Pathwise volatility in a long-memory pricing model: estimation and asymptotic behavior

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Outline

We introduce a class of pricing models that is somehow "invariant" to the Black–Scholes model pricing model. We say $(\Omega, \mathcal{F}, (S_t), (\mathcal{F}_t), \mathbb{P})$ is in the model class $\mathcal{M}_\sigma$ if

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1. S takes values in $\mathcal{C}_{s_0,+}$,
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2. the pathwise quadratic variation $\langle S \rangle$ of S is of the form

$$\mathrm{d}\langle S \rangle_t = \sigma^2 S_t^2 \,\mathrm{d}t,$$

3. for all $\varepsilon > 0$ and $\eta \in \mathcal{C}_{s_0,+}$ we have the small ball property

$$\mathbb{P}\left[\|S - \eta\|_\infty < \varepsilon\right] > 0.$$

- $\mathcal{M}_\sigma$ contains non-semimartingale models. So, we cannot use stochastic integration theory for semimartingale. However, the forward integral is economically meaningful:

$$\int_0^t \Phi_r \mathrm{d} S_r \overset{\mathbb{P}-a.s.}{=} \lim_{n\to\infty} \sum_{\substack{t_k \in \pi n \\ t_k \leq t}} \Phi_{t_{k-1}} \left(S_{t_k} - S_{t_{k-1}} \right).$$

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- Even in the classical Black–Scholes model one restricts to ‘admissible’ strategies to exclude arbitrage. We shall restrict the ‘admissible’ strategies a little more in a careful and elegant way that still many interesting options can be hedged.

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Suppose a continuous option $G : \mathcal{C}_{s_0,+} \to \mathbb{R}$. If $G(S)$ can be hedged in one model $S \in \mathcal{M}_\sigma$ with an allowed strategy then $G(S)$ can be hedged in any model $S \in \mathcal{M}_\sigma$.

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Conclusion: In the Black–Scholes model hedges for European, Asian, and lookback-options can be constructed by using the Black–Scholes partial differential equation. These hedges hold for any model that is continuous, satisfies the small ball property, and has the same quadratic variation as the Black–Scholes model.

- Let X be a semimartingale. Then, it is well-known that:

$$[X,X]_T = \mathbb{P}\text{-}\lim_{|\pi|\to 0} \sum_{t_k \in \pi} \left(X_{t_k} - X_{t_{k-1}}\right)^2,$$

where $\pi = \{t_k : 0 = t_0 < t_1 < \cdots < t_n = T\}$ is a partition of the interval $[0,T]$, $|\pi| = \max\{t_k - t_{k-1} : t_k \in \pi\}$.

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- Barndorff-Nielsen & Shephard (2002) studied precision of the realized quadratic variation estimator for a special class of continuous semimartingales. They showed that sometimes the realized quadratic variation estimator can be rather noisy estimator.

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- Although the consistency result does not depend on a specific choice of the sampling scheme, the asymptotic distribution does Jacod (1994), Barndorff-Nielsen & Shephard (2002), Fukasawa (2010), Hayashi, Jacod & Yoshida (2011), and the recent book by Aït-Sahalia & Jacod (2014).

Continuous observation

- Dzhaparidze and Spreij [4] suggested another estimator of the bracket $[X, X]$ using the spectral characterization.
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- For $\lambda \in \mathbb{R}$, define the periodogram $I_\tau(X;\lambda)$ of X at τ

$$I_\tau(X;\lambda) := \Big| \int_0^\tau e^{i\lambda s} \mathrm{d}X_s \Big|^2$$

$$= 2\,\mathbf{Re} \int_0^\tau \int_0^t e^{i\lambda(t-s)} \mathrm{d}X_s \mathrm{d}X_t + [X,X]_\tau \quad \text{(by Itô formula)}.$$

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- Now, the idea is to randomize λ. To this end, let ξ be a symmetric random variable independent of $\mathbb{F}^X$ with a density g_ξ, and a **real-valued characteristic function** φ_ξ. For given $L > 0$, define the randomized periodogram

$$\mathbb{E}_\xi I_\tau(X; L\xi) = \int_{\mathbf{R}} I_\tau(X; Lx)\, g_\xi(x) \mathrm{d}x.$$

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Theorem (Dzhaparidze & Spreij - 1994)

If the characteristic function φ_ξ is of bounded variation, then

$$\mathbb{E}_\xi I_\tau (X; L\xi) \xrightarrow{\mathbf{P}} [X,X]_\tau.$$

- Consider mixed Brownian-fractional Brownian motion X:

$$X_t = W_t + B_t^H \qquad t \in [0, T],$$

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- Cheridito (2001) showed that the process X is a $\mathbb{F}^X$ semimartingale, if $H \in (\frac{3}{4}, 1)$, and for $H \in \left(\frac{1}{2}, \frac{3}{4}\right]$, X is not a semimartingale with respect to its own filtration $\mathbb{F}^X$.

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- In both cases, since fBm has zero quadratic variation for $H > 1/2$, we have

$$[X, X]_t = \mathbb{P}\text{-} \lim_{|\pi| \to 0} \sum_{t_k \in \pi} \left(X_{t_k} - X_{t_{k-1}}\right)^2 = t, \quad \forall t \in [0, T].$$

If the partitions are nested, i.e. $\pi^{(n)} \subset \pi^{(n+1)}$, then the convergence can be strengthened to almost sure convergence. Hereafter, we always assume that the sequence of partitions are nested.

- Similarly as in semimartingale setup, for mixed BfBm, we define the randomized periodogram as

$$\mathbb{E}_{\xi} I_T\left(X; L\xi\right) = \int_{\mathbf{R}} I_T\left(X; Lx\right) g_{\xi}(x)\mathrm{d}x.$$

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- Note that in general we are in non-semimartingale world, so one has to take care of stochastic integrals. Here, for any given $\lambda \in \mathbb{R}$, we define the periodogram of X at deterministic time T

$$I_T(X;\lambda) = \Big| \int_0^T e^{i\lambda t}\mathrm{d}X_t\Big|^2 = \Big|e^{i\lambda T}X_T - i\lambda \int_0^T X_t e^{i\lambda t}\mathrm{d}t\Big|^2.$$

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Theorem (A & Valkeila - 2013)

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Here, the main tools are: (1) a version of stochastic Fubini's theorem for fBm (2) using generalized Lebesgue–Stieltjes integration theory of Zähle (1998) and developed further by Nualart & Rascanu (2002) that makes stochastic integral w.r.t fBm a continuous operator.

- Let $X = \{X_t\}_{t\in[0,T]}$ be a "nice" Gaussian process. The *Wiener–Itô chaotic decomposition* tells us that for any $F \in L^2(\sigma(X_t; t \in [0,T]))$

$$F = \mathbb{E}(F) + \sum_{p\geq 1} I_p(f_p),$$

$$I_p(f_p) = \int_{[0,T]^p} f_p(t_1, \cdots, t_p) dX_{t_1} \cdots dX_{t_p},$$

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- The vector space $\mathcal{H}_p$ generated by the elements $I_p(f)$ is called the pth Wiener chaos associated to X.
i.e. $L^2(\Omega, \sigma(X_t; t \in [0,T]), \mathbb{P}) = \mathbb{R} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_p \oplus \cdots$.

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Theorem (Nualart-Peccati - 2005)

Let $\{F_n\}_{n\geq 1} = \{I_q(f_n)\}_{n\geq 1}$ be a sequence of random variables in the qth Wiener chaos, $q \geq 2$, such that $\mathbb{E}(F_n^2) \to \sigma^2$. Then, the following asymptotic statements are equivalent:

(i) F_n *converges in law to* $\mathcal{N}(0,\sigma^2)$.

(ii) $\mathbb{E}(F_n)^4 \to 3\sigma^4 = \mathbb{E}(\mathcal{N}(0,\sigma^2))^4$, $\quad (\kappa_4(F_n) \to 0)$.

(iii) $\|DF_n\|^2_{\mathfrak{H}}$ *converges in* L^2 *to* $q\,\sigma^2$, $\quad (Var\|DF_n\|^2_{\mathfrak{H}} \to 0)$.

(iv) $\|f_n \otimes_r f_n\|_{\mathfrak{H}^{(2q-2r)}} \to 0$ *for all* $r = 1,\cdots,q-1$.

Theorem

Assume that $\varphi_\xi \in L^1(\mathbb{R})$*. Then, as* $L \to \infty$*, we have the following asymptotic statements:*

1. *if* $H \in \left(\frac{3}{4}, 1\right)$*, then*

$$\sqrt{L}\Big(\mathbb{E}_\xi I_T(X; L\xi) - [X,X]_T\Big) \xrightarrow{law} \mathscr{N}(0, \sigma_T^2).$$

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2. *if $H = \frac{3}{4}$, then*

$$\sqrt{L}\Big(\mathbb{E}_\xi I_T(X; L\xi) - [X, X]_T\Big) \xrightarrow{law} \mathcal{N}(\mu, \sigma_T^2),$$

where $\mu = 2\alpha_H T \int_0^\infty \varphi_\xi(x) x^{2H-2} \mathrm{d}x$, and $\alpha_H = H(2H-1)$.

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3. *if $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$, then*

$$L^{2H-1}\Big(\mathbb{E} I_T(X; L\xi) - [X,X]_T\Big) \xrightarrow{\mathbf{P}} \mu.$$

Notice that when $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$, we have $2H - 1 < \frac{1}{2}$.

Sketch of the proof:

- Using the generalized Lebesgue-Stieltjes integration theory to obtain

$$\mathbb{E}_\xi I_T(X; L\xi) = [X, X]_T + 2\int_0^T \underbrace{\int_0^t \varphi_\xi(L(t-s))\mathrm{d}X_s}_{:=u_t}\,\mathrm{d}X_t \qquad (1)$$

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- Using the link between pathwise and the Skorokhod integrals to obtain

$$\int_0^T u_t \mathrm{d}X_t = \int_0^T u_t \delta X_t + \alpha_H \int_0^T \int_0^T D_s^{(B^H)} u_t |t-s|^{2H-2}\mathrm{d}s\mathrm{d}t,$$

  where δX stands for the Skorokhod integral w.r.t mixed BfBm X, and $D^{(B^H)}$ denote the Malliavin derivative with respect to B^H.

Main results: asymptotic normality

- Set $\psi_L(s,t) := \varphi_\xi(L|t-s|)$, we have:

$$\mathbb{E} I_T(X; L\xi) - [X,X]_T = I_2^X(\psi_L) + \alpha_H \int_0^T \int_0^T \varphi_\xi(L|t-s|)|t-s|^{2H-2} \mathrm{d}s\mathrm{d}t.$$

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$$\mathbb{E}(I_2^X(\psi_L))^2 = 2\|\psi_L\|^2_{\mathfrak{H}^{\otimes 2}} \sim \frac{1}{L}.$$

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- Now, using 4th moment criterion to infer that for any $H > 1/2$:

$$\sqrt{L}\, I_2^X(\psi_L) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma_T^2), \quad \sigma_T^2 := 2\,T \int_0^\infty \varphi_\xi^2(x)\mathrm{d}x.$$

- Set $\psi_L(s,t) := \varphi_\xi(L|t-s|)$, we have:

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- And for the deterministic correction term we have

$$\sqrt{L}\,\alpha_H \int_0^T \int_0^T \varphi_\xi(L|t-s|)|t-s|^{2H-2} \mathrm{d}s\mathrm{d}t \to \begin{cases} 0 \text{ if } H > 3/4, \\ \mu \text{ if } H = 3/4, \\ +\infty \text{ if } H < 3/4, \end{cases}$$

where $\mu = 2\alpha_H T \int_0^\infty \varphi_\xi(x) x^{2H-2}\mathrm{d}x$.

The following general Berry-Esseen type estimate is obtained by combining the Stein's method for normal approximation with Malliavin calculus.

Theorem

Let $\{F_n\}_{n\geq 1}$ *be a sequence of elements in the second Wiener chaos such that* $\mathbb{E}(F_n^2) \to \sigma^2$ *and* $Var\|DF_n\|^2_{\mathfrak{H}} \to 0$. *Then,* $F_n \overset{law}{\to} Z \sim \mathcal{N}(0, \sigma^2)$ *and*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}(F_n < x) - \mathbb{P}(Z < x) \right| \leq \frac{2}{\mathbb{E}(F_n^2)} \sqrt{Var\|DF_n\|^2_{\mathfrak{H}}} + \frac{2|\mathbb{E}(F_n^2) - \sigma^2|}{\max\{\mathbb{E}(F_n^2), \sigma^2\}}.$$

Using this result, in the **semimartingale case** we obtain:

Proposition

Let $H \in \left(\frac{3}{4}, 1\right)$, and $Z \sim \mathcal{N}(0, \sigma_T^2)$. Then there exists a constant C (independent of L) such that for sufficiently large L, we have

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sqrt{L}(\mathbb{E}_\xi(I_T(X; L\xi) - [X,X]_T) < x \right) - \mathbb{P}(Z < x) \right| \leq C\rho(L)$$

where

$$\rho(L) = \max\left\{ L^{\frac{3}{2}-2H}, \int_L^\infty \varphi_\xi^2(Tz)\mathrm{d}z \right\}.$$

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where

$$\rho(L) = \max\left\{ L^{\frac{3}{2}-2H}, \int_L^\infty \varphi_\xi^2(Tz)\mathrm{d}z \right\}.$$

In many cases of interest the leading term in $\rho(L)$ is the polynomial term $L^{\frac{3}{2}-2H}$. For example, if φ_ξ admits an exponential decay, i.e.

$$|\varphi_\xi(t)| \leq C_1 e^{-C_2 t}.$$

Example: if ξ is a standard normal random variable with $\varphi_\xi(t) = e^{-\frac{t^2}{2}}$ or if ξ is a standard Cauchy random variable with $\varphi_\xi(t) = e^{-|t|}$.

Thank you for your attention !

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