Linear Credit Risk Models

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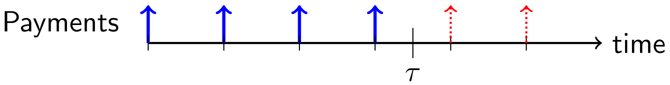
Overview

Credit Risk(s)

Default time τ



Default likelihood variation

$$\mathbb{P}_{t+1}[\tau \leq T]$$

→ 5%

→ 1%

New regulatory requirements (Basel III, IFRS 9)

Expected losses (12-month, lifetime), deterioration of credit quality, valuation adjustments (CVA, DVA), etc.

$\Rightarrow$ Tractable models ?

Doubly-stochastic default time

Ingredients

$\mathbb{F} = (\mathcal{F}_t)$	risk-factors filtration (no default)
$S_t = \mathrm{e}^{-\int_0^t \lambda_s ds}$	$\mathbb{F}$-measurable survival process with the hazard-rate $\lambda_t \geq 0$
U	uniform r.v. on $(0,1)$ independent from $\mathbb{F}$

Default time construction

$$\tau = \inf \{t \geq 0 : S_t \leq U\}$$

$\mathcal{H}_t = \sigma(\mathbb{1}_{\{\tau \leq s\}} : s \leq t)$	in which τ is a $\mathbb{H}$ stopping time
$\mathcal{G}_t = (\mathcal{F}_t \vee \mathcal{H}_t)$	all the observable information

Survival process

$$S_t = \mathbb{P}[\tau > t \mid \mathcal{F}_t] = \mathbb{P}[\tau > t \mid \mathcal{F}_\infty]$$

The Survival Process

$X_t \in E \subset \mathbb{R}^m$ risk-factor process and vector $\gamma \in \mathbb{R}^m$.

Standard approach

Model the hazard-rate process: $\lambda_t = \gamma^\top X_t \geq 0$.

A new approach

Model the survival process directly!

$$S_t = \mathbb{P}[\tau > t \mid \mathcal{F}_t] = 1 - \int_0^t \gamma^\top X_s \, ds > 0 \qquad (1)$$

which implies $dS_t = -\gamma^\top X_t dt$ and such that

$$\lambda_t = \frac{\gamma^\top X_t}{S_t} \geq 0 \qquad (2)$$

How to construct X_t such that (1) and (2) are verified?

One Risk-Factor

W.l.o.g. let $\gamma = 1$ such that $dS_t = -X_t dt$.

Assume that there exists a constant $L > 0$ such that

$$0 \leq X_t \leq LS_t$$

then $S_t \geq \mathrm{e}^{-Lt} > 0$ and $L \geq \lambda_t \geq 0$.

Assume that (X_t, S_t) is a polynomial preserving diffusion, then

$$dX_t = (b + \beta X_t + BS_t)dt + \sigma\sqrt{X_t(LS_t - X_t)}dW_t$$

for some reals b, β, B and $\sigma \geq 0$.

Lemma

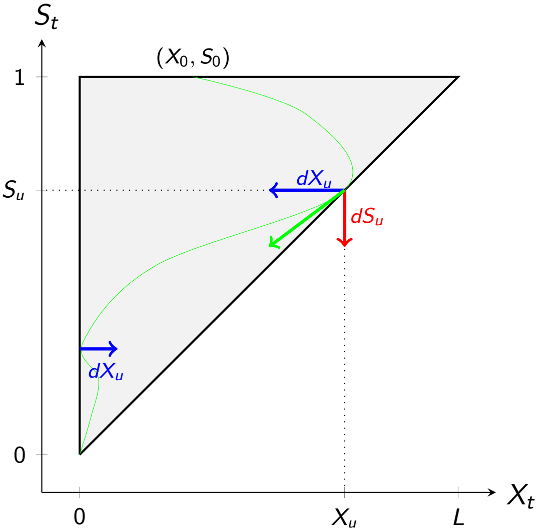
The process (X_t, S_t) is well-defined if and only if $b = 0$, $B \geq 0$, and

$$L^2 + \beta L + B \leq 0.$$

One Risk-Factor II

Inward pointing condition

The state space E is of the form



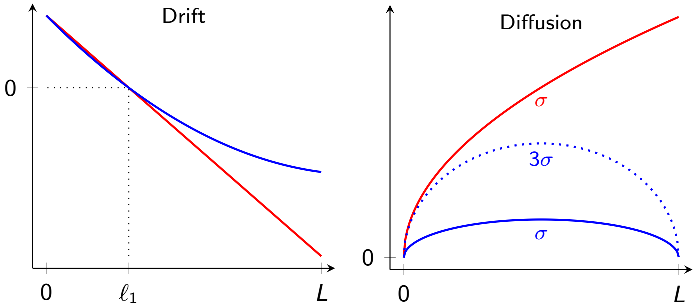
One Risk-Factor III

Hazard-rate dynamics

$$d\lambda_t = (\ell_1 - \lambda_t)(\lambda_t - \ell_2)\,dt + \sigma\sqrt{\lambda_t(L-\lambda_t)}\,dW_t$$

One risk-factor affine model (CIR process)

$$d\lambda_t = \ell_2(\lambda_t - \ell_1)\,dt + \sigma\sqrt{\lambda_t}\,dW_t$$



Multiple Risk-Factors

W.l.o.g. let $\gamma^\top \mathbf{1} = 1$ and assume $X_t \in [0, LS_t]^m$

$$-dS_t/dt = \gamma^\top X_t \leq \gamma^\top \mathbf{1} S_t = LS_t$$

such that $0 \leq \lambda_t \leq L$.

The dynamic of the multivariate process X_t rewrites

$$dX_t = (b + \beta X_t + BS_t)dt + \Sigma(X_t, S_t)dW_t$$

$$\Sigma(X_t, S_t) = \text{diag}(\sigma_1\sqrt{X_{1t}(LS_t - X_{1t})}, \ldots, \sigma_m\sqrt{X_{mt}(LS_t - X_{mt})})$$

and with $b, B \in \mathbb{R}^m$, $\beta \in \mathbb{R}^{m\times m}$, and $\sigma \in \mathbb{R}^m_+$.

Lemma

The process (X_t, S_t) is well defined if and only if $b = 0$,

$$B \geq \sum_{j\neq i}(-L\beta_{ij})^+ \quad \text{and} \quad -L\beta_{ii} - B_i \geq \sum_{j\neq i}(L(\gamma_i L + \beta_{ij}))^+$$

Preliminaries

Risk-neutral valuation

There exists an equivalent risk-neutral martingale measure $\mathbb{Q}$.

The discount process is $D_t = \mathrm{e}^{-\int_0^t r_s ds}$ with the short-rate r_s.
D_t and S_t have the same properties !

Process moments

(X_t, S_t)	polynomial preserving process with state space E
$\mathrm{Pol}_n(E)$	space of polynomials of order at most n on E
H_n	vector of polynomials forming a basis of $\mathrm{Pol}_n(E)$
G_n	matrix representation of $\mathcal{A}\,\|_{\mathrm{Pol}_n(E)}$ w.r.t. H_n
$\vec{p}$	vector representation of $p(x,s) \in \mathrm{Pol}_n(E)$ w.r.t. H_n

Lemma

(Cuchiero et al. (12), Filipović and Larsson (15))

$$\mathbb{E}\left[p(X_T, S_T) \,|\, \mathcal{F}_t\right] = H_n(X_t, S_t)^\top \mathrm{e}^{G_n(T-t)} \vec{p}.$$

Defaultable bond

Security B pays one if $\tau > T$, zero otherwise.
Assume for simplicity $r_t = 0$, and consider the monomial basis $H_1(x,s) = \{1, x_1, \dots, x_m, s\}$.

$$\begin{aligned} B(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}} \left[\mathrm{e}^{-\int_t^T \lambda_u du} \,|\, \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}} \left[\frac{S_T}{S_t} \,|\, \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} \{1, X_t, S_t\}^\top \mathrm{e}^{G_1(T-t)} \{0, \mathbf{0}, 1/S_t\} \end{aligned}$$

where the matrix G_1 is given by

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta^\top & -\gamma \\ 0 & B^\top & 0 \end{pmatrix}.$$

Only the first $\mathcal{F}_t$-conditional moment of S_T is needed !

Contingent Cash-Flow

Security C pays one at τ if and only if $t < \tau < T$.

Assume for simplicity that $r_t = r$ constant.

$$
\begin{aligned}
C(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{t<\tau<T\}}\mathrm{e}^{-r(\tau-t)} \mid \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\int_t^T \mathrm{e}^{-r(s-t)}\mathbb{E}^{\mathbb{Q}}\left[\lambda_s \mathrm{e}^{-\int_t^s \lambda_u du} \mid \mathcal{F}_t\right] ds \\
&= \mathbb{1}_{\{\tau>t\}}\int_t^T \mathrm{e}^{-r(s-t)}\mathbb{E}^{\mathbb{Q}}\left[\frac{\gamma^\top X_s}{S_t} \mid \mathcal{F}_t\right] ds \\
&= \mathbb{1}_{\{\tau>t\}}\{1, X_t, S_t\}^\top (G_1^*)^{-1}\left(\mathrm{e}^{G_1^*(T-t)} - I\right)\{0, \gamma/S_t, 0\}
\end{aligned}
$$

with $G_1^* = G_1 - \mathrm{diag}(r)$ and using $\int_0^t e^{As} ds = A^{-1}(e^{At} - I)$.

No numerical integration over $[t, T]$!

Credit-Default-Swap $\equiv$ sum of C and many Bs

$$
\pi_t^{\text{cds}} = \{1, X_t/S_t\}^\top \vec{p}_{\text{cds}} \overset{(m=1)}{=} \{1, \lambda_t\}^\top \vec{p}_{\text{cds}}
$$

Modeling Choices

Set $r_t = r$ constant over the estimation period.

A cascading structure

Consider $m = 2$ and dynamics of the form:

$$dS_t = -X_{1t}\, dt$$
$$dX_{1t} = \kappa_1(\theta_1 X_{2t} - X_{1t})\, dt + \sigma_1 \sqrt{X_{1t}(LS_t - X_{1t})}\, dW_{1t}$$
$$dX_{2t} = \kappa_2(\theta_2 S_t - X_{2t})\, dt + \sigma_2 \sqrt{X_{2t}(LS_t - X_{2t})}\, dW_{2t}$$

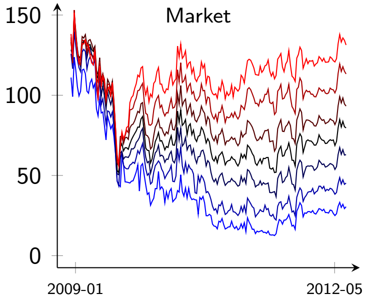
Market price of risk $dW_t^* = dW_t + \Lambda_t dt$

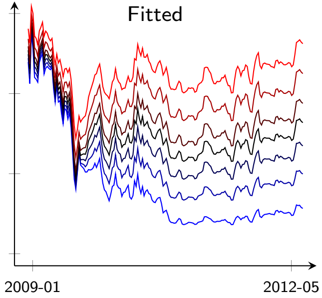
Assume (X_t, S_t) polynomial preserving process under $\mathbb{P}$ an $\mathbb{Q}$

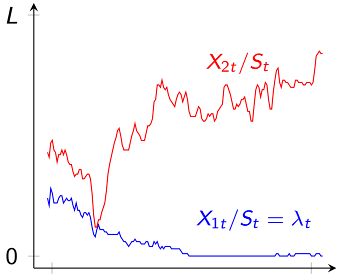
$$\Lambda_{1t} = \frac{\delta_1 \sqrt{X_{1t}}}{\sigma_1 \sqrt{LS_t - X_{1t}}} \quad \Lambda_{2t} = \frac{\delta_2 X_{2t} + \delta_3 S_t}{\sigma_2 \sqrt{X_{2t}(LS_t - X_{2t})}}$$

Stronger conditions on dynamics parameters ! 10 parameters.

Estimation Results on AT&T



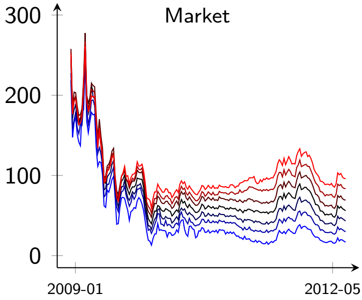


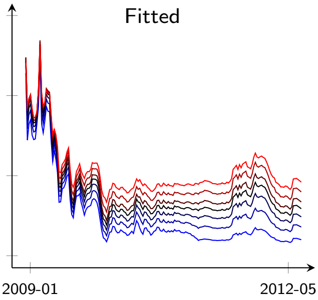


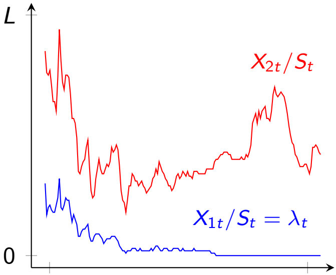
QML using UKF
CDS spreads in basis points
1 to 10-year maturity

$L \approx 8\%$
RMSE $<$ 5 b.p.

Estimation Results on Boeing







QML using UKF
CDS spreads in basis points
1 to 10-year maturity

$L \approx 12\%$
RMSE < 4 b.p.

Single-name CDS Option

European CDS option with maturity T, $r_s = 0$

$$\pi_t^{\text{opt}} = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau > T\}}\left(\pi_T^{\text{cds}}\right)^+ \,|\, \mathcal{G}_t\right]$$
$$= \frac{\mathbb{1}_{\{\tau > t\}}}{S_t}\mathbb{E}^{\mathbb{Q}}\left[\left(\{X_T, S_T\}^\top \vec{p}_{\text{cds}}\right)^+ \,|\, \mathcal{F}_t\right]$$

$Z = \{X_T, S_T\}^\top \vec{p}_{\text{cds}}/S_t \in [a, b]$ with known $\mathcal{F}_t$-cond moments.

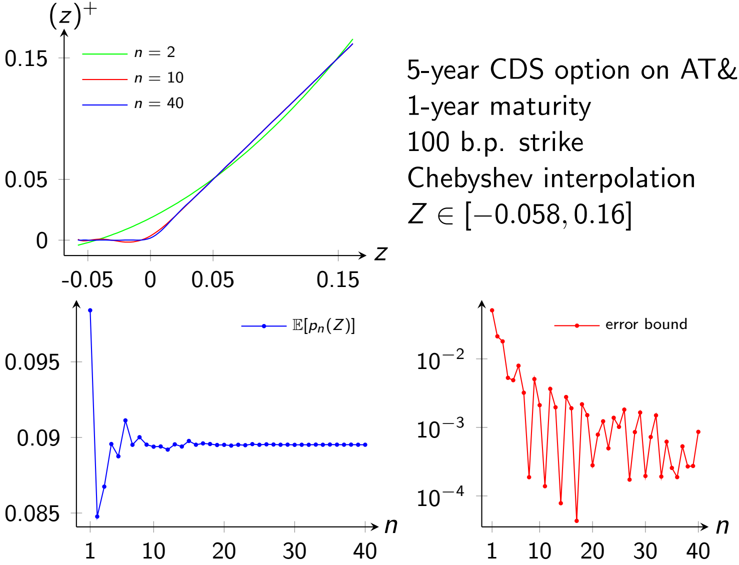
Payoff approximation

Polynomial series $p^n(y)$ converging to $(y)^+$ on $[a, b]$ such that

$$\mathbb{E}\left[p^n(Z)\right] \xrightarrow[n\to\infty]{} \pi_t^{\text{opt}}$$

with non-tight error upper bound $\|p^n(z) - (z)^+\|_\infty$ on $[a, b]$.

CDS Option Example



5-year CDS option on AT&T

1-year maturity

100 b.p. strike

Chebyshev interpolation

$Z \in [-0.058, 0.16]$

Conclusion

- New class of reduced-form models for credit-risk
- Survival process modeling $S_t = \mathbb{P}[\tau > t \mid \mathcal{F}_t]$ with PPP
- Analytical formulas for defaultable bonds and CDS prices
- Straightforward approximation of CDS options prices
- Promising directions: CVA, multi-name models, ...