *Ann. Appl. Probab.* **18**, 1825–1830.
- Gasbarra, D., Sottinen, T., and van Zanten, H. (2011) Conditional full support of Gaussian processes with stationary increments. *J. Appl. Probab.* **48**, 561–568.
- Guasoni, P., Rasonyi, M., and Schachermayer, W. (2008) Consistent price systems and face-lifting pricing under transaction costs. *Ann. Appl. Probab.* **18**, 491–520.
- Guasoni, P., Rasonyi, M., and Schachermayer, W. (2010) The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Ann. Finance* **6**, 157–191.
- Pakkanen, M. S. (2010) Stochastic integrals and conditional full support. *J. Appl. Probab.* **47**, 650–667.

Thank you for your attention

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