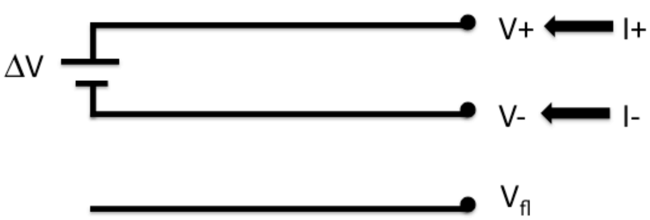
Exercise I: Considerations around the design of a Langmuir probe? (1 hour)

A) TRIPLE PROBE: FORMULE DERIVATION



Prove the formula:

$$T_e = \frac{e}{k\ln(2)}\left(V_+ - V_{fl}\right)$$

for the Triple probe configuration, assuming that the double probe circuit bias $\Delta V >> T_e$.

Recalling that $I = I_{sat}\left(1 - e^{\frac{V - V_{fl}}{kT_e/e}}\right)$ where $I_{sat} = A_p J_{sat} = A_p e \frac{n}{2} c_s$ and considering that the double probe circuit is floating, we have $I_+ = -I_-$.

$$I_{sat}\left(1 - e^{\frac{V_+ - V_{fl}}{kT_e/e}}\right) = -I_{sat}\left(1 - e^{\frac{V_- - V_{fl}}{kT_e/e}}\right) \Rightarrow 1 - e^{\frac{V_+ - V_{fl}}{kT_e/e}} = -1 + e^{\frac{V_- - V_{fl}}{kT_e/e}} \Rightarrow 2 = e^{\frac{V_+ - V_{fl}}{kT_e/e}} + e^{\frac{V_- - V_{fl}}{kT_e/e}}$$

$$\Rightarrow 2 = e^{\frac{V_+ - V_{fl}}{kT_e/e}}\left(1 + e^{\frac{V_- - V_{fl} - V_+ + V_{fl}}{kT_e/e}}\right), \text{ using } V_- - V_+ = -\Delta V \Rightarrow 2 = e^{\frac{V_+ - V_{fl}}{kT_e/e}}\left(1 + e^{\frac{-\Delta V}{kT_e/e}}\right).$$

Now we assume that $\Delta V >> kT_e/e$, therefore $e^{-\frac{\Delta V}{kT_e/e}} << 1$ and so $2 = e^{\frac{V_+ - V_{fl}}{kT_e/e}}$ and we finally obtain: $\ln(2) = \frac{e\left(V_+ - V_{fl}\right)}{kT_e} \Rightarrow T_e = \frac{e}{k\ln(2)}\left(V_+ - V_{fl}\right)$

B) TRIPLE PROBE WITH CORRECTION FOR SHEATH EXPANSION

How is modified the previous formula when the sheath expansion is taken into account?

We model the sheath expansion with the following formula: $I = I_{sat}\left[1-\alpha\left(V-V_{fl}\right)-\exp\left(\frac{V-V_{fl}}{kT_e/e}\right)\right]$. Now we consider the floating double probe circuit, where $I_+ = -I_-$ must hold.

$$I_+ = I_{sat}\left[1-\alpha\left(V_+-V_{fl}\right)-\exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right)\right] \quad \text{and} \quad I_- = I_{sat}\left[1-\alpha\left(V_--V_{fl}\right)-\exp\left(\frac{V_--V_{fl}}{kT_e/e}\right)\right]$$

$$\Rightarrow \quad 1-\alpha\left(V_+-V_{fl}\right)-\exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right) = -1+\alpha\left(V_--V_{fl}\right)+\exp\left(\frac{V_--V_{fl}}{kT_e/e}\right)$$

$$\Rightarrow \quad 2 = \alpha\left(V_+-V_{fl}\right)+\alpha\left(V_--V_{fl}\right)+\exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right)+\exp\left(\frac{V_--V_{fl}}{kT_e/e}\right)$$

$$\Rightarrow \quad 2 = \alpha V_+ + \alpha V_- - 2\alpha V_{fl} + \exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right)\left[1+\frac{\exp\left(\frac{V_--V_{fl}}{kT_e/e}\right)}{\exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right)}\right].$$

Now adding and substituting αV_+, and using $V_+ - V_- = \Delta V$, we have:

$$\Rightarrow \quad 2 = \alpha V_+ + \alpha V_+ - \alpha V_+ + \alpha V_- - 2\alpha V_{fl} + \exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right)\left[1+\exp\left(\frac{-\Delta V}{kT_e/e}\right)\right]$$

Using the fact that $\Delta V >> kT_e/e$,

$$\Rightarrow \quad 2 = 2\alpha\left(V_+-V_{fl}\right)-\alpha\Delta V+\exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right) \Rightarrow \quad 2-2\alpha\left(V_+-V_{fl}\right)+\alpha\Delta V = \exp\left(\frac{V_+-V_{fl}}{kT_e/e}\right)$$

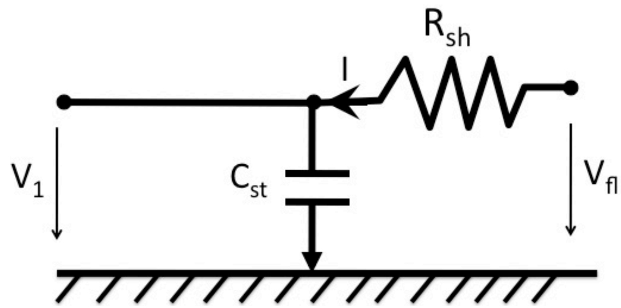
$$\ln\left[2-2\alpha\left(V_+-V_{fl}\right)+\alpha\Delta V\right] = \frac{V_+-V_{fl}}{kT_e/e} \quad \Rightarrow \quad T_e = \frac{e\left(V_+-V_{fl}\right)}{k\ln\left[2-2\alpha\left(V_+-V_{fl}\right)+\alpha\Delta V\right]}$$

C) THE EFFECT OF THE SHEATH

Prove that the presence of a sheath in front of a probe, which is biased at the floating potential, results in a limitation of the frequency response of the probe itself.

Using the formula for the current driven by a LP $I = I_{sat}\left(1 - e^{\frac{V-V_{fl}}{kT_e/e}}\right)$ and assuming that the probe is biased close to the floating potential, we have V ~ V_{fl}, therefore $\frac{V - V_{fl}}{T_e}$ << 1. We can then expand the exponential:

$$I \approx I_{sat}\left(1 - 1 + \frac{V - V_{fl}}{kT_e/e}\right) \approx \frac{I_{sat}}{kT_e/e}\Delta V \quad \Rightarrow \quad R_{sh} = \frac{kT_e}{eI_{sat}} \text{ where } I_{sat} = A_p J_{sat} = A_p e \frac{n}{2} c_s.$$



We can now consider a real LP whose signal is transported and acquired via a cable with a stray capacitance to the ground (typically C_{st} ~ 100 pF/m). A resistance and a capacitance in series can model this effect. This circuit behaves like a low-pass filter.

The current circulating through the resistance is given by: $V_{fl} - V_1 = R_{sh} I$ and the current through the capacitance is given by: $I = C_{st}\frac{dV_1}{dt}$ (since the impedance is infinite).

Since it is a series circuit, the Kirshhoff's law states that currents must be equal:

$I = \frac{V_{fl} - V_1}{R_{sh}} = C_{st}\frac{dV_1}{dt} \quad \Rightarrow (1 + j\omega R_{sh} C_{st})\hat{V}_1 = \hat{V}_{fl}$ where we have used the Fourier transform of the signal $\hat{V}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(t) e^{j\omega t} dt$

The cut-off frequency is defined as the frequency for which the amplitude of the output voltage is attenuated by $\sqrt{2}$ (which corresponds to -3 dB) w.r.t. the amplitude of the input voltage:

$$G_C = \left|\frac{\hat{V}_1}{\hat{V}_{fl}}\right| = \frac{1}{\sqrt{1+\left(\omega C_{st} R_{sh}\right)^2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad 2 = 1 + \omega^2 \left(C_{st} R_{sh}\right)^2 \Rightarrow$$

$$\omega_{cutoff} = \frac{1}{R_{sh} C_{st}} \Rightarrow f_{cutoff} = \frac{1}{2\pi R_{sh} C_{st}}$$

We note also that a sheath capacitance may be present, which can be estimated as: $C_{sh} \sim \frac{\varepsilon_0 A}{\lambda_D}$. This is usually very small (for TORPEX, C_{sh} ~ 0.5 pF).

Examples:

TORPEX: H_2 plasma, T_e ~ 5 eV – n_e ~ 10^{16} m^{-3} – A ~ 10^{-5} m^2 – C_{sh} ~ 0.5 pF – 3 m cables:

→ $R_{sh} = eT_e/I_{sat}$ with $I_{sat} = Ae\frac{n}{2}C_s$ and $C_s = 9.79\times10^3(\gamma Z T_e/\mu)^{1/2}$ m.s^{-1} → Cs ~ 2.8×10^4 m.s^{-1} → I_{sat} ~ 2.3×10^{-4} A → R_{sh}= 2.2×10^4 Ω → f_{cutoff} = 2.4×10^4 Hz (C_{st} = 300 pF)

TCV: D_2 plasma, T_e ~ 50 eV – n_e ~ 10^{19} m^{-3} – A ~ 10^{-5} m^2 – C_{sh} ~ 0.5 pF – 10 m cables: R_{sh}? f_{cutoff}?

→ Z=1, μ=2 → C_s ~ 6.3×10^4 m.s^{-1} → I_{sat} ~ 0.5 A → R_{sh} ~ 98 Ω → f_{cutoff} ~ 1.6×10^6 Hz (C_{st} ~ 1000 pF)

Note that this limits the sweep frequency and it also introduces a phase delay, which may leads to error in the evolution of the particle flux.

Reference: *Practical solutions for reliable triple probe measurements in magnetized plasmas* by C. Theiler, I. Furno, A. Kuenlin, P. Marmillod and A. Fasoli in Review of Scientific Instruments, vol. 82, p. 013504, 2011.

https://en.wikipedia.org/wiki/RC_circuit