# Completely Positive and Copositive Type Functions for Packings and Distance-Avoiding Sets

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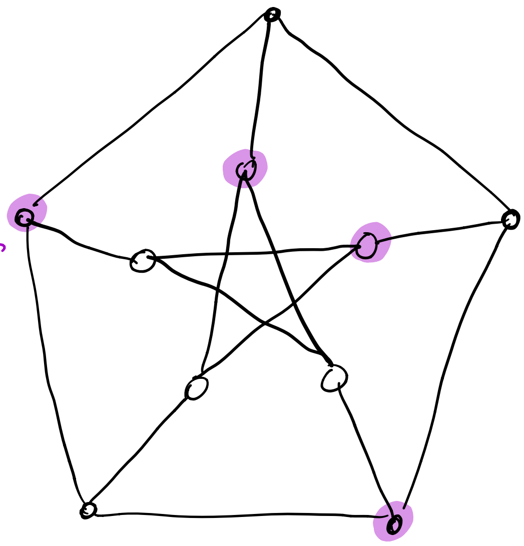
# Independent Sets & Independence Number

$G = (V, E)$ graph (finite / infinite)



ind. set

$$\alpha(G) = \sup \{ |I| : I \subseteq V \text{ ind.} \}$$

$\downarrow$

independence number (NP-hard)

## Binary Codes

- $V = H_n = \{0,1\}^n$
- $xy \in E$ if $0 < d_H(x,y) < d$.

Ind. sets $\equiv$ binary codes

$\alpha(G) = A(n, d)$.

# Witsenhausen's Problem

- $V = S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$
- $xy \in E$ if $x \cdot y = 0$.

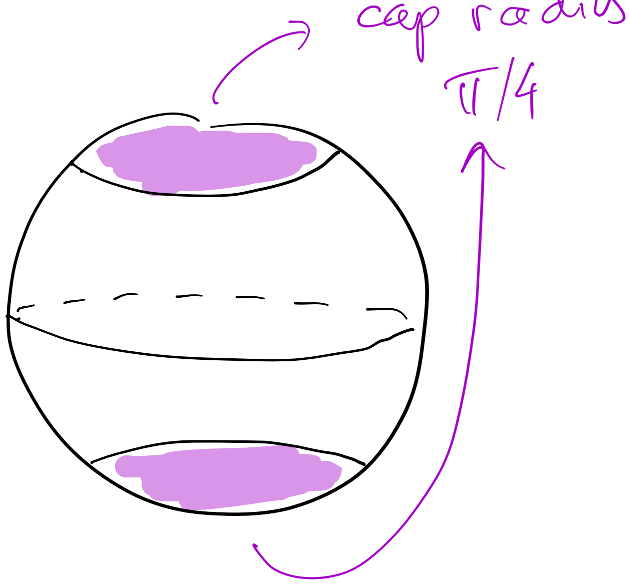
Ind. sets $\equiv$ sets w/ orthogonal points

$\alpha(G) = \infty$. (meaningless!)

$\alpha_m(G) = \sup \{ \omega(I) : I \subseteq V$ measurable and independent $\}$

What is $\alpha_m(G)$?

Double-cap conjecture



## Spherical Codes

- $V = S^{n-1}$
- $xy \in E$ if $x \cdot y > \cos\theta$

Ind. sets $\equiv$ spherical codes $\angle\, \theta$

$\alpha(G) = A(n, \theta)$

## 1-avoiding sets

- $V = \mathbb{R}^n$
- $xy \in E$ if $\|x - y\| = 1$

Unit-distance graph (Hadwiger-Nelson)

Ind. sets $\equiv$ sets avoiding distance 1

$\alpha(G) = \infty$

Again: $\alpha_m(G) = \sup\{\bar{\delta}(I) : I \subseteq \mathbb{R}^n$ measurable, independent$\}$

## Sphere packing

- $V = \mathbb{R}^n$
- $xy \in E$ if $0 < \|x-y\| < 1$

Ind. sets $\equiv$ centers of packings of radius $1/2$

$\alpha(G) = \infty$

$\alpha_\delta(G) = \sup\{\delta(P(I)) : I$ independent$\}$

(or center density)

# Different kinds of graphs

- $V$ is a Hausdorff space

Locally independent: Every compact ind. set is a subset of an open ind. set.

Topo. Packing: Every finite clique is a subset of an open clique

| | Locally ind. | Topo. Packing |
|---|---|---|
| Compact $V$ | Wit. Problem<br>Bin. codes | Spherical codes<br>Bin. codes |
| Noncompact $V$ | 1-avoid. sets | Sphere packing |

# The Lovász $\vartheta$ number (1979)

## Primal

$$\vartheta(G,\mathcal{K}) = \max \langle J, A\rangle \quad \left[= \sum_{x,y\in V} A(x,y)\right]$$

$$\operatorname{tr} A = 1$$

$$A(x,y) = 0 \text{ if } xy \in E$$

$$A \in \mathcal{K} \qquad [A : V\times V \to \mathbb{R}]$$

## Dual

$$\min \lambda$$

$$Z(x,x) \leq \lambda \quad \forall\, x\in V$$

$$Z(x,y) \leq 0 \text{ if } xy \notin E,\ x\neq y.$$

$$Z - J \in \mathcal{K}^*$$

## Cones

- $\mathcal{K} = \mathrm{PSD}(V) = \{A \in \mathbb{R}^{V\times V} : A \text{ PSD}\}$ Lovász $\vartheta$
- $\mathcal{K} = \mathrm{PSD} \cap \mathrm{NN} = \{A : A \text{ PSD}, A \geq 0\}$ $\vartheta'$.

- $\mathcal{K} = CP(V) = \operatorname{cone}\{xx^T : x \in \mathbb{R}^V, x \geq 0\}$

  $\left[\mathcal{K}^* = COP(V) = \{A : x^T A x \geq 0 \ \forall\ x \geq 0\}\right]$

- $\mathcal{K} = BQC(V) = \operatorname{cone}\{xx^T : x \in \{0,1\}^V\}$

  $\left[\mathcal{K}^* = \{A : x^T A x \geq 0 \ \forall\ x \in \{0,1\}^V\}\right]$

$$BQC \subseteq CP \subseteq PSD \cap NN \subseteq PSD \subseteq PSD + NN \subseteq COP \subseteq BQC^*$$

**Theorem.** $\vartheta(G, BQC) \geq \alpha(G)$.

**Pf.** $I \subseteq V$ indep., $I \neq \emptyset$.

- $A = \frac{1}{|I|} \chi_I \chi_I^T$ $\left[ A(x,y) = \begin{cases} \frac{1}{|I|} & \text{if } x,y \in I \\ 0 & \text{otherwise.} \end{cases} \right]$

- $\operatorname{tr} A = 1$

- $A(x,y) = 0$ if $xy \in E$

- $\langle J, A \rangle = |I|^2 / |I| = |I|$. $\underline{QED}$

**Theorem.** $\vartheta(G, CP) = \alpha(G)$. [Motzkin & Straus]

**Pf.** "$\geq$" proved above since $\vartheta(G,CP) \geq \vartheta(G,BQC)$.

Let's show "$\leq$".

- Take $A$ feasible solution.
- $A = \alpha_1 f_1 f_1^T + \dots + \alpha_k f_k f_k^T$

  $\alpha_i \geq 0$, $f_i \geq 0$ $\forall i$

  Assume $\|f_i\| = 1$ $\forall i$ (rescale $\alpha_i$)
- $\operatorname{tr} A = 1 \Rightarrow \alpha_1 + \dots + \alpha_k = 1$
- $\langle J, A \rangle = \alpha_1 \langle J, f_1 f_1^T \rangle + \dots + \alpha_k \langle J, f_k f_k^T \rangle$.

  $\Rightarrow$ assume $\langle J, A \rangle \leq \langle J, f_1 f_1^T \rangle$.
- $I = \operatorname{supp} f_1$
- $A(x,y) = 0$ if $xy \in E$, $f_i \geq 0$ $\forall i \Rightarrow I$ is indep.

$$\langle J, A \rangle \leq \langle J, f_1 f_1^T \rangle = \langle \chi_V \chi_V^T, f_1 f_1^T \rangle = (\chi_V^T f_1)^2$$

$$= (\chi_I^T f_1)^2 \leq \|\chi_I\|^2 \|f_1\|^2 = |I|.$$

QED

# Conclusions

- $\vartheta(G, BQC) = \vartheta(G, CP) = \alpha(G)$
- BQC, CP are hard cones [separation NP-hard]
- Duality $\Rightarrow$ $BQC^*$, COP are hard cones.

Use BQC, CP to strengthen $\vartheta$:

$$\vartheta(G, \mathcal{K}) = \max \langle J, A \rangle$$

$$\operatorname{tr} A = 1$$

$$A(x,y) = 0 \text{ if } xy \in E$$

$$A \in PSD \quad [A : V \times V \to \mathbb{R}]$$

$$\langle A, F \rangle \geq 0 \quad \forall F \in \mathcal{F}$$

$\mathcal{F} \subseteq BQC^*$ or $\mathcal{F} \subseteq COP$

# Extension to compact spaces

- $V$ Hausdorff compact space
- $\mu$ Borel finite measure

$$CP(V) = \text{cl cone}\{ f \otimes f^* : f \in L^2(V),\ f \geq 0 \}$$

$$(f \otimes f^*)(x,y) = f(x) f(y)$$

$$COP(V) = CP(V)^* = \left\{ A \in L^2(V \times V) : \iint A(x,y) f(x) f(y)\, dy\, dx \geq 0 \quad \forall f \in L^2(V),\ f \geq 0 \right\}$$

## Topo. Packing Graphs

$$\alpha(G) = \inf \lambda$$

[Dobre, Dür, Frick, Vallentin]

$$Z(x,x) \leq \lambda \quad \forall x \in V$$

$$Z(x,y) \leq 0 \quad \forall x \neq y,\ xy \notin E$$

$$Z - J \in COP(V)$$

$Z : V \times V \to \mathbb{R}$ is continuous

$$PSD(V) = \{A \in L^2(V\times V) : \iint A(x,y)\, f(x) f(y)\, dy\, dx \geq 0 \quad \forall f \in L^2(V)\}$$

- $G$ = spherical code graph
- PSD instead of COP $\Rightarrow$ LP bound.

## Locally Independent Graphs

$$\alpha_m(G) = \sup \iint A(x,y)\, dy\, dx$$

[DeCorte, Oliveira, Vallentin]

$$\int A(x,x)\, dx = 1$$

$$A(x,y) = 0 \quad \text{if } xy \in E$$

$$A \in CP(V), \quad A \text{ continuous}$$

# Extra Constraints

$$\alpha_m(G) \leq \sup \iint A(x,y)\, dy\, dx$$

$$\int A(x,x)\, dx = 1$$

$$A(x,y) = 0 \text{ if } xy \in E$$

$$A \in PSD(V), \; A \text{ continuous}$$

Continuous $A \in CP(V)$

$\underline{\text{iff}}$ $(A(x,y))_{x,y \in U} \in CP(U)$ $\forall$ finite $U \subseteq V$

Fix $U \subseteq V$ finite. Take $F: U \times U \to \mathbb{R}$ copositive.

Add the constraint: $\displaystyle\sum_{x,y \in U} A(x,y)\, F(x,y) \geq 0.$

## Extension to $\mathbb{R}^n$

$$PSD(\mathbb{R}^n) = \{ f : f : \mathbb{R}^n \to \mathbb{R} \text{ is continuous, } (f(x-y))_{x,y \in \mathcal{U}} \in PSD(\mathcal{U}) \ \forall \text{ finite } \mathcal{U} \subseteq \mathbb{R}^n \}$$

funcs. of positive type

$f$ Schwartz $\in PSD(\mathbb{R}^n)$ iff $\hat{f} \geq 0$

**Theorem (Bochner).** $f \in PSD(\mathbb{R}^n)$ iff $\exists$ measure $\nu$ s.t.

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i u \cdot x} \, d\nu(u).$$

$$COP(\mathbb{R}^n) = \{ f : f : \mathbb{R}^n \to \mathbb{R} \text{ is continuous, } (f(x-y))_{x,y \in \mathcal{U}} \in COP(\mathcal{U}) \ \forall \text{ finite } \mathcal{U} \subseteq \mathbb{R}^n \}$$

funcs. of copositive type

## Theorem (de Laat, O.)

$$\Delta_n = \inf \quad f(0) + g(0)$$
$$\hat{f}(0) = 1$$
$$f(x) + g(x) \leq 0 \quad \text{if } \|x\| \geq 1$$
$$f \in PSD(\mathbb{R}^n), \quad g \in COP(\mathbb{R}^n)$$
$$f, g \text{ are admissible}$$

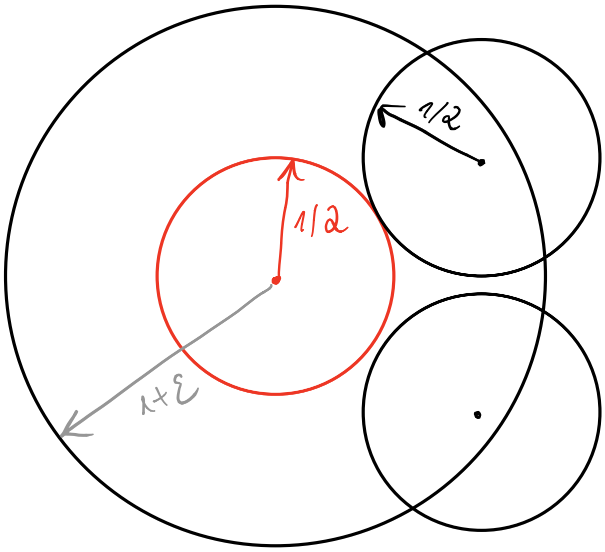
Admissable = continuous and ...

- bounded support ($\mathbb{R}^n$ is limit of tori)
- $L^1$ (reduce to bounded support)
- Schwartz (harder proof)

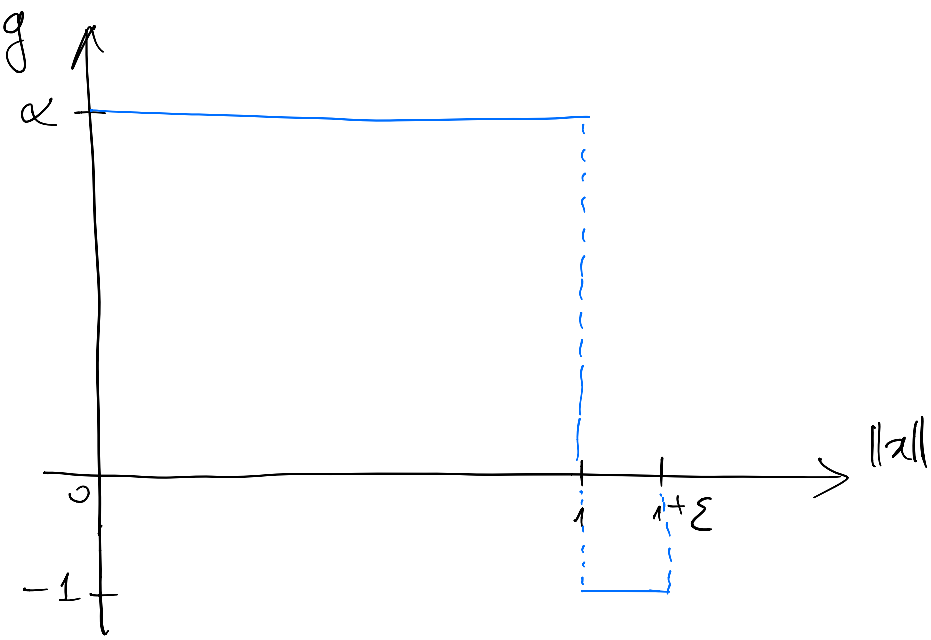
Cohn - Elkies: take $g = 0$.

Exercise: Prove " $\leq$ " ; same as Cohn-Elkies.

# A copositive $g$




$\alpha \geq$ max # of nonoverlapping balls at dist. in $[1, 1+\varepsilon]$ from central.




**Prop.** $g$ is of copositive type.

Leads to only known (?) improvement of Cohn-Elkies [de Laat, Oliveira, Vallentin]