Packing, coding, and ground states

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Today: introductory examples

Later this week:

- continuous spaces
- discrete spaces
- physics (CFT)

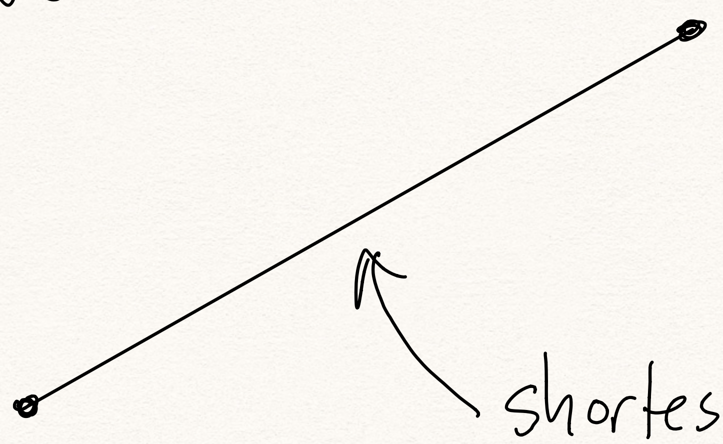
We'll focus on directions for future research.

Lectures will be mostly independent.
Nothing from today will be needed.

Goal: optimal geometry

Optimize some objective function within a moduli space of geometric objects.

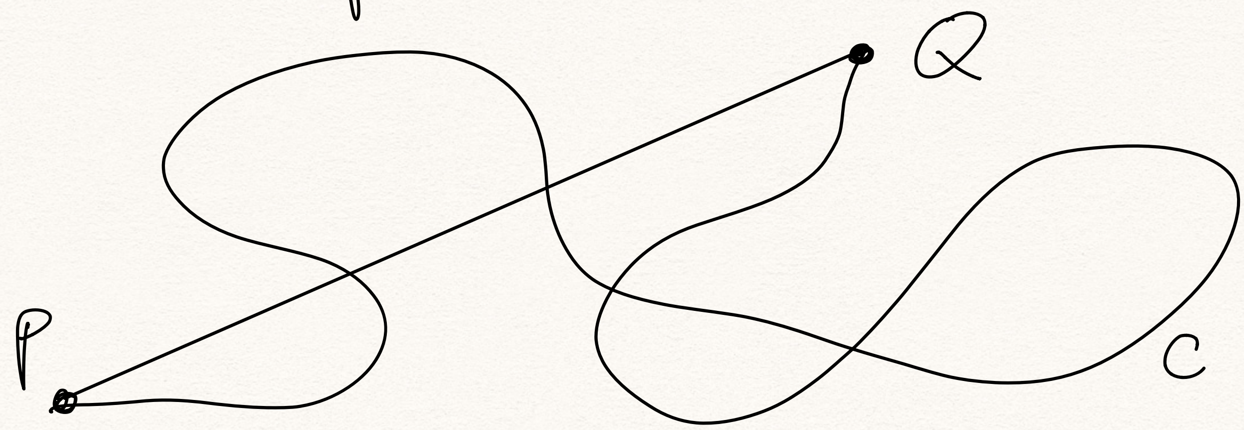
E.g., minimize length over curves.



shortest distance

(easy: Δ ineq.)

Alternate proof:



Let $u = (u_1, u_2)$ be unit vector in $P \to Q$ direction. Consider the $\boxed{\text{differential form } \omega = u_1\,dx + u_2\,dy}$, and let C be a piecewise smooth curve from P to Q.

① $\int_C \omega$ is independent of C since ω is closed.

② $\int_C \omega \leq \text{length}(C)$, w/ equality for a straight line

(since $\langle u, v\rangle \leq 1$ for all unit vectors v, w/ equality iff $u = v$).

This proof was overkill, but the idea is much more general.

Wirtinger's Theorem: Complex submanifolds of a Kähler manifold are homologically volume-minimizing.

E.g., $\mathbb{C}^n$ or $\mathbb{CP}^n$.

I.e., they minimize volume within homology classes, including specifying boundaries of patches.

Proof: let $\omega =$ Kähler form.

On $\mathbb{C}^n$ w/ coordinates $(z_1, \dots, z_n)$

and $z_j = x_j + y_j i$,

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j .$$

Claim:

$$\left| \frac{\omega^k}{k!} \right| \leq \begin{array}{c} 2k\text{-dim'l} \\ \text{volume,} \end{array}$$

with equality exactly for complex subspaces.

Then for any real $2k$-manifold M,

$\int_M \frac{\omega^k}{k!}$ is independent of M within its homology class,

and

$$\int_M \frac{\omega^k}{k!} \leq \mathrm{vol}_{2k}(M),$$

with equality when M is a complex manifold.

Only remaining question: why does Wirtinger's inequality hold?

Proof for $k=1$, $n=2$:

In coordinates (x_1, y_1, x_2, y_2),

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 .$$

Consider applying ω to the 2-dim'l subspace spanned by orthogonal unit tangent vectors $(a_1, a_2, a_3, a_4) = a$ and $(b_1, b_2, b_3, b_4) = b$.

Want $|\omega(a,b)| \leq 1$ with equality iff a, b span complex subspace.

Key identity:

Hermitian form

$$(a_1 - a_2 i)(b_1 + b_2 i) + (a_3 - a_4 i)(b_3 + b_4 i)$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

Riemannian metric

$$+ (a_1 b_2 - a_2 b_1 + a_3 b_4 - a_4 b_3) i$$

$\omega(a, b)$

We assumed a, b are orthogonal unit vectors, so Hermitian Cauchy-Schwarz implies

$$|\omega(a,b)| \leq 1,$$

with equality iff $(a_1 + a_2 i, a_3 + a_4 i)$ and $(b_1 + b_2 i, b_3 + b_4 i)$ are proportional over $\mathbb{C}$ (i.e., a and b span a complex subspace).

Linear algebra exercise:

Prove Wirtinger's inequality for all k.

What we've seen is a common pattern. We get a lower bound via some auxiliary construction like $\omega^k/k!$.. In especially nice cases, the bound is sharp, but we can't expect that to happen always.

We'll focus mainly on configurations of discrete points.

Not because they're the most interesting or important, but because they are already really hard.

Sometimes annotate w/ add'l info. (weight, tangent vector, frame, etc.).

Why point configurations?

- packing
- covering
- error-correcting codes
- discretization
- numerical integration
- sampling
- states of particle systems
- etc.

Different spaces:

- $\mathbb{R}^n$
- S^n
- $\mathbb{H}^n$
- projective space
- Grassmannian
- $\{0,1\}^n$
- $\{x \in \{0,1\}^n : \sum x_i = w\}$

and many others...

Let (X,d) be compact metric space.

$G =$ isometry group

Suppose G acts transitively,

$e \in X$ base point, $H = \mathrm{Stab}_G(e)$.

Then $X \cong G/H$.

E.g., $X = S^{n-1}$

$G = O(n)$

$H \cong O(n-1)$

We'll make use of representations of G.

A code in X is a finite subset of X, with minimum distance $r = \min_{\substack{x,y \in X \\ x \neq y}} d(x,y)$.

Goals:

- maximize r given $|X|$.
- maximize $|X|$ given $r \geq r_0$.

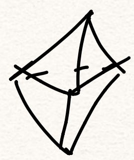
(not quite equivalent, but closely related)

How to distribute N points on S^2.

$N =$ 2 antipodal

3 Δ on equator

4 reg. tetrahedron

5 

6 regular octahedron

8 <u>not</u> cube, but square antiprism

12 regular icosahedron

What happens in general?

Thm. For $N \leq n+1$, best N-point code in S^{n-1} is regular simplex w/ origin as centroid.

($\nexists$ regular simplex in $\mathbb{R}^n$ for $N > n+1$.)

Proof: $x_1, \ldots, x_N \in S^{n-1} = \{x \in \mathbb{R}^n : |x|^2 = 1\}$.

Key inequality: $|x_1 + \cdots + x_N|^2 \geq 0$

$$\Longrightarrow \frac{1}{\binom{N}{2}} \sum_{i<j} \langle x_i, x_j \rangle \geq -\frac{1}{N-1}.$$

$$\max_{i \neq j} \langle x_i, x_j \rangle \geq -\frac{1}{N-1}$$

$=$ iff regular simplex, $\sum x_i = 0$.

Q.E.D.

Even for $N > n+1$, still get

$$\max_{i \neq j} \langle x_i, x_j \rangle \geq -\frac{1}{N-1},$$

but this bound is no longer sharp.

What about $N = 2n$?

Best code: regular cross polytope ($\pm$ orthonormal basis vectors).

But how can we prove it?

More sophisticated harmonic analysis on S^{n-1}.

We'll show that every code in S^{n-1} w/ minimal angle $\geq \pi/2$ has at most $2n$ points.

Key technique: embedding into representations of $O(n)$.

$$S^{n-1} = O(n)/O(n-1)$$

Suppose V is representation of $O(n)$, with $v \in V \setminus \{0\}$ fixed by $O(n-1)$.

Get embedding

$$S^{n-1} \xrightarrow{\varphi} V$$

$$ge \longmapsto gv$$

$$g \in O(n)$$

$$O(n-1) = \mathrm{Stab}(e)$$

$$S^{n-1} \xrightarrow{\varphi} V$$
$$ge \longmapsto gv$$

Let $\langle \cdot, \cdot \rangle$ be an $O(n)$-inv't inner product on V.

S^{n-1} is 2-point homogeneous: orbits of pairs def'd by distance.

$\Longrightarrow$

$\langle \varphi(x), \varphi(y) \rangle =$ some function of $\langle x, y \rangle$

Let's do an example.

std. representation $\mathbb{R}^n$ of $O(n)$

$$\mathrm{Sym}^2(\mathbb{R}^n) \cong \{ M \in \mathbb{R}^{n \times n} : M^t = M \}$$

$A \in O(n)$ acts via $A \cdot M = AMA^t$.

$$\mathbb{R}^n \longrightarrow \mathrm{Sym}^2(\mathbb{R}^n)$$
$$x \longmapsto xx^t$$

preserves this action.

Inner product on $\mathrm{Sym}^2(\mathbb{R}^n)$:

$$\langle M_1, M_2 \rangle = tr(M_1 M_2).$$

This is $O(n)$-inv't:

$$\langle A \cdot M_1, A \cdot M_2 \rangle = \langle M_1, M_2 \rangle.$$

Define

$$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n)$$

$$x \longmapsto (x, xx^t)$$

Then

$$\begin{aligned}\langle \varphi(x), \varphi(y) \rangle &= \langle x, y \rangle + \langle xx^t, yy^t \rangle \\ &= \langle x, y \rangle + \mathrm{tr}(xx^t yy^t) \\ &= \langle x, y \rangle + \mathrm{tr}(x^t y\, y^t x) \\ &= \langle x, y \rangle + \langle x, y \rangle^2 \\ &= \langle x, y \rangle (\langle x, y \rangle + 1).\end{aligned}$$

so for $x, y \in S^{n-1}$,

$$\langle \varphi(x), \varphi(y) \rangle \leq 0$$

$$\Longleftrightarrow$$

$$\langle x, y \rangle \leq 0.$$

Suppose $\mathcal{C} \subseteq S^{n-1}$ with $\langle x, y \rangle \leq 0$ for all $x, y \in \mathcal{C}$ with $x \neq y$. How big can $|\mathcal{C}|$ be?

<u>Thm.</u> $|\mathcal{C}| \leq 2n$

<u>Proof:</u> Look at

$\langle x, y \rangle (\langle x, y \rangle + 1)$

$$S = \sum_{x, y \in \mathcal{C}} \langle \varphi(x), \varphi(y) \rangle.$$

We have

$$S \leq \sum_{x \in \mathcal{C}} \langle \varphi(x), \varphi(x) \rangle$$
$$= 2|\mathcal{C}|.$$

How small can S be?

$$S = \underbrace{\sum_{x,y \in \mathcal{C}} \langle x,y\rangle}_{\geq 0} + \underbrace{\sum_{x,y \in \mathcal{C}} \langle x,y\rangle^2}_{\text{better lower bound}}$$

Why better? $\mathrm{Sym}^2(\mathbb{R}^n)$ contains a trivial subrepresentation.

$$A \cdot M = M \iff AMA^t = M$$
$$\iff AM = MA$$

This holds for all $A \in O(n)$ iff $M = \lambda I_n$ w/ $\lambda \in \mathbb{R}$.

$$\mathrm{Sym}^2(\mathbb{R}^n) = V_0 \oplus V_0^\perp$$
$$V_0 = \{\lambda I_n : \lambda \in \mathbb{R}\}$$

For $x \in S^{n-1}$,

$$\langle xx^t, I_n \rangle = tr(xx^t)$$
$$= tr(x^t x)$$
$$= 1$$

so $xx^t = \frac{1}{n} I_n + \overset{\text{in } V_0^\perp}{v_x}$

and $\langle xx^t, yy^t \rangle = \frac{1}{n} + \langle v_x, v_y \rangle$.

Thus,

$$\sum_{x,y \in \mathcal{C}} \langle x,y \rangle^2 = \sum_{x,y \in \mathcal{C}} \langle xx^t, yy^t \rangle$$
$$= \frac{|\mathcal{C}|^2}{n} + \sum_{x,y \in \mathcal{C}} \langle v_x, v_y \rangle$$
$$\geqslant \frac{|\mathcal{C}|^2}{n}.$$

We have found that

$$\frac{|\mathcal{C}|^2}{n} \leq \sum_{x,y \in \mathcal{C}} \langle \varphi(x), \varphi(y) \rangle \leq 2|\mathcal{C}|$$

so

$$|\mathcal{C}| \leq 2n.$$

Q.E.D.

<u>Exercise</u>: check that the regular cross polytope is the only case with $|\mathcal{C}| = 2n$.

General methodology:

Embed
$$S^{n-1} \xrightarrow{\varphi} V$$
$$V = O(n)\text{-rep.}$$

Choose φ so that
$$\langle \varphi(x), \varphi(y) \rangle \leq 0$$
for $x \neq y$
(due to min. dist.).

Then

ind. of x

$$\sum_{x,y \in \mathcal{C}} \langle \varphi(x), \varphi(y) \rangle \leq |\mathcal{C}| \cdot |\varphi(x)|^2$$

Use trivial representations
to bound below.

Key Questions

What sort of bounds do we get?

Which problems do they apply to?

When are they sharp?

How can we choose V optimally?

Suggested exercises

- Prove Wirtinger's inequality for all k. (linear algebra)

- Show that the cross polytope is the only $2n$-point code in S^{n-1} w/ minimal angle $\geq \pi/2$. (understanding today's proof)

- The <u>hemicube</u> in S^4 consists of half the vertices of a cube (alternating). Prove that it is an optimal code. (serious generalization, so this will take considerable effort)