Modular forms and their applications IV

Danylo Radchenko

ETH Zurich

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Eisenstein series of weight 2

Let us look once again at the Eisenstein series of weight 2.

$$G_2(\tau) = -\frac{1}{24} + \sum_{n\geq 1} \sigma(n) q^n$$

The fact that it is not modular may appear as an issue at first, but in fact it points to some interesting directions.

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - \frac{c(c\tau + d)}{4\pi i}$$

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6c(c\tau + d)}{\pi i}$$

Quasimodular forms

So far we did not say anything about what happens when one differentiates a modular form.

$$D := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$$

If f is a modular form of weight k, then differentiating the transformation law we get

$$Df\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-2} = Df(\tau)(c\tau+d)^{k} + f(\tau)\frac{kc(c\tau+d)^{k-1}}{2\pi i}$$

so f' can be modular only for $k = 0$. Nevertheless, the combination

$$\vartheta_k f := Df - \frac{k}{12}E_2 f$$

is a modular form of weight $k+2$. The operator $\vartheta = \vartheta_k$ is called the *Serre derivative*.

Quasimodular forms

Using this fact it is not hard to prove the following statement.

Theorem (Ramanujan)

We have

$$DE_2 = \frac{E_2^2 - E_4}{12}, \qquad DE_4 = \frac{E_2 E_4 - E_6}{3}, \qquad DE_6 = \frac{E_2 E_6 - E_4^2}{2}$$

Proof.

Exercise. □

This means that $\widetilde{M}_*(\Gamma_1) := \mathbb{C}[E_2, E_4, E_6]$ is closed under differentiation.

Let us call f a quasimodular form of weight k and depth $\leq p$ if it is a polynomial of degree p in E_2 with modular coefficients of homogeneous weight k.

Quasimodular forms

Is there an intrinsic description of $\widetilde{M}_*(\Gamma_1)$?

Definition

A quasimodular form of depth p and weight k for $\mathrm{PSL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that there exist functions $f_0, \ldots, f_p : \mathbb{H} \to \mathbb{C}$ (of polynomial growth at the boundary) such that

$$(f|_k\gamma)(\tau) = \sum_{j=0}^{p} f_j(\tau)\left(\frac{c}{2\pi i(c\tau + d)}\right)^j, \qquad \gamma \in \mathrm{PSL}_2(\mathbb{Z})$$

Quasimodular forms were introduced by Kaneko and Zagier, together with the related concept of *almost modular forms*: these are functions

$$f(\tau) = \sum_{j=0}^{p} f_j(\tau)(-4\pi y)^{-j}$$

with f_j holomorphic, such that $f|_k\gamma = f$ for all $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$.

Quasimodular forms

Quasimodular forms appear naturally as components of vector-valued modular forms for symmetric power representations.

Given a representation $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ one can consider vector valued modular forms $F\colon \mathbb{H} \to \mathbb{C}^n$ such that $F|_k\gamma = \rho(\gamma)F$.

Theorem (Kaneko, Nagatomo)

Let $f = (f_0, \dots, f_p)$ be a vector-valued modular form of weight k for the symmetric power representation of $\mathrm{SL}_2(\mathbb{Z})$. Then each

$$g_m(\tau) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \tau^{m-j} f_j(\tau)\,, \qquad 0 \le m \le p$$

is a quasimodular form of weight $k + p - 2m$ and depth $p - m$.

Symmetric power representation

$$\begin{pmatrix} (c\tau + d)^n \\ (a\tau + b)(c\tau + d)^{n-1} \\ \vdots \\ (a\tau + b)^{n-1}(c\tau + d) \\ (a\tau + b)^n \end{pmatrix} = \rho(\gamma) \begin{pmatrix} 1 \\ \tau \\ \vdots \\ \tau^{n-1} \\ \tau^n \end{pmatrix}$$

Modular integrals

Let us come back to the transformation law

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{c(c\tau+d)}{4\pi i}$$

once more. We can rewrite it in the form

$$G_2(\tau) - (c\tau+d)^{-2} G_2\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{1}{4\pi i}\frac{c}{c\tau+d}$$

or equivalently,

$$G_2 - G_2|_2\gamma = \varphi_\gamma\,,$$

where

$$\varphi_\gamma(\tau) = \frac{1}{4\pi i}\frac{c}{c\tau+d}\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$$

Modular integrals

Note that, since

$$G_2 - G_2|\gamma_1\gamma_2 = G_2 - G_2|\gamma_2 + G_2|\gamma_2 + G_2|\gamma_1\gamma_2\,,$$

we necessarily have

$$\varphi_{\gamma_1\gamma_2} = \varphi_{\gamma_2} + \varphi_{\gamma_1}|_k\gamma_2 \qquad \text{for all } \gamma_1, \gamma_2 \in \mathrm{PSL}_2(\mathbb{Z}) \tag{*}$$

We call a family of holomorphic functions φ_γ that satisfies (*) a 1-cocycle for $\mathrm{PSL}_2(\mathbb{Z})$. A holomorphic function $F\colon \mathbb{H} \to \mathbb{C}$ is called a modular integral of a 1-cocycle φ_γ if

$$F - F|_k\gamma = \varphi_\gamma\,, \qquad \gamma \in \mathrm{PSL}_2(\mathbb{Z})$$

Usually, we require both φ_γ and F to have at most polynomial growth at the boundary. Note that if $\varphi_\gamma = 0$ for all γ, then F is a modular form of weight k.

Eichler integrals

A nontrivial construction of modular integrals is given by Eichler integrals.
A key role in this construction is played by the following elementary identity.

Theorem (Bol's identity)

Let $F \colon \mathbb{H} \to \mathbb{C}$ be any smooth function. Then

$$D^{k-1}(F|_{2-k}\gamma) = (D^{k-1}F)|_k\gamma\,, \qquad \gamma \in \mathrm{PSL}_2(\mathbb{R})\,.$$

From Bol's identity it follows that if we take $(k-1)$-st primitive of a weight k modular form, then we obtain a modular integral of weight $2-k$ with a polynomial cocycle.

Eichler integrals: example

Let us look at the Eichler integral of E_4:

$$\widetilde{E_4}(\tau) = \frac{(2\pi i\tau)^3}{1440} + \sum_{n\geq 1} \sigma_{-3}(n) q^n$$

Then

$$\widetilde{E_4} - \widetilde{E_4}|_{-2}T = -\frac{(2\pi i)^3}{1440}(3\tau^2 + 3\tau + 1)$$

$$\widetilde{E_4} - \widetilde{E_4}|_{-2}S = \frac{\pi^3\tau}{36i} + \frac{\zeta(3)}{2}(\tau^2 - 1)$$

Note that the polynomials on the right must generate a 1-cocycle.

"Magic functions" for sphere packing

Cohn and Elkies have proved that if there exists a radial function $F\colon \mathbb{R}^8 \to \mathbb{R}$ such that

$$\begin{aligned} F(x) &\leq 0\,, \quad |x| \geq \sqrt{2}\,, \\ \widehat{F}(x) &\geq 0\,, \quad x \in \mathbb{R}^8\,, \\ F(0) &= \widehat{f}(0) = 1\,, \end{aligned}$$

then E_8 is the optimal sphere packing in 8 dimensions.
Assuming that $F(x) = f(|x|)$, f must also satisfy

$$\begin{aligned} f(\sqrt{2n}) &= 0\,, \quad n \geq 1\,, \\ f'(\sqrt{2n}) &= 0\,, \quad n \geq 2\,, \\ \widehat{f}(\sqrt{2n}) &= 0\,, \quad n \geq 1\,, \\ \widehat{f}'(\sqrt{2n}) &= 0\,, \quad n \geq 1\,. \end{aligned}$$

"Magic functions" for sphere packing

Viazovska has found a beautiful construction of such a function using Laplace transforms of weakly-holomorphic quasi-modular forms.

Theorem (Viazovska)

The E_8 lattice is an optimal sphere packing in $\mathbb{R}^8$.

Using the same strategy a "magical function" was then also found in dimension 24.

Theorem (Cohn-Kumar-Miller-R.-Viazovska)

The Leech lattice gives an optimal sphere packing in $\mathbb{R}^{24}$.

Fourier Interpolation

To prove universal optimality of the E_8 lattice using LP bounds, one needs to construct more general "magic functions" that now have prescribed values of $f(\sqrt{2n})$, $f'(\sqrt{2n})$ while $\widehat{f}(\sqrt{2n}) = \widehat{f'}(\sqrt{2n}) = 0$, for $n \geq 1$.

Theorem (CKMRV)

For $d \in \{8, 24\}$ there exist two sequences of radial Schwartz functions $a_n, b_n \in \mathcal{S}(\mathbb{R}^d)$, $n \geq 0$ such that for any radial Schwartz function f we have

$$f(x) = \sum_{n \geq n_0} a_n(x) f(\sqrt{2n}) + \sum_{n \geq n_0} b_n(x) f'(\sqrt{2n}) + \sum_{n \geq n_0} \widehat{a}_n(x) \widehat{f}(\sqrt{2n}) + \sum_{n \geq n_0} \widehat{b}_n(x) \widehat{f}'(\sqrt{2n})$$

Here $n_0 = 1$ for $d = 8$ and $n_0 = 2$ for $d = 24$.

Theorem (CKMRV)

The E_8 lattice and the Leech lattice are universally optimal.

Fourier Interpolation: reformulation

How do modular forms appear?

We want to verify

$$f(x) = \sum_{n \geq n_0} a_n(x) f(\sqrt{2n}) + \sum_{n \geq n_0} b_n(x) f'(\sqrt{2n}) + \sum_{n \geq n_0} \widehat{a}_n(x) \widehat{f}(\sqrt{2n}) + \sum_{n \geq n_0} \widehat{b}_n(x) \widehat{f}'(\sqrt{2n})$$

for all Schwartz functions. Let

$$f_\tau(x) = e^{i\pi\tau x^2}$$

so that

$$\widehat{f_\tau}(\xi) = \tau^{-d/2} f_{-1/\tau}(\xi)$$

Fourier Interpolation: reformulation

Applying the identity to $f_\tau(x)$ leads to

$$e^{i\pi\tau x^2} = F(\tau) + \tau^{-k} G(-1/\tau)$$

where

$$F(\tau) = F(\tau, x) = \sum_{n \geq n_0} a_n(x) e^{2\pi i n\tau} + (2\pi i\tau) \sum_{n \geq n_0} \sqrt{2n} b_n(x) e^{2\pi i n\tau},$$
$$G(\tau) = G(\tau, x) = \sum_{n \geq n_0} \widehat{a_n}(x) e^{2\pi i n\tau} + (2\pi i\tau) \sum_{n \geq n_0} \sqrt{2n} \widehat{b_n}(x) e^{2\pi i n\tau}.$$

Equivalently, F and G satisfy

$$F(\tau+2) - 2F(\tau+1) + F(\tau) = 0, \quad G(\tau+2) - 2G(\tau+1) + G(\tau) = 0$$

together with a growth condition at $i\infty$.

Reduction to modular integrals

$$\begin{cases} F(\tau+2) - 2F(\tau+1) + F(\tau) = 0\,, \\ G(\tau+2) - 2G(\tau+1) + G(\tau) = 0\,, \\ F(\tau) + \tau^{-k} G(-1/\tau) = \varphi(\tau) := e^{i\pi\tau x^2} \end{cases}$$

To turn this into an equation for a modular integral we consider the vector $\mathcal{F}\colon \mathbb{H} \to \mathbb{C}^6$

$$\mathcal{F} = (F,\ F|_k T,\ F|_k TS,\ G,\ G|_k T,\ G|_k TS)\,,$$

in terms of which the system of equations becomes

$$\begin{cases} \mathcal{F}(\tau) - A_T^{-1}\mathcal{F}(\tau+1) & = \ \psi_T(\tau)\,, \\ \mathcal{F}(\tau) - A_S^{-1}\tau^{-k}\mathcal{F}(-1/\tau) & = \ \psi_S(\tau)\,. \end{cases}$$

6D representation

$$A_T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{pmatrix}, \quad A_S = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\psi_T = \begin{pmatrix} 0 \\ 0 \\ 2\varphi|S - \varphi|T^{-1}S \\ 0 \\ 0 \\ 2\varphi - \varphi|TST \end{pmatrix}, \quad \psi_S = \begin{pmatrix} \varphi \\ 0 \\ 0 \\ \varphi|S \\ 0 \\ 0 \end{pmatrix}.$$

Modular integrals

How to find modular integrals? To make life easier let's look at the scalar version.

$$\begin{cases} F(\tau) - F(\tau+1) & = \psi_T(\tau)\,, \\ F(\tau) - \tau^{-k}F(-1/\tau) & = \psi_S(\tau)\,. \end{cases}$$

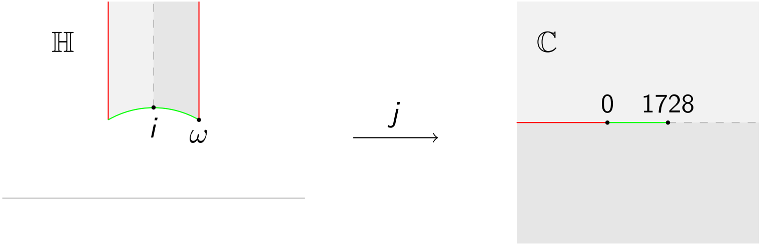
A solution can be written using modular Green's functions:

$$F(\tau) = \int_i^{\omega} K(\tau,z)\psi_S(z)dz + \int_{\omega}^{i\infty} K(\tau,z)\psi_T(z)dz\,, \qquad \tau \in \mathcal{D}$$

- $K(\tau,z)$ is modular of weight k in τ
- $K(\tau,z)$ is modular of weight $2-k$ in z
- $K(\tau,z)$ has simple poles only at $z \in \mathrm{PSL}_2(\mathbb{Z})\tau$ with residue $1/(2\pi i)$ at $z=\tau$
- "good behavior at the cusps"

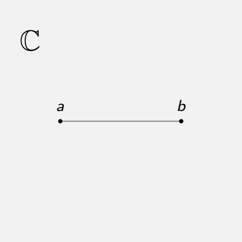
Modular integrals as a boundary value problem

For $k = 0$ we have $K(\tau, z) = \frac{1}{2\pi i}\frac{j'(z)}{j(z)-j(\tau)} = \frac{E_{14}(z)/\Delta(z)}{j(\tau)-j(z)}$



- Enough to satisfy the equations for F on the closure of the fundamental domain.
- Change of variable $w = j(\tau)$ gives $\widetilde{F}\colon \mathbb{C} \smallsetminus (-\infty, 1728] \to \mathbb{C}$ with prescribed jumps along $(-\infty, 0)$ and $(0, 1728)$.
- After the change of variables $K(\tau, z)$ becomes the Cauchy kernel.
- The reason why the jump conditions are satisfied is the Sokhotski-Plemelj formula

The Sokhotski-Plemelj formula



Suppose we want to construct a holomorphic function $f\colon \mathbb{C} \setminus [a, b] \to \mathbb{C}$ such that

- $f(z) = O(1/z)$ at $z \to \infty$;
- $f(t + i\varepsilon) - f(t - i\varepsilon) \to w(t)$ as $\varepsilon \to 0+$, $t \in (a, b)$, where $w\colon [a, b] \to \mathbb{C}$ is given.

Then the Sokhotsky-Plemelj formula says that

$$f(z) = \frac{1}{2\pi i} \int_a^b \frac{w(t)}{t - z} dt$$

has the required properties.

Vector-valued case

Going back to the vector-valued case, we need to know the space of modular forms for the $6D$ representation given by A_T and A_S.

That representation decomposes into two $3D$-subrepresentations. One of them is the symmetric square, and thus involves quasimodular forms of depth 2.

The other is a bit harder to describe, but one can show that vector-valued modular forms for it involve Eichler integrals of weight 2 Eisenstein series for $\Gamma(2)$. These are

$$\log \lambda(\tau)\,, \qquad\qquad \log(1-\lambda(\tau))$$

where $\lambda(\tau) = 1 - \frac{\theta^4((\tau+1)/2)}{\theta^4(\tau/2)}$ is the modular lambda invariant.

What about 2-dimensional case?

Why does this not work for the hexagonal lattice?

Instead of $\{\sqrt{2n}\}_{n\geq 1}$ one needs to interpolate from $\{(4/3)^{1/4}\sqrt{a^2+ab+b^2}\}_{a,b\in\mathbb{Z}}$.
The set S of integers represented by a^2+ab+b^2

$$0, 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, 39, \ldots$$

is rather sparse, since Paul Bernays has shown that

$$|S \cap [0,x]| \sim c\frac{x}{\sqrt{\log x}}$$

which is much smaller than $x/2$ corresponding to $\{\sqrt{2n}\}_{n\geq 1}$.