

Semi-Static Completeness and Model-independent Pricing by Informed Investors

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joint work with Martin Larsson

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Outline

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- 2 Setup
- 3 Semi-static completeness and the Jacod-Yor theorem
- 4 Semi-static completeness and filtration structure
- 5 Pricing by informed investors
- 6 Conclusions

Model-independent framework

- Model-independent framework:
 - \mathcal{X} : path-space, S : canonical process on \mathcal{X}
 - Ψ : set of claims ψ available for buy-and-hold trading
 - \mathcal{M} : martingale measures consistent w/ the market price of ψ 's
 - Φ : a given derivative, robust pricing: $\sup_{\mathbf{Q} \in \mathcal{M}} \mathbb{E}_{\mathbf{Q}}[\Phi]$

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$$\sup_{\mathbf{Q} \in \mathcal{M}} \mathbb{E}_{\mathbf{Q}}[\Phi] = \inf \left\{ c \in \mathbb{R} : \begin{array}{l} \Phi \text{ can be hedged pathwise} \\ \text{starting with initial capital } c \end{array} \right\}$$

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Beiglböck, H.-Labordère, Penkner (2013); Galichon, H.-Labordère, Touzi (2014); Acciaio, Beiglböck, Penkner, Schachermayer (2013); Bouchard, Nutz (2013); Dolinsky, Soner (2014a, 2014b); Beiglböck, Cox, Huesmann (2014); Biagini, Bouchard, Kardaras, Nutz (2014); Beiglböck, Nutz, Touzi (2015); Guo, Tan, Touzi (2015); Hou, Obłój (2015); Beiglböck, Cox, Huesmann, Perkowski, Prömel (2015), ...

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- **Note:** \mathcal{M} clearly depends on the underlying filtration, as does the set of available trading strategies.
- **Question:** What can be said about the [relation between the super-hedging price and the choice of filtration](#)? In particular, when passing from \mathbb{F} to $\mathbb{G} \supseteq \mathbb{F}$?

Insider information

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- Informed agent has less pricing measures: $\mathcal{M}(\mathbb{G}) \subseteq \mathcal{M}(\mathbb{F})$, so

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- Question:** Which measures in $\mathcal{M}(\mathbb{F})$ are still relevant for pricing for the informed agent?

Setup

- $(\Omega, \mathbb{F}, \mathcal{F})$: Filtered measurable space with $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ right-continuous.
↳ Later we will consider other filtrations.
- $S = (S_t)_{0 \leq t \leq T}$: càdlàg \mathbb{F} -adapted discounted price process of an asset available for **dynamic trading**. We assume $S_0 = 0$. (Everything works the same for multiple assets.)
- A risk-free asset with price $\equiv 1$ available for **dynamic trading**.
- $\Psi = \{\psi_1, \dots, \psi_n\}$ a set of \mathcal{F}_T -measurable payoffs available for **buy-and-hold trading**. Today's price of ψ_i is zero for each i .

Martingale measures

Calibrated martingale measures:

$$\mathcal{M}(\mathbb{F}) = \left\{ \mathbf{Q} \in \mathcal{P}(\mathcal{F}_T) : \begin{array}{l} S \text{ is an } \mathbb{F}\text{-martingale, } \mathbb{E}_{\mathbf{Q}}[S_T^2] < \infty, \\ \mathbb{E}_{\mathbf{Q}}[\psi | \mathcal{F}_0] = 0, \mathbb{E}_{\mathbf{Q}}[\psi^2] < \infty \text{ for all } \psi \in \Psi \end{array} \right\}$$

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- $\mathcal{M}(\mathbb{F})$ is “huge”
 - ↳ Can we reduce to the study of a special subset?

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- $\mathcal{M}(\mathbb{F})$ is “huge”
 - ↪ Can we reduce to the study of a special subset?
 - ↪ For example, if $\mathcal{P}(\mathcal{F}_T)$ is endowed with a topology s.t. $\mathcal{M}(\mathbb{F})$ is compact, then

$$\mathcal{M}(\mathbb{F}) = \overline{\text{conv}(\text{ext } \mathcal{M}(\mathbb{F}))},$$

where $\text{ext } \mathcal{M}(\mathbb{F})$ is the set of all extreme points in $\mathcal{M}(\mathbb{F})$.

Extreme points

- **Extreme points:** $\mathbf{Q} \in \mathcal{M}(\mathbb{F})$ is called an extreme point if

$$\mathbf{Q} = \lambda \mathbf{Q}^1 + (1 - \lambda) \mathbf{Q}^2 \quad \implies \quad \mathbf{Q}^1 = \mathbf{Q}^2 = \mathbf{Q}$$

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- Consider an \mathcal{F}_T -measurable payoff Φ and endow $\mathcal{P}(\mathcal{F}_T)$ with a topology such that
 - $\mathcal{M}(\mathbb{F})$ is compact and $\mathbf{Q} \mapsto \mathbb{E}_{\mathbf{Q}}[\Phi]$ is continuous.

Then
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- **Note:** The notion of extreme point is purely algebraic, independent of any topology we may put on the space of probability measures.

Examples

Example (Discrete time and bounded prices)

- ▶ $\Omega = [a, b]^T$, S is the coordinate process,
- ▶ each $\omega \mapsto \psi_i(\omega)$ is continuous,
- ▶ \mathbb{F} is generated by S

Then $\mathcal{M}(\mathbb{F})$ is weakly compact.

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Example (Continuous time and bounded volatility)

- ▶ $\Omega = \{\omega \in C_0[0, T] : \langle \omega \rangle \text{ and } \langle \omega \rangle' \text{ exist and are bounded by } 1\}$,
- ▶ S coordinate process, $\omega \mapsto \psi_i(\omega)$ bounded and continuous,
- ▶ \mathbb{F} is generated by S

Then $\mathcal{M}(\mathbb{F})$ is weakly compact.

Examples

Example (Jakubowski topology)

- ▶ $\Omega = D_0([0, T], [-1, 1])$ with Jakubowski's S-topology,
- ▶ S is the coordinate process, ψ_i suitable continuity conditions,
- ▶ \mathbb{F} is generated by S

Then $\mathcal{M}(\mathbb{F})$ is sequentially S-compact. Cf. Jakubowski (1997) and Guo, Tan, Touzi (2015).

Semi-static completeness and the Jacod-Yor theorem

The classical Jacod-Yor theorem

- Suppose $\Psi = \emptyset$ (no static claims).
- For $\mathbf{Q} \in \mathcal{M}(\mathbb{F})$, by the classical **Jacod-Yor (1977) theorem**:

$$\mathbf{Q} \in \text{ext } \mathcal{M}(\mathbb{F}) \iff \underbrace{L^2(\mathcal{F}_T) = \{x + (H \cdot S)_T : H \in L^2(S)\}}_{\text{classical completeness (in } L^2)}$$

- This result can be generalized to the semi-static case.

Generalization of the Jacod-Yor theorem

Definition

For $\mathbf{Q} \in \mathcal{M}(\mathbb{F})$, we say that **semi-static completeness** holds if any $X \in L^2(\mathcal{F}_T)$ can be represented as

$$X = x + a_1\psi_1 + \cdots + a_n\psi_n + (H \cdot S)_T$$

for some $x, a_1, \dots, a_n \in \mathbb{R}$ and $H \in L^2(S)$.

Notation:

$$\mathbf{SSC}(\mathbb{F}) = \{\mathbf{Q} \in \mathcal{M}(\mathbb{F}) : \text{semi-static completeness holds}\}$$

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Theorem (semi-static Jacod-Yor theorem)

The extreme martingale measures are exactly the semi-statically complete models, i.e.

$$\text{ext } \mathcal{M}(\mathbb{F}) = \mathbf{SSC}(\mathbb{F}).$$

Generalization of the Jacod-Yor theorem

About the proof.

- The proof is very close to the classical case . . .
- . . . but uses duality for random variables ($L^1 - L^\infty$) instead of processes ($\mathcal{H}^1 - BMO$):

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- ... but uses duality for random variables ($L^1 - L^\infty$) instead of processes ($\mathcal{H}^1 - BMO$):

1. Fix $\mathbf{Q} \in \text{ext } \mathcal{M}(\mathbb{F})$ and show that this set is **dense in $L^1(\mathcal{F}_T)$**

$$\left\{ x + \sum_i a_i \psi_i + (H \cdot S)_T : x, a_i \in \mathbb{R}, H \in L^2(S) \right\}.$$

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2. Prove it is **dense and closed in $L^2(\mathcal{F}_T)$** using Hahn-Banach and a result by Yor (see also Delbaen/Schachermayer, 1999):

Theorem (Yor (1978))

Let $H^n \in L(S)$ be such that $H^n \cdot S$ is a martingale for each n , and suppose $\lim_n (H^n \cdot S)_T = X$ in L^1 for some r.v. X . Then there is $H \in L(S)$ such that $H \cdot S$ is a martingale with $(H \cdot S)_T = X$.

Generalization of the Jacod-Yor theorem

Remarks.

- Infinitely many ψ_i 's would allow to treat the case of a fixed (by the market) marginal law $S_T \sim \mu$
- But the arguments we use in the above proof break down in this case – for the moment we are only able to deal with finitely many ψ_i 's

Generalization of the Jacod-Yor theorem

Can we say more?

- In the classical case ($\Psi = \emptyset$), completeness is a strong property — but still allows for many “unstructured” models.
- For instance, completeness holds if \mathbb{F} is generated by S , and S is a strong solution to a possibly path-dependent SDE of the form

$$dS_t = \sigma(t; S_u : u \leq t) dW_t$$

where W is a Brownian motion and σ is never zero.

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Question: Should we expect some additional structure in the semi-static case?

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- We are going to show an interesting consequence of semi-static completeness
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Notation: For any martingale N , denote

$$\mathcal{S}(N) = \{H \cdot N : H \in L^2(N)\}.$$

This is a closed subspace of \mathcal{H}^2 (stable subspace generated by N).

A curious consequence of semi-static completeness

- Let $K \cdot S$ be the orthogonal projection of $\mathbb{E}_{\mathbf{Q}}[\psi \mid \mathcal{F}_t]$ onto $\mathcal{S}(S)$ and define

$$M_t = \mathbb{E}_{\mathbf{Q}}[\psi \mid \mathcal{F}_t] - (K \cdot S)_t$$

Note: M_T is the part of ψ which is not replicable by trading on S

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- Then $H \cdot M \perp \mathcal{S}(S)$ for any $H \in L^2(M)$
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$$\mathcal{H}^2 = \text{span}\{1\} \oplus \text{span}\{M\} \oplus \mathcal{S}(S)$$

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- Consequently,

$$\mathcal{S}(M) = \text{span}\{M\},$$

which is **one-dimensional!**

A curious consequence of semi-static completeness

We will use the following result on ψ :

Lemma

Let N be a square-integrable martingale null at zero. The following are equivalent:

- (i) $\mathcal{S}(N) = \text{span}\{N\}$
- (ii) $N = N_T \mathbf{1}_{B \times [t^*, T]}$ for some $t^* \in (0, T]$ and some atom B of \mathcal{F}_{t^*-}

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And the following one on S , when S is **continuous**:

Lemma

Let N be a continuous local martingale, and let B be an atom of \mathcal{F}_{t^*} for some $t^* \in (0, T]$. Then $N_t = N_0$ on B for all $t < t^*$.

A curious consequence of semi-static completeness

Therefore, for S continuous, $\Psi = \{\psi\}$, $\mathbf{Q} \in \text{SSC}(\mathbb{F})$, we have

$$M = M_T \mathbf{1}_{B \times [t^*, T]} \quad \text{and} \quad S_t = S_0 \text{ on } B \text{ for } t \leq t^*$$

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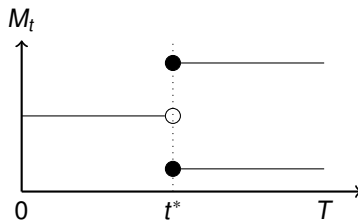
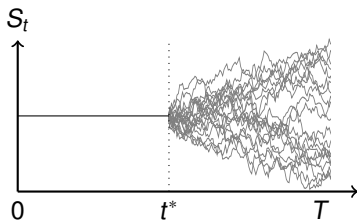
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Semi-static completeness and filtration structure

Atomic tree

- Fix $\mathbf{Q} \in \mathcal{M}(\mathbb{F})$
- For $A \in \mathcal{F}_T$, denote by $t(A)$ the first time A becomes measurable,

$$t(A) = \inf\{t \in [0, T]: A \in \mathcal{F}_t\}.$$

Definition

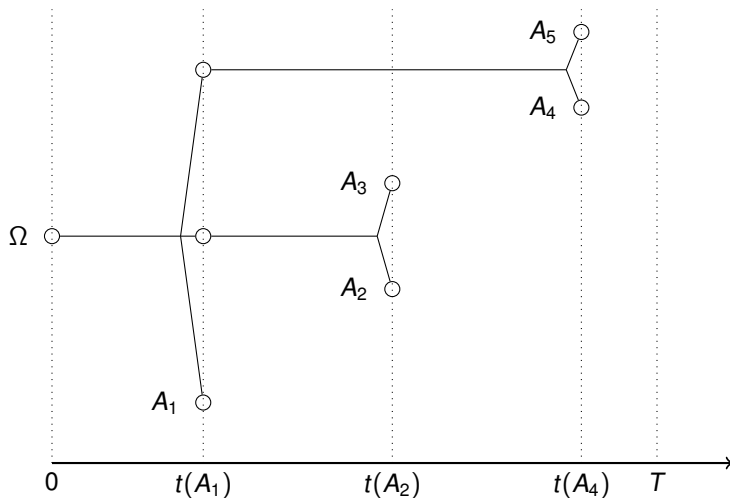
An **atomic tree** is a finite collection \mathbf{T} of events in \mathcal{F}_T s.t.:

- (i) every $A \in \mathbf{T}$ is a non-null atom of $\mathcal{F}_{t(A)}$;
- (ii) $\forall A, A' \in \mathbf{T}$ s.t. $t(A) < t(A')$, either $A \supseteq A'$ or $A \cap A' = \emptyset$;
- (iii) $\forall A, A' \in \mathbf{T}$ such that $A \supseteq A'$, $\mathbf{Q}(A \setminus A') > 0$;
- (iv) the leaves form a partition of Ω (up to nullsets), and A is an atom of $\mathcal{F}_{t(A')-}$ whenever A' is a child of A .

leaf: $A \in \mathbf{T}$ s.t. there is no $A' \in \mathbf{T}$ s.t. $A' \subsetneq A$; **dim \mathbf{T} :** # leaves

child: A' is a child of A if $A, A' \in \mathbf{T}$ satisfy $A' \subsetneq A$ and there is no $A'' \in \mathbf{T}$ such that $A' \subsetneq A'' \subsetneq A$

Atomic tree



Atomic tree

Remarks.

- $\sigma(\mathbf{T})$ is well-defined. It can be described as $\sigma(\mathbf{T}) = \mathcal{F}_{\zeta(\mathbf{T})}$, where the stopping time $\zeta(\mathbf{T})$ is the “end” of the tree:

$$\zeta(\mathbf{T}) = \sum_{A \in \mathbf{T} \text{ is a leaf}} t(A) \mathbf{1}_A.$$

- Note that $\dim \mathbf{T} = \dim L^2(\sigma(\mathbf{T}))$.

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Definition

We say that S is **complete on** $A \times [t, T]$ for given $t \in [0, T]$ and $A \in \mathcal{F}_t$ if any $X \in L^2(\mathcal{F}_T)$ can be dynamically replicated there:

$$X = x + (H \cdot S)_T \quad \text{on } A$$

for some $x \in \mathbb{R}$ and some $H \in L^2(S)$ with $H = 0$ on $\llbracket 0, t \rrbracket$.

Semi-static completeness for continuous price processes

Recall: $\mathbf{Q} \in \mathcal{M}(\mathbb{F})$ is fixed.

Theorem

Assume S is continuous. Then semi-static completeness holds if and only if there exists an **atomic tree** \mathbf{T} such that

1. the set $\{\mathbb{E}_{\mathbf{Q}}[\psi_i \mid \sigma(\mathbf{T})] : i = 1, \dots, n\}$ contains $\dim \mathbf{T} - 1$ linearly independent elements,
2. S is **complete** on $A \times [t(A), T]$ for each leaf $A \in \mathbf{T}$.

In this case, S is constant on $\llbracket 0, \zeta(\mathbf{T}) \rrbracket$ and

$$L^2(\mathcal{F}_T) = \text{span}\{1, \Psi\} + \mathcal{S}(S) = L^2(\sigma(\mathbf{T})) \oplus \mathcal{S}(S).$$

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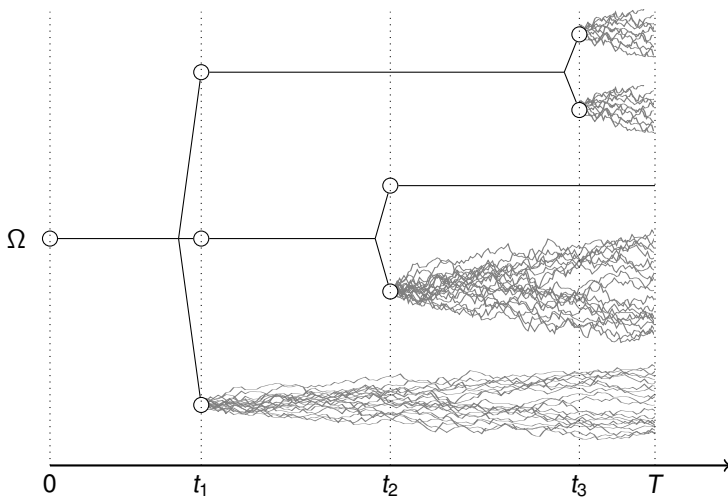
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Remark: $\psi_i = \mathbb{E}_{\mathbf{Q}}[\psi_i \mid \sigma(\mathbf{T})] + \underbrace{(H^i \cdot S)_T}_{\text{orthog. proj.}}, \quad i = 1, \dots, n.$

Semi-static completeness for continuous price processes



The filtration \mathbb{F} under $\mathbf{Q} \in \text{SSC}(\mathbb{F})$. Each set of lines emanating from the leaves of T corresponds to a dynamically complete stock price model.

Semi-static completeness for continuous price processes

Example (Semi-statically complete continuous model)

One static claim $\psi = \langle S, S \rangle_T - K$ with zero value at $t = 0$.

- Pick $t^* \in (0, T)$, $\sigma_1, \sigma_2 > 0$ with $\sigma_1 \neq \sigma_2$.
- Set $\mathbf{Q} = \lambda \mathbf{Q}^1 + (1 - \lambda) \mathbf{Q}^2$ where

$$S_t = \sigma_i W_{t-t^*} \mathbf{1}_{\{t \geq t^*\}} \quad \text{under } \mathbf{Q}^i,$$

where W is Brownian motion, and λ is determined by calibration:

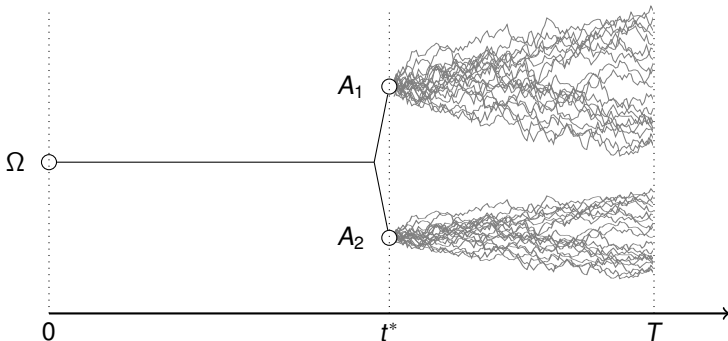
$$0 = \mathbb{E}_{\mathbf{Q}}[\psi \mid \mathcal{F}_0] = \lambda \sigma_1^2 (T - t^*) + (1 - \lambda) \sigma_2^2 (T - t^*) - K.$$

- Define $A_i = \{\partial^+ \langle S, S \rangle_{t^*} = \sigma_i^2\}$ and set $\mathbf{T} = \{\Omega, A_1, A_2\}$.
- \mathbf{T} is an atomic tree with $\dim \mathbf{T} = 2$ and

$$\mathbb{E}_{\mathbf{Q}}[\psi \mid \sigma(\mathbf{T})] = \sigma_1^2 (T - t^*) \mathbf{1}_{A_1} + \sigma_2^2 (T - t^*) \mathbf{1}_{A_2} - K \neq 0.$$

- By the theorem, $\mathbf{Q} \in \text{SSC}(\mathbb{F})$.

Semi-static completeness for continuous price processes



The leaves A_1, A_2 correspond to Bachelier models with volatilities $\sigma_1 > \sigma_2$. Thus the “variance swap” $\psi = \langle S \rangle_T$ is priced differently under the two models, and can be used to hedge against A_1 or A_2 .

Semi-static completeness for continuous price processes

Example (Semi-statically complete jump model, but no atomic tree)

- $\psi = [S, S]_T - K$

- $$S_t = \begin{cases} -t & t < \theta \wedge t^* \\ 1 - \theta + f(\theta)W_{t-\theta} & t \geq \theta, \theta < t^* \\ -t^* + \mathbf{1}_{A_1}\sigma_1 W_{t-t^*} + \mathbf{1}_{A_2}\sigma_2 W_{t-t^*} & t \geq t^*, t^* \leq \theta \end{cases}$$

with $\theta \sim \text{Exp}(1)$, $W, t^*, \sigma_1, \sigma_2 > 0$ as above, $f(t) : [0, t^*) \rightarrow \mathbb{R}_+$.

Conclusion: When the asset is allowed to jump, we do not have anymore the tree structure.

Pricing by informed investors

Setup

- $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$: right-continuous filtration (of the informed agent) with

$$\mathcal{F}_t \subseteq \mathcal{G}_t, \quad 0 \leq t \leq T.$$

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Question: How are $\text{SSC}(\mathbb{G})$ and $\text{SSC}(\mathbb{F})$ related?

Progressive filtration enlargement

Specification of \mathbb{G} : Progressive enlargement with a random time

- Let τ be a random time: $[0, T] \cup \{\infty\}$ -valued random variable.
- Define

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma(u \wedge \tau)$$

Smallest right-continuous filtration that contains \mathbb{F} and makes τ a stopping time.

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- For this kind of filtration enlargement there are clear-cut results between $SSC(\mathbb{G})$ and $SSC(\mathbb{F})$.

Progressive filtration enlargement

Theorem

Assume S is continuous and $S(\omega)$ is not constant on $[0, \tau(\omega)]$ for all $\omega \in \Omega$ such that $\tau(\omega) > 0$. Then

$$\mathbf{SSC}(\mathbb{G}) = \{ \mathbf{Q} \in \mathbf{SSC}(\mathbb{F}) : \mathbb{F} = \mathbb{G} \text{ up to nullsets} \}$$

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In the proof we use the classical Jeulin-Yor theorem.

- Fix $\mathbf{Q} \in \mathbf{SSC}(\mathbb{G})$
- Let Z be the Azéma supermartingale associated with τ :

$$Z_t = \mathbf{Q}(\tau > t \mid \mathcal{F}_t)$$

and let $Z = m - a$ be its Doob-Meyer decomposition

Theorem (Jeulin-Yor (1978))

The following process is a \mathbb{G} -martingale w.r.to \mathbf{Q} :

$$M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} da_s.$$

Progressive filtration enlargement

Sketch of the proof of “ \subseteq ”.

- Fix $\mathbf{Q} \in \text{SSC}(\mathbb{G})$

- Consider the process $M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} da_s \quad (1)$

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- By semi-static completeness,

$$M = M_0 + V + H \cdot S, \quad (2)$$

for some $H \in L(S)$ and martingale V with $V_T \in L^2(\sigma(\mathbf{T}))$

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- By (1), (2) and continuity of S , by considering the jumps of M :

$$\tau = \inf \left\{ t \in [0, T] : \frac{1}{Z_{t-}} \Delta a_t + \Delta V_t = 1 \right\}.$$

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- Also, by assumption $\tau > \sigma := \inf\{t > 0 : S_t \neq S_0\}$

- And V is constant on $\llbracket \sigma, \infty \llbracket$ by our Theorem

- Therefore $\tau = \inf \left\{ t \in [0, T] : \frac{1}{Z_{t-}} \Delta a_t = 1 \right\}$ \mathbb{F} -stopping time.

More general filtration enlargements

Remarks.

- From the proof it is clear that the set equivalence still holds true without any assumption on S when $\Psi = \emptyset$.
- Easy to generalize the theorem for filtration enlargements with multiple random times τ_1, \dots, τ_m . Set

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma(u \wedge \tau_i : i = 1, \dots, m).$$

Theorem

Assume S is continuous and $S(\omega)$ is not constant on $[0, \tau_i(\omega)]$ for all $\omega \in \Omega$ such that $\tau_i(\omega) > 0$, for all $i = 1, \dots, m$. Then

$$\text{SSC}(\mathbb{G}) = \{ \mathbf{Q} \in \text{SSC}(\mathbb{F}) : \mathbb{F} = \mathbb{G} \text{ up to nullsets} \}$$

More general filtration enlargement

Let \mathbb{H} be a filtration and define

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \mathcal{H}_u.$$

Assume that, for any $\mathbf{Q} \in \mathcal{M}(\mathbb{G})$, there are filtrations \mathbb{H}^k and random times τ_k such that

1. \mathbb{H} is the progressive enlargement of \mathbb{H}^k with τ_1, \dots, τ_k ;
2. S is almost surely non-constant on $[0, \tau_k]$ for all k ;
3. $\mathcal{F}_t = \bigcap_{k \geq 1} \mathcal{F}_t \vee \mathcal{H}_t^k$ up to nullsets for all $t \in [0, T]$.

Theorem

Under the above conditions,

$$\text{SSC}(\mathbb{G}) = \{ \mathbf{Q} \in \text{SSC}(\mathbb{F}) : \mathbb{F} = \mathbb{G} \text{ up to nullsets} \}$$

Conclusions

- Motivated by robust super-hedging price computation, we study extreme calibrated martingale measures
- We obtain a semi-static version of the Jacod-Yor theorem
- Description of semi-statically complete models in terms of dynamically complete models glued together using an atomic tree.
- Application to robust pricing by informed agents:
Under structural assumptions, informed agents price using only those models that render the additional information uninformative.
- Lots of things remain to be done and appear to be within reach:
 - Infinitely many static claims
 - Better understanding of price processes with jumps
 - Weaker hypotheses with infinitely many random times
 - ...

Thank you!