Completely Positive and Copositive Type Functions for Packings and Distance-Avoiding Sets

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Independent Sets & Independence Number

\[ G = (V, E) \text{ graph (finite/infinite)} \]

\[ \alpha(G) = \sup \{ |I| : I \subseteq V \text{ ind.} \} \]

\[ \downarrow \]

independence number (NP-hard)

\[ \text{ind. set} \]

\underline{Binary Codes}

- \[ V = H_n = \{0, 1\}^n \]

- \[ xy \in E \text{ if } 0 < d_4(x, y) < d. \]

\[ \text{ind. sets } \equiv \text{ binary codes} \]

\[ \alpha(G) = A(n, d). \]
Witsenhausen's Problem

- \( V = S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \)
- \( x, y \in E \) if \( x \cdot y = 0 \).

Ind. sets \( \equiv \) sets w/o orthogonal points
\( \alpha(G) = \infty \). (meaningless!)

\[ \alpha_m(G) = \sup \{ w(I) : I \subseteq V \text{ measurable and independent} \} \]

What is \( \alpha_m(G) \)?

Double-cap conjecture
Spherical Codes

\( V = S^{n-1} \)

\( x y \in E \quad \text{if} \quad x \cdot y > \cos \theta \)

Ind. sets \( \equiv \) spherical codes \( \leq \theta \)

\( \chi(G) = A(n, \theta) \)

1-avoiding sets

\( V = \mathbb{R}^n \)

\( x y \in E \quad \text{if} \quad \|x-y\| = 1 \) \hspace{1cm} \text{(Unit-distance graph, Hadwiger-Nelson)}

Ind. sets \( \equiv \) sets avoiding distance 1

\( \chi(G) = \infty \)
Again: $\alpha_m(A) = \sup \{ \bar{s}(I) : I \subseteq \mathbb{R}^n \text{ measurable, independent} \}$

Sphere packing

- $V = \mathbb{R}^n$
- $xy \in E$ if $0 < ||x - y|| < 1$

Ind. sets $\equiv$ centers of packings of radius $1/2$

$\alpha(A) = \infty$

$\alpha_s(A) = \sup \{ s(P(I)) : I \text{ independent} \}$ (or center density)
Different kinds of graphs

- $V$ is a Hausdorff space

**Locally independent**: Every compact ind. set is a subset of an open ind. set.

**Topo. Packing**: Every finite clique is a subset of an open clique

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The loudest J number (1979)

Primal

\[\psi(G, \kappa) = \max \langle J, A \rangle \left[ = \sum_{x, y \in V} A(x, y) \right] \]

\[\text{tr} \ A = 1\]

\[A(x, y) = 0 \ if \ xy \notin E\]

\[A \in \mathbb{R} \quad [A : V \times V \to \mathbb{R}]\]

Dual

\[\min \ \lambda\]

\[\varepsilon(x, x) \leq \lambda \quad \forall \ x \in V\]

\[\varepsilon(x, y) \leq 0 \ if \ xy \notin E, \ x \neq y.\]

\[\varepsilon - J \in \mathcal{K}^*\]

Cones

- \[\mathcal{K} = \text{PSD}(V) = \{ A \in \mathbb{R}^{V \times V} : A \ \text{PSD}\} \quad \text{largest } J\]
- \[\mathcal{K} = \text{PSD} \cap \text{NN} = \{ A : A \ \text{PSD}, A \geq 0 \} \quad 0\]
• $K = \mathbf{CP}(V) = \text{cone}(xx^T : x \in \mathbb{R}^V, x \geq 0)$
  
  $[K^* = \mathbf{COP}(V) = \{ A : x^TAx \geq 0 \ \forall \ x \geq 0 \}]$

• $K = \mathbf{BQC}(V) = \text{cone}(xx^T : x \in \{0,1\}^V)$
  
  $[K^* = \{ A : x^TAx \geq 0 \ \forall \ x \in \{0,1\}^V \}]$

$\mathbf{BQC} \subseteq \mathbf{CP} \subseteq \mathbf{PSD}_{\mathbb{R}} \subseteq \mathbf{PSD} \subseteq \mathbf{PSD} + \mathbf{NN} \subseteq \mathbf{COP} \subseteq \mathbf{BQC}^*$

Theorem. $\vartheta(G, \mathbf{BQC}) \geq \alpha(G)$.

\textbf{Pf.} \quad I \subseteq V \text{ independent, } I \neq \emptyset.

• $A = \frac{1}{|I||I|} x_I x_I^T$

  $[A(x,y) = \begin{cases} \frac{1}{|I|} & \text{if } x,y \in I \\ 0 & \text{otherwise.} \end{cases}]$

• $\text{tr } A = 1$

• $A(x,y) = 0 \text{ if } xy \notin E$

• $\langle J, A \rangle = |I|^2 / |I| = |I|$. \hfill QED
**Theorem.** \( \sigma(G, CP) = \alpha(G) \). [Motzkin & Straus]

**Pf.** "\( \Rightarrow \)" proved above since \( \sigma(G, CP) \geq \sigma(G, BGC) \). Let's show "\( \leq \)."

- Take a feasible solution.
- \( A = \alpha_1 f_1 f_1^T + \cdots + \alpha_k f_k f_k^T \)
  \( \alpha_i \geq 0, \ f_i \geq 0 \ \forall i \)

Assume \( \|f_i\| = 1 \ \forall i \) (rescale \( \alpha_i \))

- \( \|A\| = 1 \Rightarrow \alpha_1 + \cdots + \alpha_k = 1 \)

- \( \langle J, A \rangle = \alpha_1 \langle J, f_1 f_1^T \rangle + \cdots + \alpha_k \langle J, f_k f_k^T \rangle \)
  \( \Rightarrow \) assume \( \langle J, A \rangle \leq \langle J, f_1 f_1^T \rangle \)

- \( I = \text{supp } f_1 \)
- \( A(x, y) = 0 \) if \( xy \notin E \), \( f_i \geq 0 \ \forall i \Rightarrow I \) is indep.

\[ \langle J, A \rangle \leq \langle J, f_1 f_1^T \rangle = \langle x u x^T, f_1 f_1^T \rangle = \langle x u f_1 \rangle^2 \]
\[ = \langle x_i^T f_1 \rangle^2 \leq \|x_i\|^2 \|f_1\|^2 = \|I\|. \]
\( \square \)
Conclusions

- $\Theta(G, \text{BQC}) = \Theta(G, \text{CP}) = \chi(G)$
- BQC, CP are hard cones [separation NP-hard]
- Duality $\Rightarrow$ BQC*, COP are hard cones.

Use BQC, CP to strengthen $\Theta$.

$$\Theta(G, \kappa) = \max \langle J, A \rangle$$

\[ \text{tr } A = 1 \]

\[ A(x, y) = 0 \text{ if } xy \notin EE \]

\[ A \in \text{PSD} \quad [A: V \times V \rightarrow \mathbb{R}] \]

\[ \langle A, F \rangle \geq 0 \quad \forall F \in F \]

$F \subseteq \text{BQC}^*$ or $F \subseteq \text{COP}$
Extension to compact spaces

- $V$ Hausdorff compact space
- $\mu$ Borel finite measure

$$CP(V) = \text{cl cone} \{ f \otimes f^* : f \in L^2(V), f \geq 0 \}$$

$$\downarrow$$

$$(f \otimes f^*)(x,y) = f(x) f(y)$$

$$\text{COP}(V) = CP(V)^* = \{ A \in L^2(V \times V) : \int \int A(x,y) f(x) f(y) \, dy \, dx \geq 0 \ \forall \ f \in L^2(V), f \geq 0 \}$$

**Topo. Packing Graphs**

$$\alpha(G) = \inf \lambda$$

[Dobre, Dür, Frick, Vallentin]

$Z(\lambda, x) \leq \lambda \ \forall \ x \in V$

$Z(\lambda, y) \leq 0 \ \forall \ x \neq y, \ xy \notin E$

$Z - J \in \text{COP}(V)$

$Z : V \times V \to \mathbb{R}$ is continuous
$$\text{PSD}(V) = \{ A \in L^2(V \times V) : \int \int A(x,y) f(x)f(y) \, dy \, dx \geq 0 \}$$

- \( G = \text{spherical code graph} \)
- \( \text{PSD instead of COP} \Rightarrow \text{LP bound} \)

**Locally Independent Graphs**

$$\alpha_m(G) = \sup \int \int A(x,y) \, dy \, dx$$

[DeCorte, Oliveira, Vallentin]

\[ \int A(x,x) \, dx = 1 \]

\[ A(x,y) = 0 \quad \text{if} \quad xy \notin E \]

\[ A \in \text{CP}(V) \, , \quad A \text{ continuous} \]
Extra Constraints

$\chi_m(G) \leq \sup \left\{ \int \int A(n, y) \, dy \, dx \right\}$

$\int A(n, n) \, dx = 1$

$A(x, y) = 0 \quad \text{if} \quad xy \notin E$

$A \in \text{PSD}(V), \quad A \text{ continuous}$

Continuous $A \in \text{CP}(V)$

iff $(A(x, y))_{x, y \in \text{u}} \in \text{CP}(u)$ for finite $u \subseteq V$

Fix $U \subseteq V$ finite. Take $F : U \times U \rightarrow \mathbb{R}$ copositive.

Add the constraint: $\sum_{x, y \in \text{u}} A(x, y) F(x, y) \geq 0$. 

Extension to $\mathbb{R}^n$

$$\text{PSD}(\mathbb{R}^n) = \{ f : f : \mathbb{R}^n \to \mathbb{R} \text{ is continuous,} \quad (f(x-y))_{x,y \in U} \in \text{PSD}(U) \quad \forall \text{ finite } U \subseteq \mathbb{R}^n \}$$

$f$ Schwartz $\in \text{PSD}(\mathbb{R}^n)$ iff $\hat{f} \geq 0$

**Theorem (Bochner).** $f \in \text{PSD}(\mathbb{R}^n)$ iff $f$ measure $\nu$ s.t.

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i u \cdot x} d\nu(u).$$

$$\text{COP}(\mathbb{R}^n) = \{ f : f : \mathbb{R}^n \to \mathbb{R} \text{ is continuous,} \quad (f(x-y))_{x,y \in U} \in \text{COP}(U) \quad \forall \text{ finite } U \subseteq \mathbb{R}^n \}$$

funcs. of copositive type
Theorem (de Laat, 0).

\[ \Delta_n = \inf f \quad f(0) + g(0) \]
\[ \hat{f}(0) = 1 \]
\[ f(x) + g(x) \leq 0 \quad \text{if } |x| \geq 1 \]
\[ f \in \text{PSD}(\mathbb{R}^n), \quad g \in \text{COP}(\mathbb{R}^n) \]
\[ f, g \text{ are admissible} \]

Admissible = continuous and...

- bounded support (\(\mathbb{R}^n\) is limit of tori)
- \(L^1\) (reduce to bounded support)
- Schwartz (harder proof)

Cohn-Ellies: take \(g = 0\).

Exercise: Prove \(\mu \leq \nu\); same as Cohn-Ellies.
A copositive $g$

$\alpha \geq \max$ # of non-overlapping balls at dist. in $[1, 1+\varepsilon]$ from central.

**Prop.** $g$ is of copositive type.

Leads to only known (?) improvement of Cohn-Elkies [de Laat, Oliveira, Vallentin]