Packing, coding, and ground states III

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LP bound for code size in $\{0,1\}^n$ with min. dist. $\geq d$:

Maximize

$$A_0 + \cdots + A_n$$

subject to

$$A_0 = 1$$

$$A_i \geq 0 \quad \text{for } 1 \leq i \leq n$$

$$A_1 = A_2 = \cdots = A_{d-1} = 0$$

$$\sum_{i=0}^{n} K_j(i) A_i \geq 0 \quad \text{for } 0 \leq j \leq n$$

(Note: $K_0(i) = 1$.)

Let's take the dual LP.

How can we get an upper bound on $A_0 + \cdots + A_n$ using linear combinations of the Delsarte inequalities?

$$\sum_{j=0}^{n} c_j \sum_{i=0}^{n} K_j(i) A_i \geq c_0 \sum_{i=0}^{n} A_i.$$

if $c_j \geq 0$ for all $j > 0$.

This amounts to

$$\sum_{i=0}^{n} \left(\sum_{j=0}^{n} c_j K_j(i) \right) A_i \geq c_0 \sum_{i=0}^{n} A_i.$$

Let

$$f(i) = \sum_{j=0}^{n} c_j \, K_j(i).$$

(Krawtchouk expansion of polynomial)

Then

$$\sum_{i=0}^{n} A_i f(i) \geq c_0 \sum_{i=0}^{n} A_i .$$

as long as coeffs $c_j \geq 0$ for $j > 0$.

Let's try to bound $\sum_{i=0}^{n} A_i f(i)$ from above.

<u>Goal</u>: bound

$$\sum_{i=0}^{n} A_i f(i)$$

from above.

$i=0$: get $f(0)$ since $A_0 = 1$.

$i = 1, 2, \ldots, d-1$: irrelevant since $A_i = 0$.

$d \leq i \leq n$: get upper bound if $f(i) \leq 0$.

This is the dual LP: don't think too hard, but rather use only the simplest bounds.

Thm. Suppose $f(i) = \sum_{j=0}^{n} c_j K_j(i)$

with $c_j \geq 0$ for all j (and $c_0 > 0$) and

$f(i) \leq 0$ for $d \leq i \leq n$. Then

every code C in $\{0,1\}^n$ with

min. dist. $\geq d$ satisfies

$$|C| \leq \frac{f(0)}{c_0}.$$

Strong LP duality says this gives the true LP bound if f is optimized.

f = certificate for bound

How should we choose f to optimize this bound?

Unclear in general. Humanity has just one good idea, based on the Christoffel-Darboux formula, and it's not optimal in general.

When can the LP bound be sharp?

To get a sharp bound, need to have no loss in inequalities:

$$f(i) = 0 \quad \text{whenever } A_i > 0 \text{ and } i > 0.$$

$$c_j = 0 \quad \text{whenever } \sum_{i=0}^{n} A_i K_j(i) > 0 \text{ and } j > 0.$$

"Complementary slackness."

Let's look at the extended binary Golay code:

block length $n = 24$

min. dist. $d = 8$

$2^{12} = 4096$ codewords

$A_0 = 1$, $A_8 = 759$, $A_{12} = 2576$,

$A_{16} = 759$, $A_{24} = 1$

(others vanish)

Need $f(i) \leq 0$ for $8 \leq i \leq 24$,

$f(8) = f(12) = f(16) = f(24) = 0$.

Note root mult's.

<u>Guess:</u>

$$f(i) = (8-i)(i-12)^2(i-16)^2(i-24)$$

This works! (Not obvious.)

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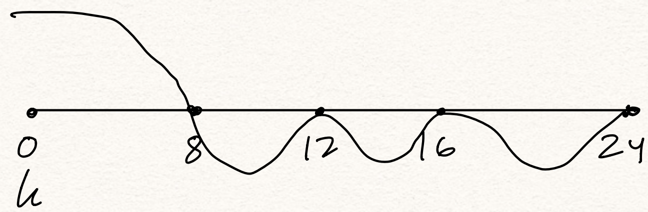


Need to chec:

$$\text{coeffs } c_j \geq 0$$

$$\frac{f(0)}{c_0} = 4096$$

Both are true.

Levenshtein gave a conceptual explanation, but one can prove it by direct calculation.

Miraculous aspects

Get an *exact* bound,

using the simplest polynomial

that could possibly work.

(This is not the only way, or
even the simplest way, to
prove the Golay code is
optimal, but it applies
more broadly.)

Given that

Hamming, Golay codes

are analogous to

E_8, Leech lattices,

is there a Euclidean version of this proof?

Ideally solve the sphere packing problem exactly in $\mathbb{R}^8$, $\mathbb{R}^{24}$ using a very simple certificate.

This part turned out to be too optimistic.

How can we build a Euclidean analogue?

sphere packing $\longleftrightarrow$ code

density $\longleftrightarrow$ code size

? $\longleftrightarrow$ Krawtchouk transform

This turns out not to be so hard.
Answer: radial Fourier transform.

$$K_j(i) = \sum_{w(y)=j} (-1)^{\langle x,y\rangle} \quad \text{with } w(x)=i$$

$$\sum_{j=0}^{n} c_j K_j(i) = \underbrace{\sum_{y\in\{0,1\}^n} c_{w(y)}\, \overbrace{(-1)^{\langle x,y\rangle}}^{\text{characters of } (\mathbb{Z}/2\mathbb{Z})^n}}_{\text{discrete Fourier transform of radial function (i.e., one that depends only on } w(y)\text{))}}$$

Analogue:

$$\underbrace{\int_{\mathbb{R}^n} c(y)\, e^{\overbrace{2\pi i\langle x,y\rangle}^{\text{unitary characters of } \mathbb{R}^n}}\, dy}_{\text{continuous Fourier transform}}$$

Euclidean LP bound

want $c : \mathbb{R}^n \to \mathbb{R}$ nice fn.

radial ($c(y)$ depends only on $|y|$)

$$\boxed{c(y) \geq 0 \quad \forall y}$$

$$f(x) = \int_{\mathbb{R}^n} c(y)\, e^{2\pi i \langle x, y \rangle}\, dy$$

(i.e., $\boxed{c = \hat{f}}$ by Fourier inversion)

$$\boxed{f(x) \leq 0 \quad \text{for } |x| \geq r}$$

Then sphere packing density $\leq$

$$\operatorname{vol}\left(B^n_{r/2}\right) \cdot \frac{f(0)}{\hat{f}(0)}.$$

Thm. (Cohn and Elkies)

WLOG

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a radial Schwartz function and $r > 0$, and suppose

can weaken this hypothesis

$$f(x) \leq 0 \quad \text{for } |x| \geq r,$$

$$\hat{f}(y) \geq 0 \quad \text{for all } y,$$

and $$\hat{f}(0) > 0.$$

Then the sphere packing density is at most

$$\mathrm{vol}(B^n_{r/2}) \cdot \frac{f(0)}{\hat{f}(0)}.$$

We didn't arrive at it via this analogy, but one could have.

Q: How can we prove this?

The double sums

$$\sum_{x,y \in \mathcal{C}} f(d_H(x,y))$$

from the discrete proof diverge.

One can renormalize them (Cohn and Zhao), but Elkies and I used Poisson summation.

Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice
(discrete subgroup of rank n).

$f : \mathbb{R}^n \to \mathbb{R}$ Schwartz function

$\Lambda^* =$ dual lattice

$$\left(\begin{array}{ll} \Lambda \text{ basis } b_1, \dots, b_n & \\ \Lambda^* \text{ basis } b_1^*, \dots, b_n^* & \langle b_i, b_j^* \rangle = \delta_{ij} \end{array} \right)$$

Poisson summation

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y)$$

Poisson summation is most easily proved by generalizing it.

Let
$$F(z) = \sum_{x \in \Lambda} f(x+z)$$
(make f periodic modulo Λ).

<u>Exercise</u> Compute the Fourier expansion of F as

$$F(z) = \frac{1}{\mathrm{Vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y) e^{2\pi i \langle y, z \rangle}$$

and set $z=0$ to obtain Poisson summation.

Let's prove the LP bound for lattice packings. (General proof similar, but a little more algebra.)

$\Lambda \subseteq \mathbb{R}^n$ lattice

$$\min_{x \in \Lambda \setminus \{0\}} |x| = r$$

$$\text{density} = \mathrm{vol}(B^n_{r/2}) \cdot \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$$

radius $r/2$ spheres fit w/o overlap

one sphere in each fundamental cell

(scale invariance $\Longrightarrow$ choice of r is not significant)

$f : \mathbb{R}^n \to \mathbb{R}$ Schwartz fn.

$f(x) \le 0$ for $|x| \ge r$

$\hat{f}(y) \ge 0$ for all y

Poisson summation:

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y)$$

$$\le \qquad\qquad\qquad\qquad \ge$$

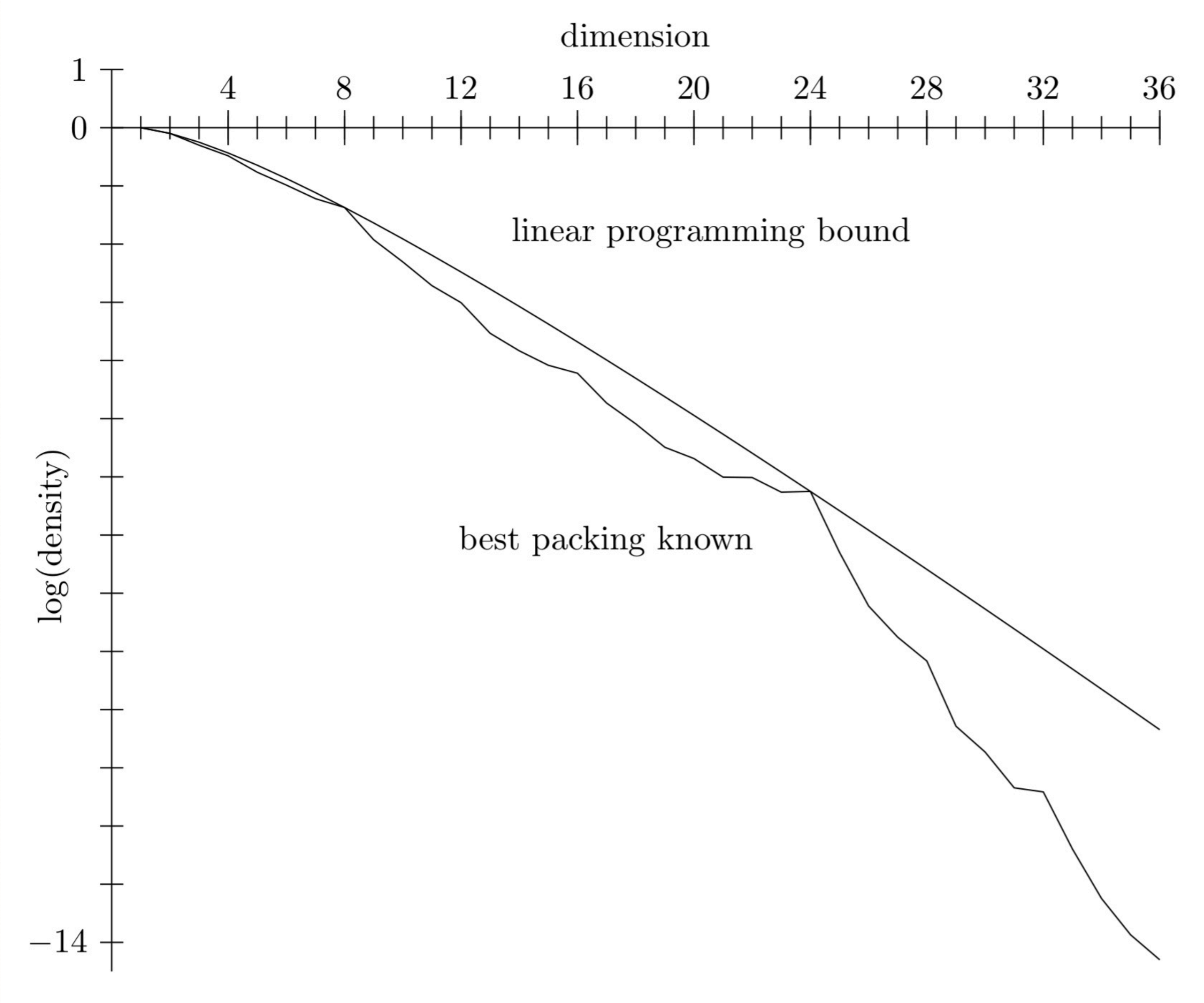
$$f(0) \qquad\qquad\qquad \frac{\hat{f}(0)}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$$

$\Longrightarrow$

$$\frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \le \frac{f(0)}{\hat{f}(0)}$$

QED.

Numerical results



Looks sharp in 1, 2, 8, 24 dimensions.

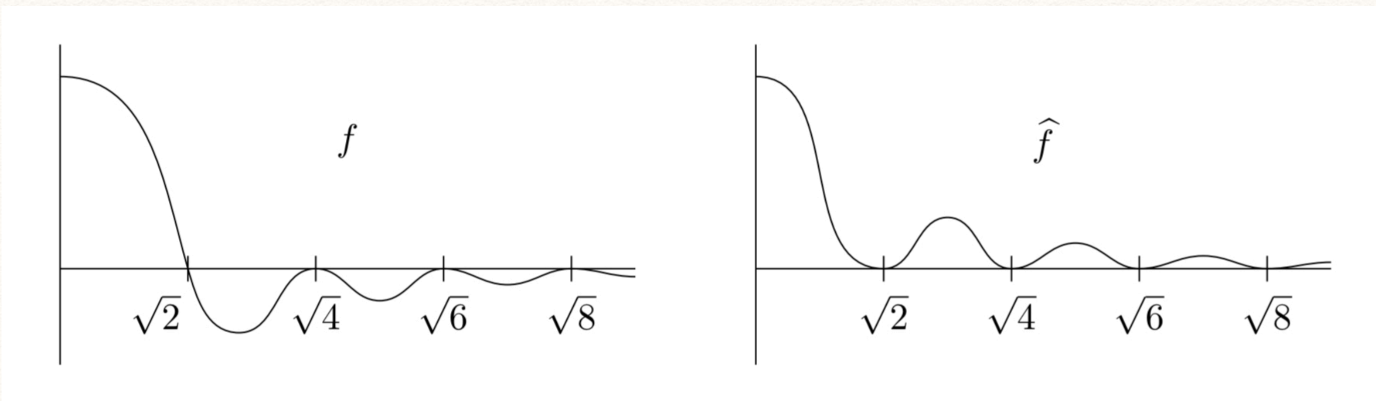
not hard

still unsolved!

To get a sharp bound, must have

$$f(x)=0 \text{ for } x\in\Lambda\setminus\{0\},$$
$$\hat{f}(y)=0 \text{ for } y\in\Lambda^*\setminus\{0\}.$$

Vector lengths in E_8, Λ_{24} are simple: $\sqrt{2k}$, with $k\neq 1$ in Λ_{24}.



But how can we guess the right function?

It's hard to control both f and $\hat{f}$ at once (uncertainty principles).

<u>Viazovska</u>: modular forms, E_8 case

(extended to Λ_{24} w/ Cohn, Kumar, Miller, Radchenko)

Interpolation theorem

(Cohn, Kumar, Miller, Radchenko, Viazovska)

$$n \in \{8, 24\}, \quad k_0 = \begin{cases} 1 & n = 8 \\ 2 & n = 24 \end{cases}$$

A radial Schwartz function

$$f : \mathbb{R}^n \to \mathbb{R}$$

is uniquely det'd by

$$f(\sqrt{2k}),\ f'(\sqrt{2k}),$$
$$\hat{f}(\sqrt{2k}),\ \hat{f}'(\sqrt{2k})$$

radial deriv.

for $k \geq k_0$.

We also prove universal optimality for E_8, Λ_{24}.

In $\mathbb{R}^2$, best sphere packing known (not by LP bounds), but universal optimality is not known.

Why can't we settle $\mathbb{R}^2$?

Questions

- What about $\mathbb{R}^2$?
- Are there universally optimal Euclidean quasicodes in every dimension?
- Do they correspond to interpolation formulas?

  ($\mathbb{R}^2$ fails?)
- Discrete analogues for $\{0,1\}^n$?
- $\mathbb{R}^n$ as $n \to \infty$?

Exercises

- Prove Poisson summation.
- Find an optimal f in $\mathbb{R}^1$.
  (I don't know how to get a Schwartz function.)

Further reading

H. Cohn, A conceptual breakthrough in sphere packing

H. Cohn, A. Kumar, S.D. Miller, D. Radchenko, and M. Viazovska, Universal optimality of the E_8 and Leech lattices and interpolation formulas

Particularly the intro. and open questions.