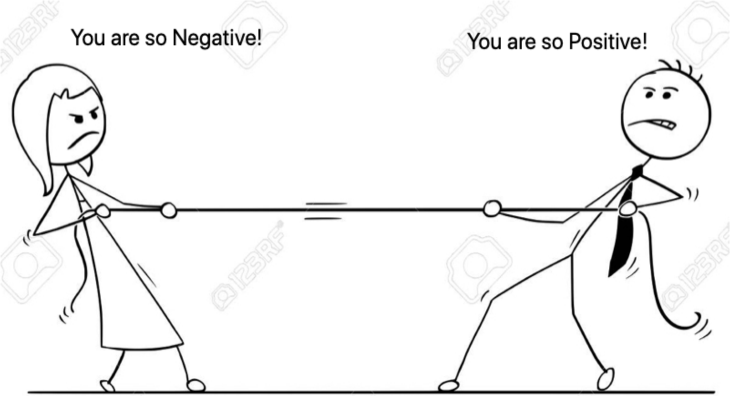
Sign Uncertainty Principle ≈ Tug-of-war



△ Sign uncertainty principle
□ LP bounds for sphere packing
◯ Promise: **Show why these problems are mysterious, hard and exciting.**

We say $f : \mathbb{R}^d \to \mathbb{R}$ is eventually nonnegative (E.NN.) if

$$f(x) \geq 0, \quad \text{for all sufficiently large } |x|.$$

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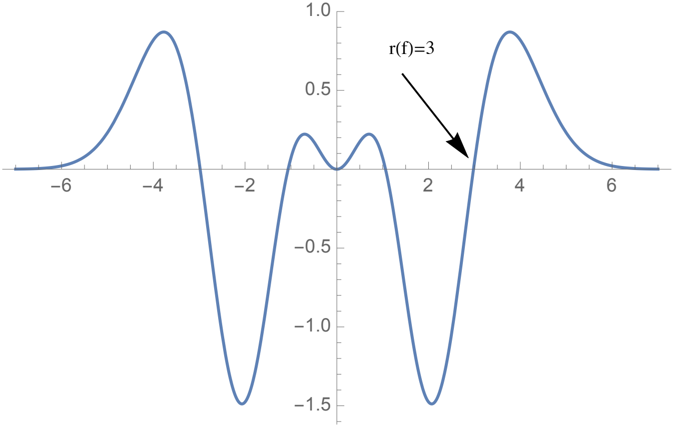

$$\mathcal{A}_+(d) = \begin{cases} f,\ \widehat{f} \in L^1(\mathbb{R}^d) \text{ both even and real-valued} \\ \widehat{f}(0) \le 0,\ f(0) \le 0 \\ f \text{ and } \widehat{f} \text{ E.NN.} \end{cases}$$

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Thm (Bourgain, Clozel, Kahane, 2010)

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Thm (Cohn, G. 2019)

$$\mathbb{A}_+(12) = \sqrt{2}$$

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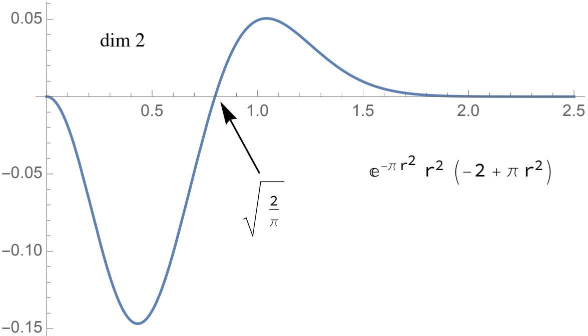

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$$\frac{1}{\sqrt{2\pi e}} \le \frac{\mathbb{A}_{-}(d)}{\sqrt{d}} \le (1+o(1))\left(\frac{1}{\sqrt{2\pi}} - 0.079\ldots\right).$$

$$\mathbb{A}_{-}(d) = \inf\{r(f) : f \text{ radial}, \widehat{f} = -f, f(0) = 0, f \text{ E.NN.}\}$$

$\mathbb{A}_{\pm}(d) \gtrsim \sqrt{d} \rightsquigarrow \pm 1$ **Uncertainty Principles**

Thm (Gonçalves, Oliveira e Silva, Steinerberger, 2016; Cohn, Gonçalves, 2018)

Existence of Optimal: $\exists f \in \mathcal{A}_\pm(d)$ such that

$$r(f) = \mathbb{A}_\pm(d),$$

we can assume f radial, $\widehat{f} = \pm f$ and $f(0) = 0$.

Multiple Roots: $f(|x|)$ has infinitely many double roots for $|x| > r(f)$.

Sphere Packing Problem

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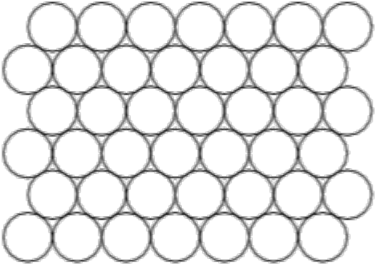
(Thue, 1910)

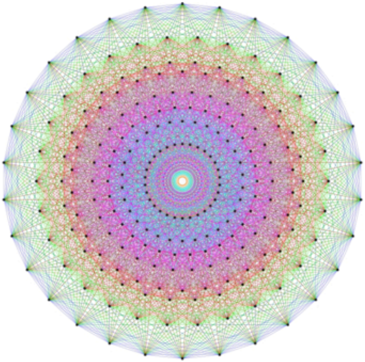


honeycomb ~ 91%

(Hales, 1998)

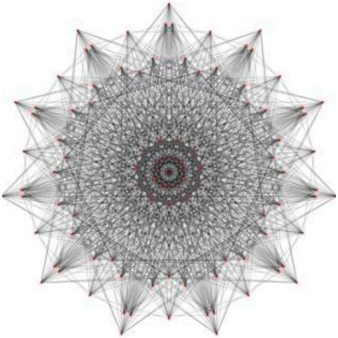
Cannonball Packing ~ 70%

(Viazovska, 2016)



E8 Lattice ~ 25%

(Viazovska, et al 2016)



Leech Lattice Λ_{24} ~ 0.2%

Thm (Cohn, Elkies, 2003) [Linear Programming Bounds]

Let

$$\mathcal{A}_{LP}(d) = \begin{cases} f,\ \widehat{f} \in L^1(\mathbb{R}^d) \text{ radial and real-valued} \\ f(0) = \widehat{f}(0) = 1 \\ f \text{ E.NN. and } \widehat{f} \geq 0. \end{cases}$$

and $\mathbb{A}_{LP}(d) = \inf_{f \in \mathcal{A}_{LP}(d)} r(f)$. Then

$$\Delta(d) \leq vol(\tfrac{1}{2}B^d)\mathbb{A}_{LP}(d)^d.$$

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Viazovska showed $\mathbb{A}_{LP}(8) = \sqrt{2}$ and $\mathbb{A}_{LP}(24) = \sqrt{4}$ with a construction using modular forms.

The Link between Sign Uncertainty and LP bounds

- The map $f \in \mathcal{A}_{LP}(d) \to \widehat{f} - f \in \mathcal{A}_{-}(d)$ shows $\mathbb{A}_{LP}(d) \geq \mathbb{A}_{-}(d)$.

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and presented numerical evidence for the conjecture
$$\mathbb{A}_{LP}(d) = \mathbb{A}_{-}(d) \sim \mathbb{A}_{+}(d) \sim c\sqrt{d}.$$

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- Recent numerical evidence in connection with modular bootstraps for CFTs indicates $c = \frac{1}{\pi} = .31...$

d	Best Packing	$\mathbb{A}_{LP}(d)$	$\mathbb{A}_{-}(d)$	$\mathbb{A}_{+}(d)$
1	$\mathbb{Z}$	1	1	??surprise
2	Honeycomb	$? = (4/3)^{\frac{1}{4}}$	$?= (4/3)^{\frac{1}{4}}$	?
8	$E8$	$\sqrt{2}$	$\sqrt{2}$	?
12	?	?	?	$\sqrt{2}$
24	Leech	$\sqrt{4}$	$\sqrt{4}$	?

New Sign Uncertainty Principles (G., Oliveira e Silva, Ramos – arXiv March 2020)

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For $f : \mathbb{Z}_Q \to \mathbb{C}$ we define the DFT

$$\widehat{f}(n) = \frac{1}{\sqrt{Q}} \sum_{m=-\frac{1}{2}(Q-1)}^{\frac{1}{2}(Q-1)} f(m) e^{-2\pi i\, m\, n/Q} \quad (Q \text{ Odd}).$$

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Thm (2020)

$$\mathcal{A}_{\pm}^{disc}[Q] = \begin{cases} f,\ \widehat{f} : \mathbb{Z}_Q \to \mathbb{R} \text{ both even and real-valued} \\ \widehat{f}(0) \leq 0,\ \pm f(0) \leq 0. \end{cases}$$

Then

$$\mathbb{A}_{\pm}^{disc}[Q] := \min \left\{ \sqrt{k(f)\, k(\pm\widehat{f})} \right\} \gtrsim \sqrt{Q},$$

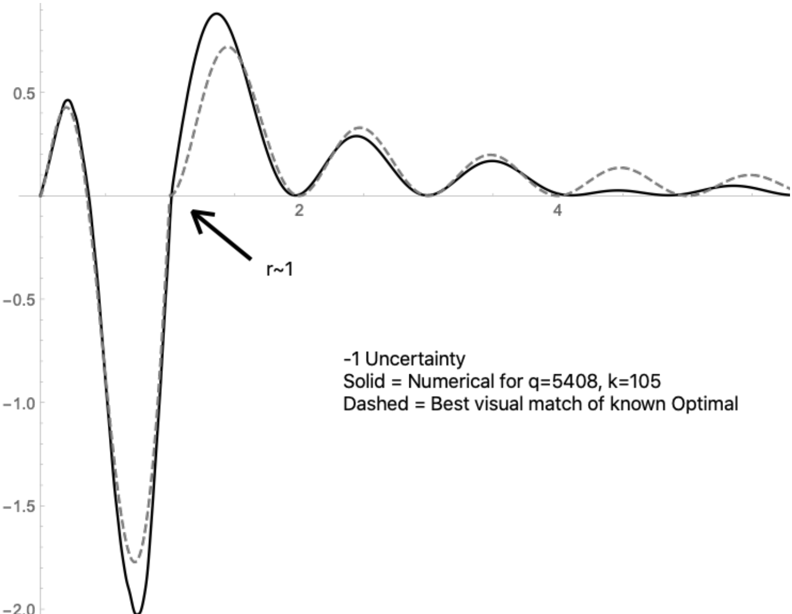
where $k(f) = \min\{k > 0 : f(n) \geq 0 \text{ if } k \leq n \leq Q\}$.

One expects/wishes that

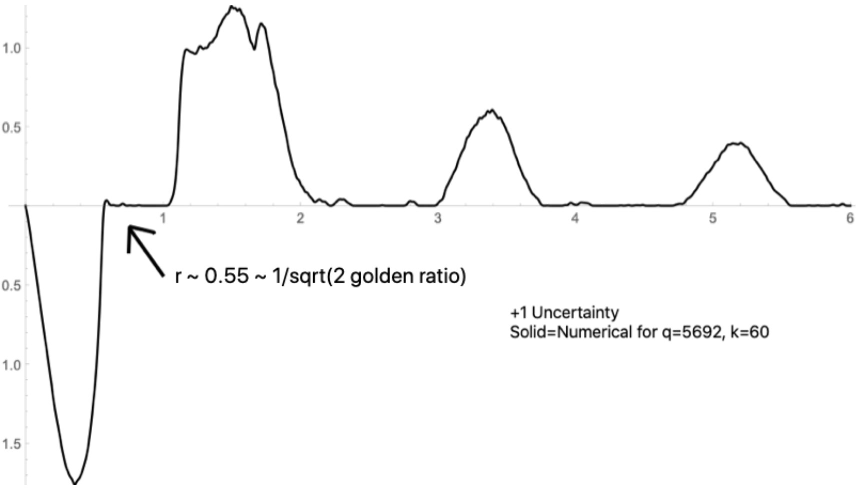
$$\text{DISCRETE} \longrightarrow \text{CONTINUOUS} \quad (Q \to \infty).$$

$$\frac{\mathbb{A}_{\pm}^{disc}[Q]}{\sqrt{Q}} \longrightarrow \mathbb{A}_{\pm}(1) \quad (Q \to \infty).$$

$$\frac{\mathbb{A}_{-}^{disc}[Q]}{\sqrt{Q}} \approx 1 = \mathbb{A}_{-}(1)$$



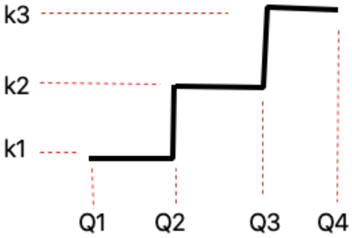
$$\frac{\mathbb{A}_+^{disc}[Q]}{\sqrt{Q}} \approx \frac{1}{\sqrt{2\,golden.ratio}}$$



The function

$$Q \mapsto k = \mathbb{A}_{\pm}^{disc}[Q]$$

is a stairway



So we can "invert" it $k \mapsto Q_{\pm}^{jump}(k)$

$$\lim_{Q \to \infty} \frac{\mathbb{A}_{\pm}^{disc}[Q]}{\sqrt{Q}} = \lim_{k \to \infty} \frac{k}{\sqrt{Q_{\pm}^{jump}(k)}}$$

if $Q_{\pm}^{jump}(k+1) \sim Q_{\pm}^{jump}(k)$.

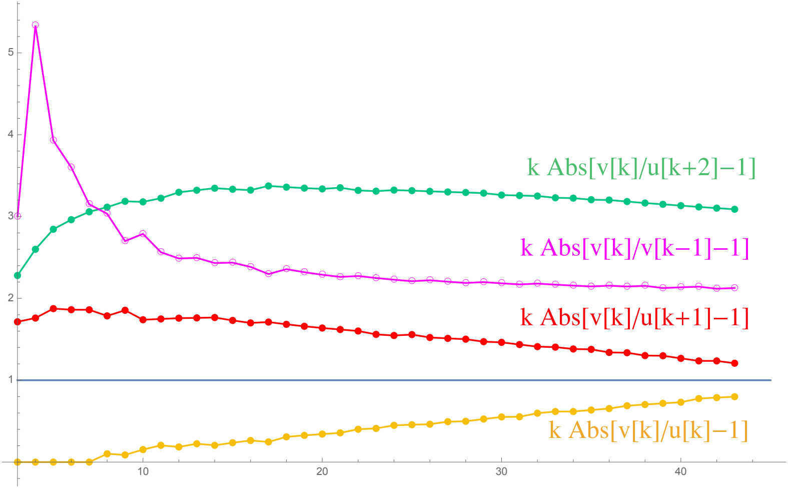
It turns out that numerically...

$$Q_-^{jump}(k) = k^2 - 2k + \tfrac{1}{2}((-1)^k + 5)$$
$$= 5, 8, 13, 18, 25, 32, \ldots$$

$$\tfrac{1}{2}(Q_+^{jump}(k) - 1) \approx \left\lfloor (k-1)^2 \times \text{golden.ratio} \right\rfloor$$
$$= 6, 14, 25, 40, 58, 79, \ldots$$

$$u[k] = \lfloor (k-1)^2 \times \text{golden.ratio} \rfloor$$

$$v[k] = \frac{1}{2}(Q_+^{jump}(k) - 1)$$



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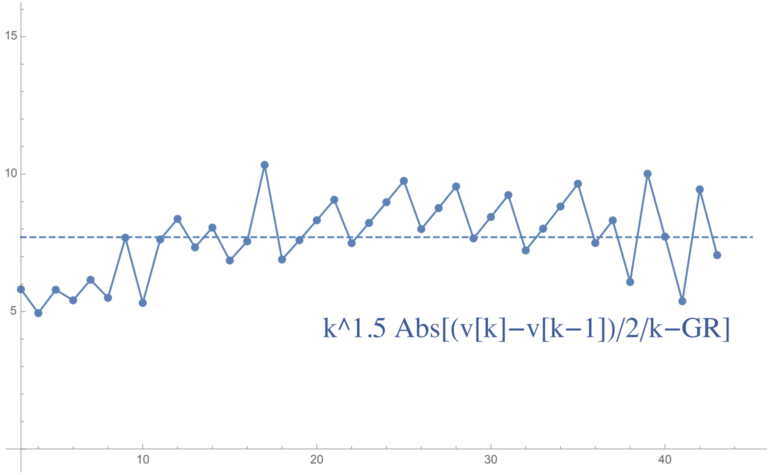

In higher dimensions we use a Disc. Hankel Transf.

$$H_d^{\text{disc}}(f)(m) = \frac{2}{j_{q+1}} \sum_{n=1}^{q} f(n) \frac{J_{d/2-1}\left(\frac{j_m j_n}{j_{q+1}}\right)}{J_{d/2}(j_n)^2} \quad [j_n = n^{th}\text{-zero of } J_{d/2-1}]$$

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$$\mathcal{A}_{\pm}^{disc}[d,q] = \begin{cases} f,\ H_d f : \{1,..,q\} \to \mathbb{R} \\ H_d f(0) \leq 0,\ \pm f(0) \leq 0. \end{cases}$$

and

$$\mathbb{A}_{\pm}^{disc}[d,q] := \min\{\sqrt{k(f)\, k(\pm H_d f)}\}.$$

Then $j_{\mathbb{A}_{\pm}^{disc}[d,q]} \gtrsim \sqrt{2\pi j_{q+1}}$

$$\frac{j_{\mathbb{A}_{\pm}^{disc}[d,q]}}{\sqrt{2\pi j_{q+1}}} \approx \mathbb{A}_{\pm}(d)$$

Since $q \mapsto k = \mathbb{A}_{\pm}^{disc}[d,q]$ is a **stairway** we can define

$$k \mapsto q_{\pm}^{jump}(d,k)$$

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$$q_-^{jump}(\mathbf{2}, k) \approx \left\lfloor \frac{\sqrt{3}(k^2-2k+2)}{4} \right\rfloor_{k\geq 4} = 4, 7, 11, 16, 21, 28, 35, 43, 52, 62, \ldots$$

$$q_-^{jump}(\mathbf{8}, k) \approx \left\lfloor \frac{k^2}{4} \right\rfloor_{k\geq 4} = 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, \ldots$$

$$q_-^{jump}(\mathbf{24}, k) \approx \left\lfloor \frac{k^2+6k-8}{8} \right\rfloor_{k\geq 4} = 4, 5, 8, 10, 13, 15, 19, 22, 26, 29, \ldots$$

$$q_+^{jump}(\mathbf{12}, k) \approx \left\lfloor \frac{k^2+2k-1}{4} \right\rfloor_{k\geq 3} = 3, 5, 8, 11, 15, 19, 24, 29, 35, 41, \ldots$$

numerically

$$\frac{j_{\mathbb{A}_-^{disc}[d,q]}}{\sqrt{2\pi j_{q+1}}} \to 1,\ (\tfrac{4}{3})^{\frac{1}{4}},\ \sqrt{2},\ \sqrt{4} \quad (d = 1, 2, 8, 24)$$

$$\frac{j_{\mathbb{A}_+^{disc}[d,q]}}{\sqrt{2\pi j_{q+1}}} \to \frac{1}{\sqrt{1+\sqrt{5}}},\ \sqrt{2} \quad (d = 1, 12)$$

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