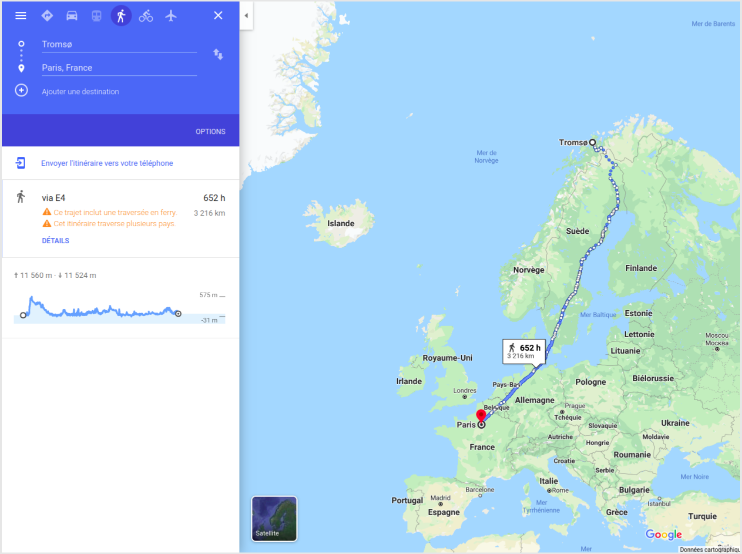
Exact semidefinite programming bounds for packing problems

Philippe Moustrou, UiT The Arctic University of Norway
Joint work with M. Dostert (EPFL) and D. de Laat (TU Delft).

Online Summer School on Optimization, Interpolation and Modular Forms
August 28, 2020

Tromsø: the Paris of the North



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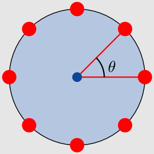
Problem:

Usually semidefinite programming provides approximate numerical bounds.

How can we turn these bounds into exact bounds?

Spherical codes and variants



Spherical codes:

$$\max\{|C|, \quad C \subset S^{n-1}, \quad x \cdot y \leq \cos\theta \text{ for all } x \neq y \in C\}$$

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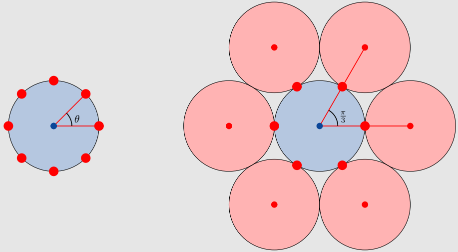

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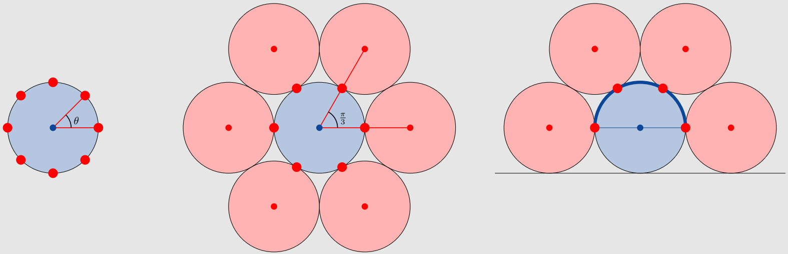

Kissing number of the hemisphere:

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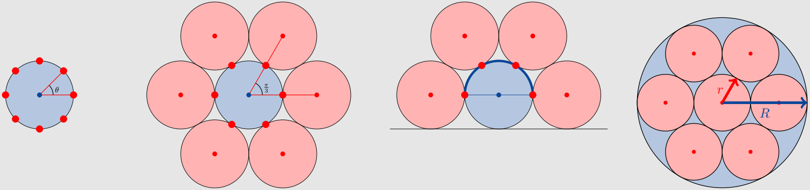

Packing spheres in spheres:

$$\max\{|C| : C \subset B(0, R-r), \|x-y\| \geq 2r \text{ for all } x \neq y \in C\}$$

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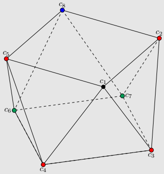

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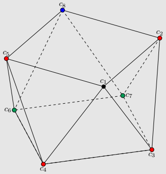
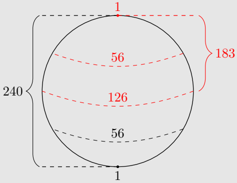


- For the Hemisphere in dimension 8: the E_8 lattice provides an optimal configuration (Bachoc-Vallentin, 2008). What about uniqueness?



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  - 2-point bound (Delsarte-Goethals-Seidel 1977)
  - 3-point bound (Bachoc-Vallentin 2008).

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- The normalized Gegenbauer polynomials $P_k^n(u)$ (with $P_k^n(1) = 1$), satisfying:

$$\text{For every } X \subset S^{n-1} \text{ finite, } \sum_{x,y \in X} P_k^n(x \cdot y) \geq 0.$$

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- there exists coefficients $\alpha_0, \ldots, \alpha_d \geq 0$ such that

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So

$$|C| \leq f(1) + 1$$

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So for every $d \geq 0$, the size of a θ-spherical code is at most

$$\begin{aligned}\min\{M \in \mathbb{R} : \alpha_0, \ldots, \alpha_d &\geq 0,\\ f(1) &\leq M - 1,\\ f(u) &\leq -1 \text{ for all } u \in [-1, \cos\theta]\}\end{aligned}$$

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This is a linear programming bound.

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- Up to symmetry, a triple of points x, y, z in a θ-spherical code is uniquely determined by

$$u = x \cdot y, \quad v = x \cdot z, \quad t = y \cdot z,$$

with (u, v, t) in

$$\begin{cases} \{(1,1,1)\} & x = y = z \\ \Delta_0 = \{(u,u,1) : u \in [-1, \cos\theta]\} & x \neq y = z \\ \Delta & x, y, z \text{ distinct} \end{cases}$$

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$$\Delta = \{(u, v, t) : u, v, t \in [-1, \cos\theta], 1 + 2uvt - u^2 - v^2 - t^2 \geq 0\}$$

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- Matrix polynomials $S_k^n(u, v, t)$ satisfying:

For every $X \subset S^{n-1}$ finite, $\displaystyle\sum_{x,y,z \in X} S_k^n(x \cdot y, x \cdot z, y \cdot t) \succeq 0.$

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So for every $d \geq 0$, the size of a θ-spherical code is at most

$$\min\{M \in \mathbb{R} : \alpha_k \geq 0, F_k \succeq 0$$

$$\sum_{k=0}^{d} \alpha_k + F(1,1,1) \leq M - 1,$$

$$\sum_{k=0}^{d} \alpha_k P_k^n(u) + 3F(u,u,1) \leq -1 \text{ for all } u \in [-1, \cos\theta],$$

$$F(u,v,t) \leq 0 \text{ for all } (u,v,t) \in \Delta\}$$

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This leads to semidefinite programming upper bounds using sums of squares.

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- Nevertheless, one can still compute the 2-point bound for these problems.
- These bounds look like the 3-point bound for spherical codes. In particular they are semidefinite programming bounds.

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- Optimization: When does a bound give the independence number?
- Geometry: Sharp bounds provide additional information on optimal configurations, leading to uniqueness proofs.

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$$\Rightarrow \text{for all } x, y \in C, x \cdot y \in \{0, \pm 1/2, \pm 1\} \quad \Rightarrow C = C_0$$

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- The Petersen code is the unique optimal $1/6$-code in dimension 4 (Bachoc-Vallentin 2009, Dostert-de Laat-M 2020).
- Numerically sharp for the square antiprism (Bachoc-Vallentin 2009) $\rightarrow$ Rigorous proof (Dostert-de Laat-M 2020)

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Many examples of exact sharp LP bounds ...

But very few cases in which SDP bound is proven to be sharp while LP is not:

- The Petersen code is the unique optimal 1/6-code in dimension 4 (Bachoc-Vallentin 2009, Dostert-de Laat-M 2020).
- Numerically sharp for the square antiprism (Bachoc-Vallentin 2009) $\to$ Rigorous proof (Dostert-de Laat-M 2020)
- E_8 gives an optimal configuration on the hemisphere in dimension 8 (Bachoc-Vallentin 2009)
  $\to$ Uniqueness (Dostert-de Laat-M 2020)

Solving an SDP: Rage against the machine precision

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with x the vector of unknowns, and $\mathcal{B}_i(x)$ the blocks of x.

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- Our context: The problems provide a candidate field to round over, either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{d})$.

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The linear system is then satisfied... But what about the PSD conditions?

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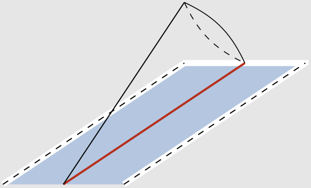

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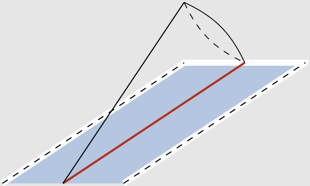

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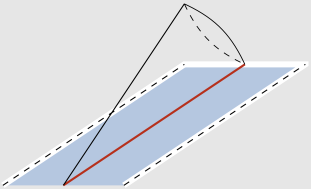

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- Sometimes not. How to force all these constraints?

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- Then $\mathcal{B}_i(x)v = 0$ provides new linear constraints on x!

Rounding over $\mathbb{Q}$: detecting kernel vectors (general case)

This is not enough in general. How to extract a nice basis from the numerical values?

$$\ker(\mathcal{B}_i(x^*)) \approx \left\langle \begin{pmatrix} 0.19550004741012542 \\ -0.10616756374846323 \\ -0.25700180101766007 \\ -0.33241916014721035 \end{pmatrix}, \begin{pmatrix} -0.8676883652023846 \\ -0.4321427618192919 \\ -0.2143699892153049 \\ -0.1054836185183479 \end{pmatrix} \right\rangle$$

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Key idea: use the LLL algorithm to detect an integer linear equation almost sastisfied by the kernel vectors...

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$$\left\{ -u_1 + 3u_2 - 2u_3 = 0 \right.$$

Rounding over $\mathbb{Q}$: detecting kernel vectors (general case)

...and another one...

$$\ker(\mathcal{B}_i(x^*)) \approx \left\langle \begin{pmatrix} \cancel{0.19550004741012542} \\ -0.1061675637484632 3 \\ -0.2570018010176600 7 \\ -0.3324191601472103 5 \end{pmatrix}, \begin{pmatrix} \cancel{-0.8676883652023846} \\ -0.432142761819291 9 \\ -0.214369989215304 9 \\ -0.105483618518347 9 \end{pmatrix} \right\rangle \begin{array}{c} \\ \mathbf{1} \\ \mathbf{-3} \\ \mathbf{2} \end{array}$$

$$\begin{cases} -u_1 + 3u_2 - 2u_3 & = 0 \\ \qquad\quad u_2 - 3u_3 + 2u_4 & = 0 \end{cases}$$

Rounding over $\mathbb{Q}$: detecting kernel vectors (general case)

With enough equations, we can compute the expected kernel basis.

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$$\begin{cases} -u_1 + 3u_2 - 2u_3 & = 0 \\ u_2 - 3u_3 + 2u_4 & = 0 \end{cases}$$

$$\ker(\mathcal{B}_i(x)) = \left\langle \begin{pmatrix} 7 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

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- Check that the only possible inner products are the ones in the candidate optimal configuration (use Sturm sequences).
- If needed compute the possible 3-point distance distribution of an optimal code.
- Use this information and a bit of geometry to prove that the candidate optimal configuration is unique!

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- What about other applications?

Thank you!

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- We also expect a solution over $\mathbb{Q}(\sqrt{d})$, so write

$$x = x_1 + \sqrt{d}x_2$$

  and work over $\mathbb{Q}$:

$$\begin{pmatrix} A_1 & dA_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Bonus: extension to quadratic fields (finding good x_1^*,x_2^*)

- From the numerical x^* satisfying $Ax^* \approx b$ we need to find x_1^* and x_2^* such that $x^* \approx x_1^* + \sqrt{d}x_2^*$ and

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- To do so, solve (in floating point) the linear system:

$$\begin{pmatrix} A_1 & dA_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} y \\ \frac{1}{\sqrt{d}}(x^* - y) \end{pmatrix} \approx \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

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- Compute the expected kernel over $\mathbb{Q}$ and add the corresponding constraints on x_1 and x_2.