# Optimization for lattices, packings, and coverings

Lecture 3

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G.F. Voronoi (1868–1908)

# 2. Voronoi’s lattice reduction theory

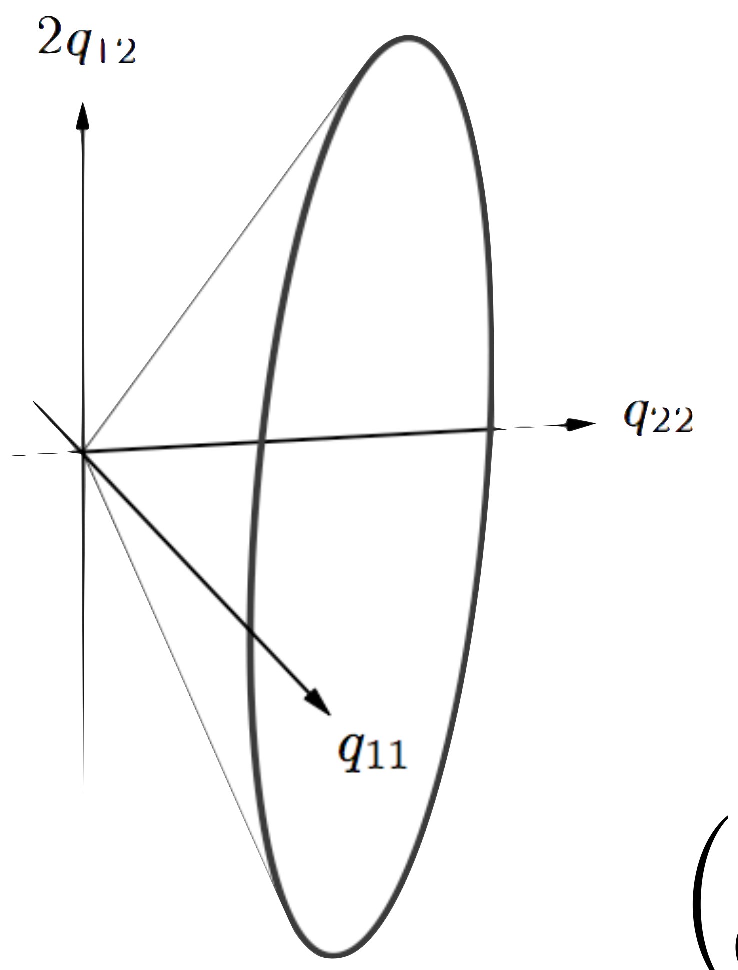
# *Parameter space of lattice & Voronoi's second reduction theory*

$\tilde{\mathbb{S}}^n_+ = \mathrm{cone}_{\mathbb{Q}}\{xx^\top : x \in \mathbb{Z}^n\} \cup \mathbb{S}^n_{++} \subset \mathbb{S}^n_+$ rational closure of $\mathbb{S}^n_{++}$

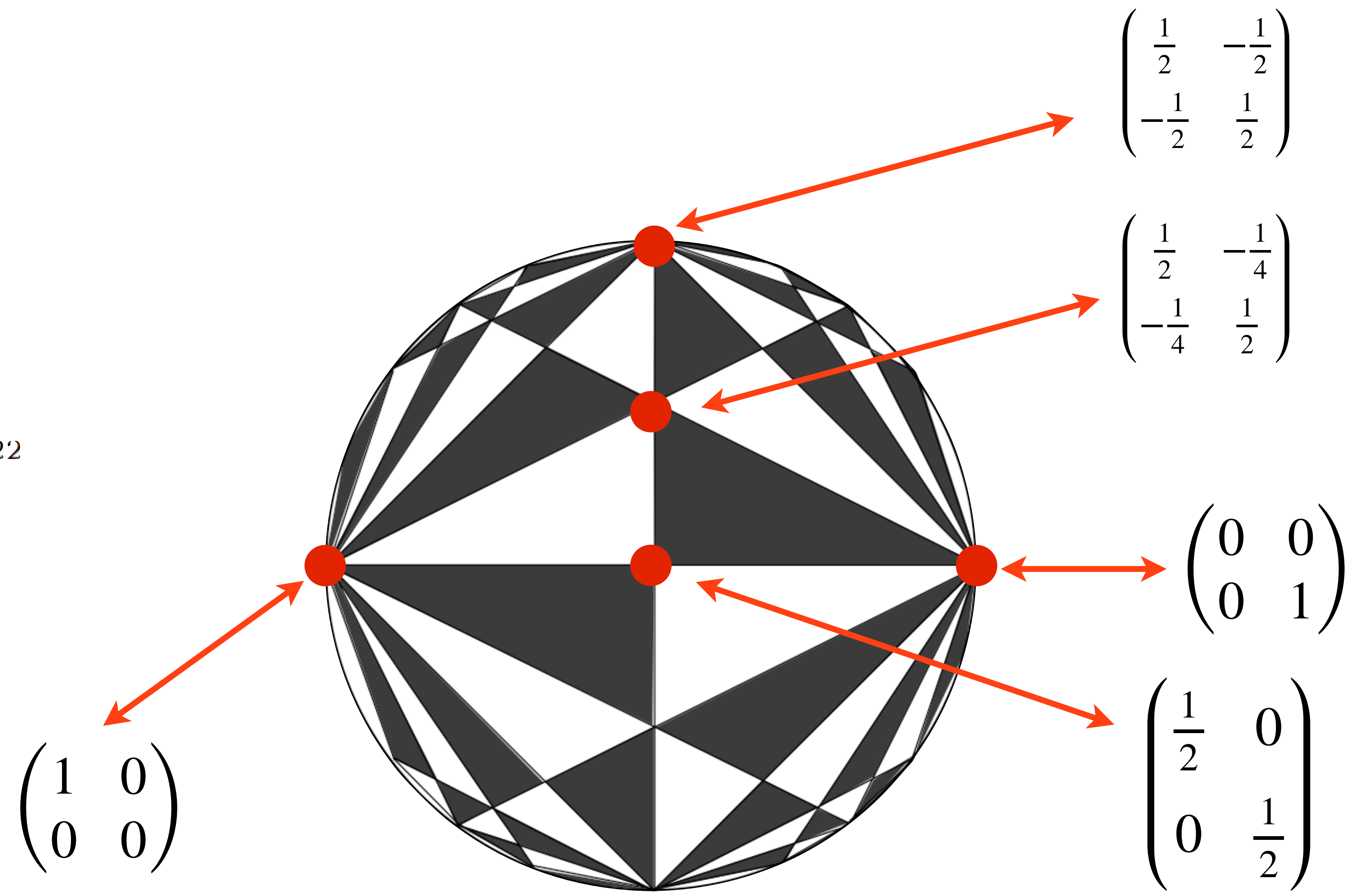
$GL_n(\mathbb{Z})$ acts on $\tilde{\mathbb{S}}^n_+$ by $(g, Q) \mapsto g^\top Q g$

reduction theory of lattices = find „nice" fundamental domain for $\tilde{\mathbb{S}}^n_+ \;/\; GL_n(\mathbb{Z})$

trace $= 1$




$\mathbb{S}^2_+$




*properties*

infinite polyhedral face-to-face tiling

all triangular polyhedra $GL_2(\mathbb{Z})$-equivalent

# Construction of Delaunay polyhedra

B.N. Delaunay
= Б.Н. Делоне
= B.N. Delone (1890–1980)

## Empty sphere construction

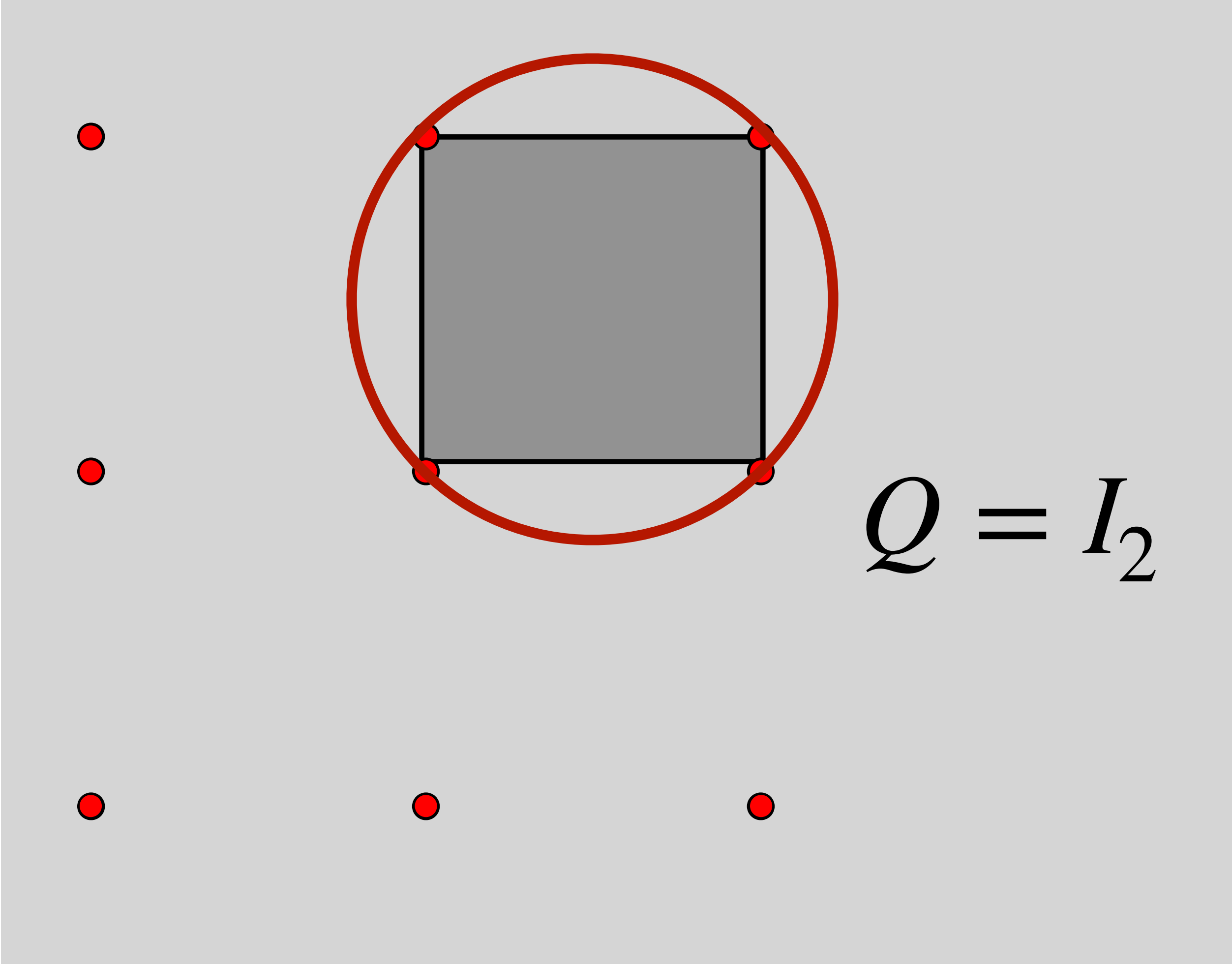
*Delaunay polyhedra* of $Q \in \tilde{\mathbb{S}}^n_+$

$$P = \text{conv}\{v_1, v_2, \ldots\}, v_i \in \mathbb{Z}^n, \text{ where}$$

there is a center $c \in \mathbb{R}^n$ and a radius $r > 0$ so that

$$Q[v_i - c] = r^2 \text{ and } Q[w - c] > r^2 \text{ for all } w \in \mathbb{Z}^n \setminus \{v_1, v_2, \ldots\}$$

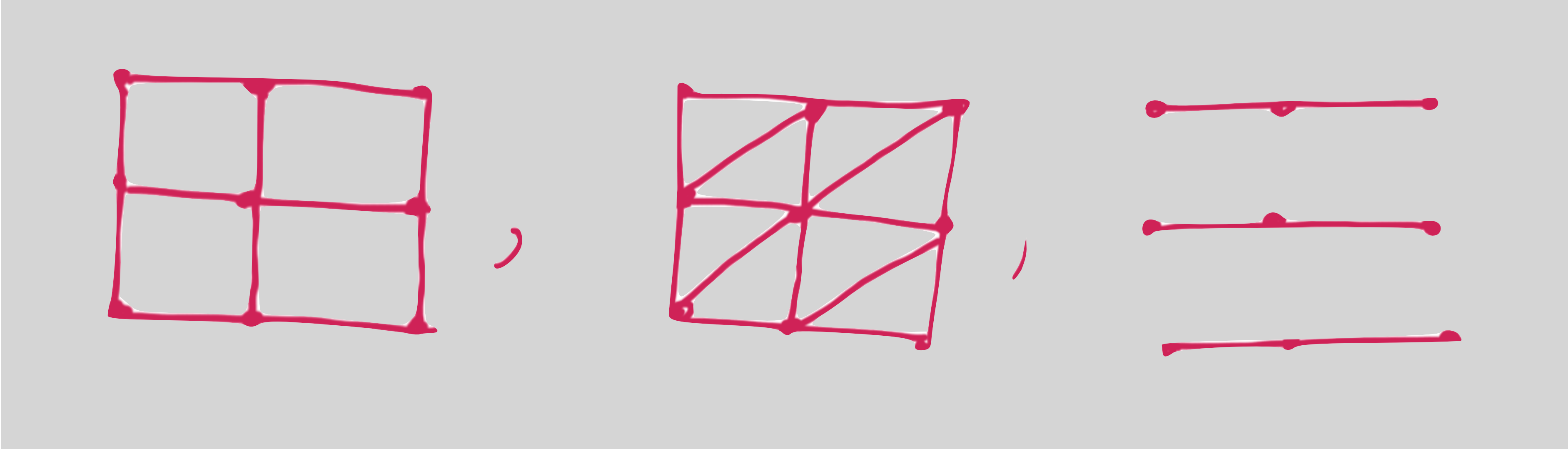
with $Q[x] = x^\mathsf{T} Q x$



# *Delaunay subdivisions & secondary cones*

*Delaunay subdivision* of $Q \in \tilde{\mathbb{S}}^n_+$

Del$(Q) = \{P : P$ is a Delaunay polyhedron of Q$\}$ is a polyhedral complex



*Secondary cone* of Delaunay subdivision

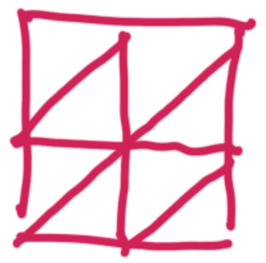
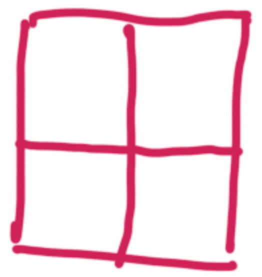
$$\Delta(\text{Del}(Q)) = \{Q' \in \tilde{\mathbb{S}}^n_+ : \text{Del}(Q') = \text{Del}(Q)\}$$

is (open) polyhedral cone in $\tilde{\mathbb{S}}^n_+$; facets of $\Delta(\text{Del}(Q))$ are determined by facets in Del$(Q)$

# *Some facts about secondary cones*

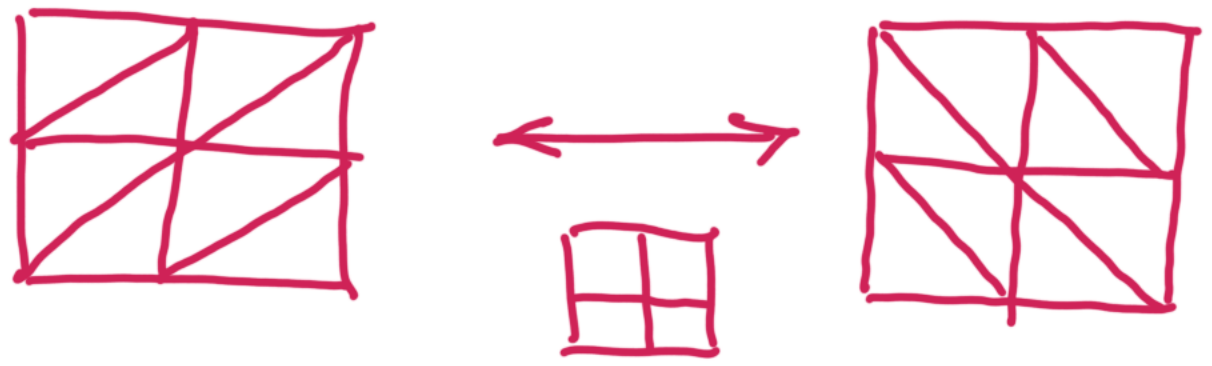
$\Delta(\text{Del}(Q))$ is full-dimensional $\iff$ $\text{Del}(Q)$ is a triangulation

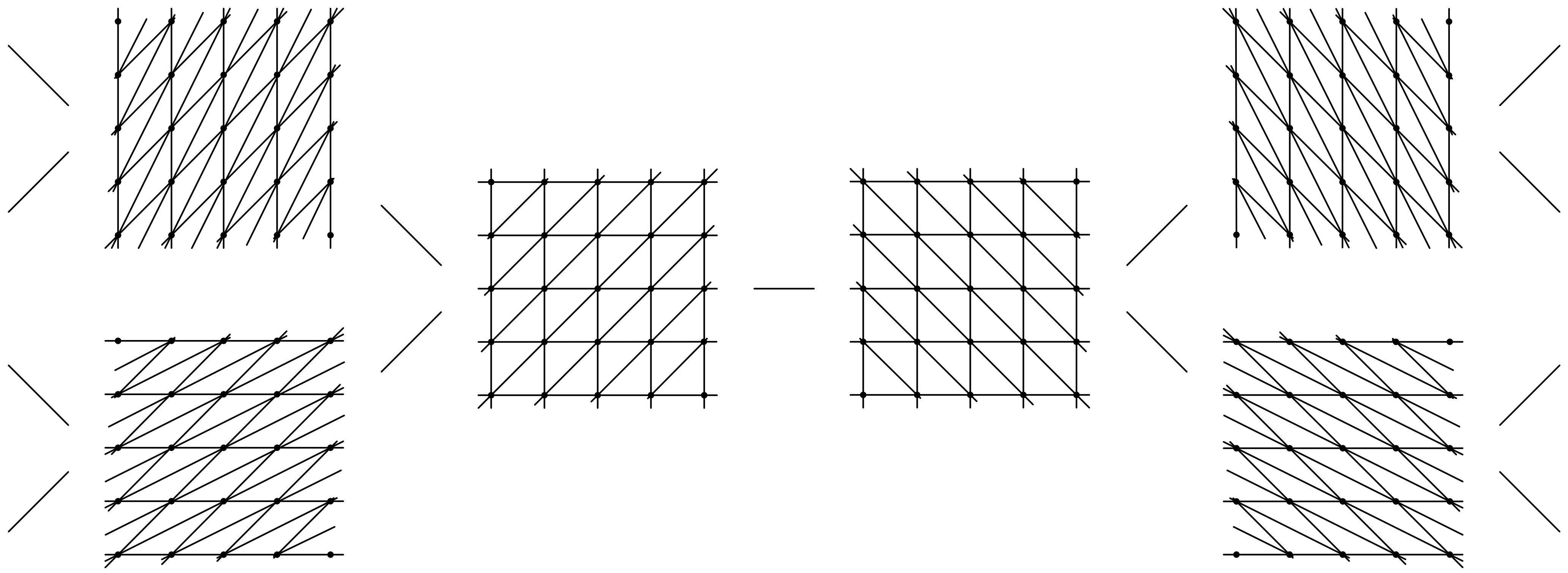
$\Delta(\text{Del}(Q)) \subseteq \overline{\Delta(\text{Del}(Q')}$ $\iff$ $\text{Del}(Q')$ is a refinement of Del(Q)

e.g.  refinement of 

$\text{Del}(Q), \text{Del}(Q')$ triangulations, $\dim\left(\overline{\Delta(\text{Del}(Q)} \cap \overline{\Delta(\text{Del}(Q')}\right) = \binom{n+1}{2} - 1$

$\iff$ $\text{Del}(Q), \text{Del}(Q')$ differ by a *bistellar flip*

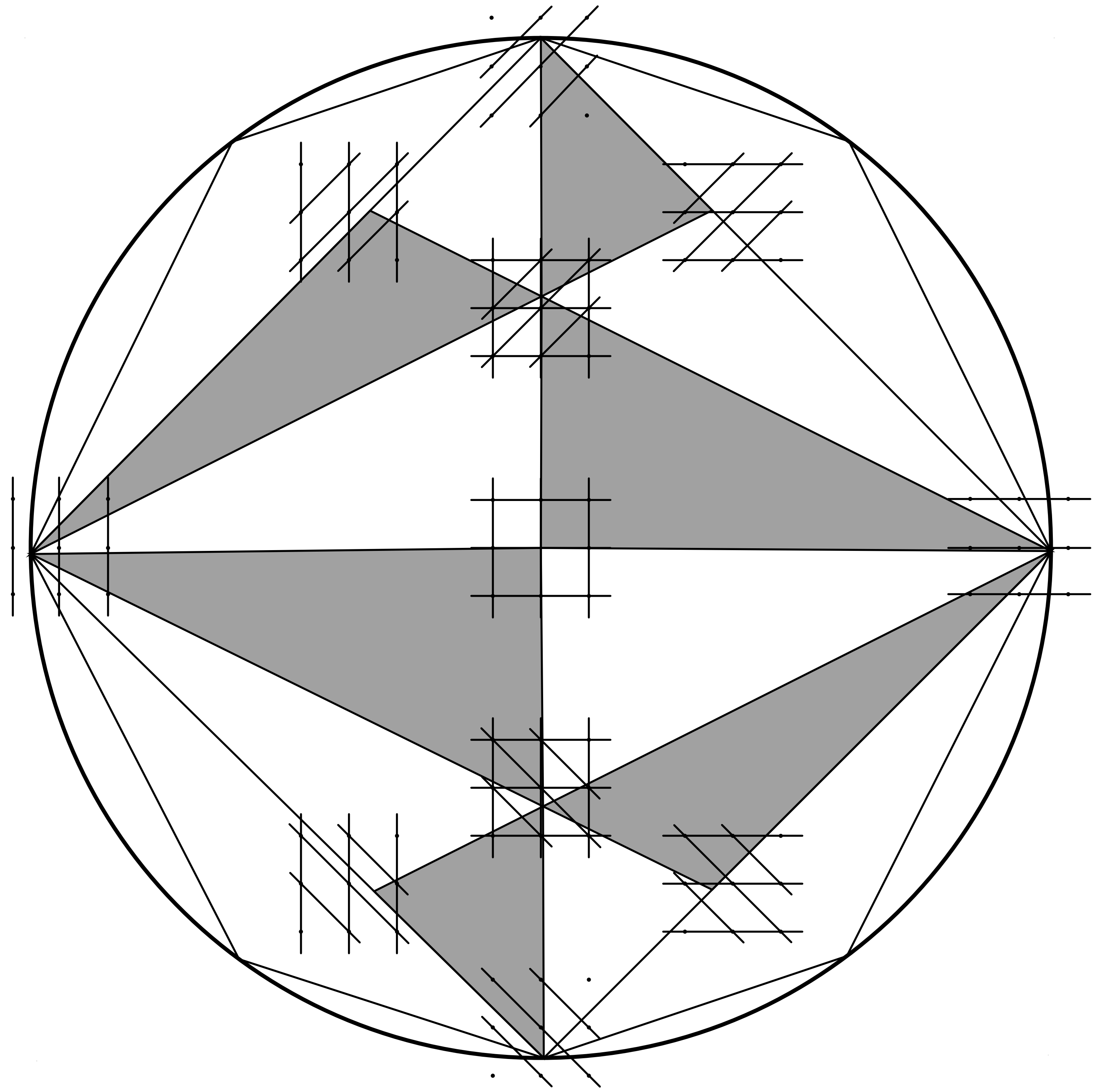
e.g. 



# *Main theorem of Voronoi's second reduction theory*

There are only finitely many $n$-dimensional Delaunay triangulations up to $GL_n(\mathbb{Z})$-equivalence.

Furthermore: $$\tilde{\mathbb{S}}^n_+ = \bigcup_{\mathscr{D} \text{ Delaunay triangulation}} \overline{\Delta(\mathscr{D})}$$



Known classification

| $n$ | # triang. | |
|---|---|---|
| 2 | 1 | Voronoi (1908) |
| 3 | 1 | |
| 4 | 3 | |
| 5 | 221+1 | Baranovski & Ryshkov (1976) |
| 6 | many... | |

# 3. Application to lattice sphere covering

# *Lattice sphere covering problem*

<u>lattice sphere covering problem</u>

<u>given</u>: dimension $n \in \mathbb{N}$

<u>find</u>: lattice $L \subseteq \mathbb{R}^n$ s.t. covering density $\Theta(L)$ is minimized

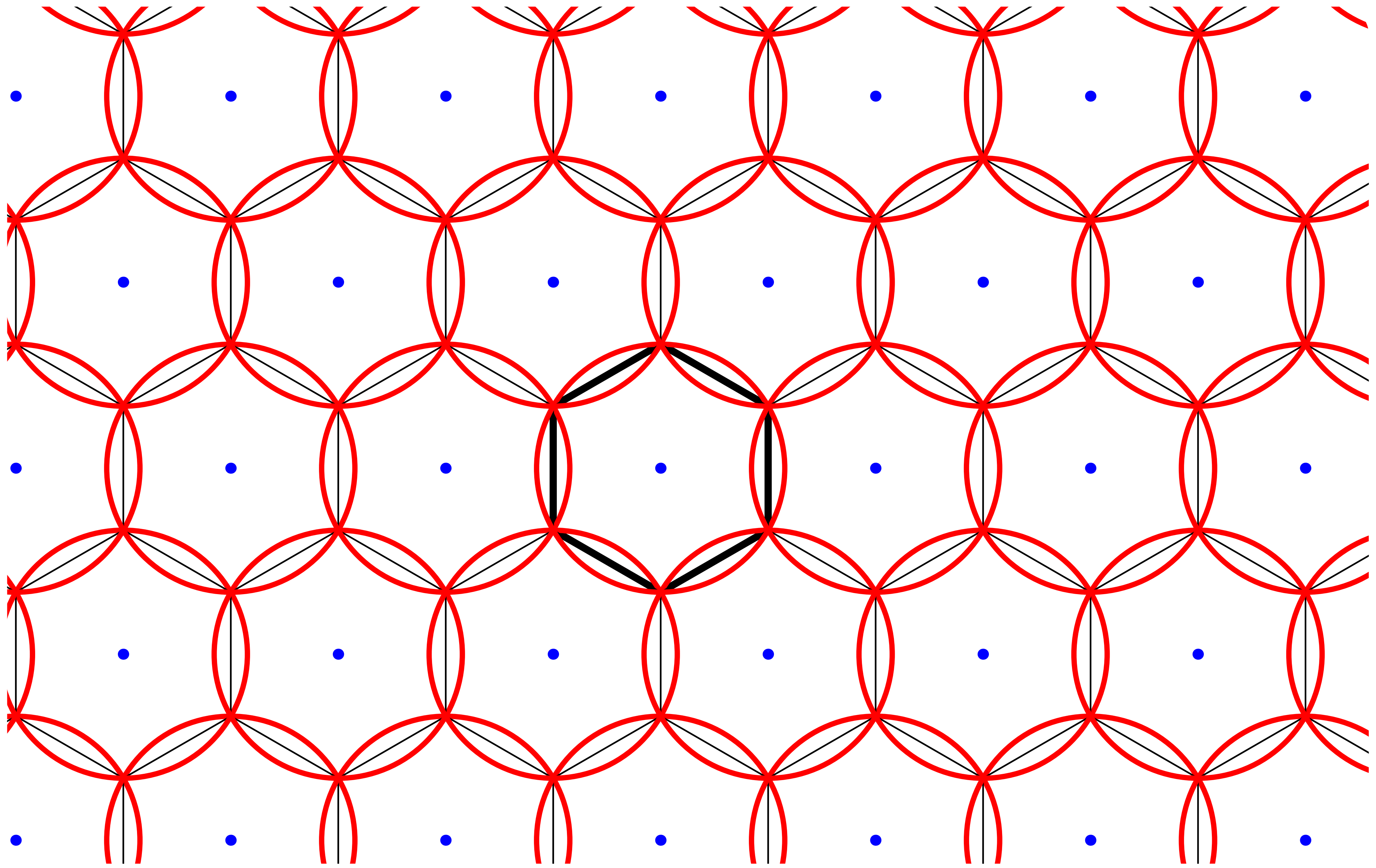
$$\Theta(L) = \frac{\mu(L)^n}{\det L} \cdot \mathrm{vol}\, B_n$$

← unit ball

covering radius

$= \max_{x \in \mathbb{R}^n} \min_{v \in L} \|x - v\|$

$=$ max. circumradius of Delaunay polytope



$$\Theta(L) = 1.20\ldots$$

# MAXDET optimization

→ see Vandenberghe, Boyd, Wu (1998)

→ here : conic formulation

$\mathcal{D}^n = \{ (X,s) \in S^n_+ \times \mathbb{R}_+ : (\det X)^{1/n} \geq s \}$ proper convex cone

computation of its dual cone by

- Minkowski's determinant inequality $\forall X,Y \in S^n_+ : \det(X+Y)^{1/n} \geq (\det X)^{1/n} + (\det Y)^{1/n}$
- AM-GM inequality $\forall X \in S^n_+ : \mathrm{Tr}(X) - n(\det X)^{1/n} \geq 0$

gives

$(\mathcal{D}^n)^* = \{ (Y,t) \in S^n_+ \times \mathbb{R} : (\det Y)^{1/n} \geq -\frac{t}{n} \}$

→ can optimize over $\mathcal{D}^n$ and $(\mathcal{D}^n)^*$ in polynomial time
(under usual conditions for the ellipsoid method)

# *Basic idea*

Fix Delaunay triangulation $\mathscr{D}$, then the restricted lattice covering problem becomes a tractable MAXDET problem.

$$\max \ (\det Q)^{1/n}$$

$$\text{s.t.} \quad Q \in \overline{\Delta(\mathscr{D})}$$

(linear cond.)

$$\mu_L(Q) \leq 1$$

for every Delaunay simplex $L = \operatorname{conv}\{0, v_1, \dots, v_n\}$

# Gadget

<u>Lemma</u> $L = \operatorname{conv}\{0, v_1, \dots, v_n\}$ simplex. Then $\mu_L(Q) \leq 1$ iff the following matrix is in $S_+^{n+1}$:

$$\begin{bmatrix} 4 & v_1^T Q v_1 & v_2^T Q v_2 & \cdots & v_n^T Q v_n \\ v_1^T Q v_1 & & & & \\ \vdots & & \boxed{v_i^T Q v_j} & & \\ v_n^T Q v_n & & & & \end{bmatrix}$$

$\rightsquigarrow$ Cayley-Menger determinant

# *Algorithmic approach to lattice covering*

Schürmann, Vallentin (2006):

C++ implementation of Voronoi's second reduction theory (scc = second cone cruiser)

finding solutions for dimensions $n \leq 5$,

reproving results of Baranovski, Ryshkov (1976)

finding new, conjecturally optimal lattices for $n \geq 6$

2020: still only conjectures

# *The $\Lambda_{24}$ sphere covering problem*

The Leech lattice gives clearly the least dense sphere covering in dimension 24.

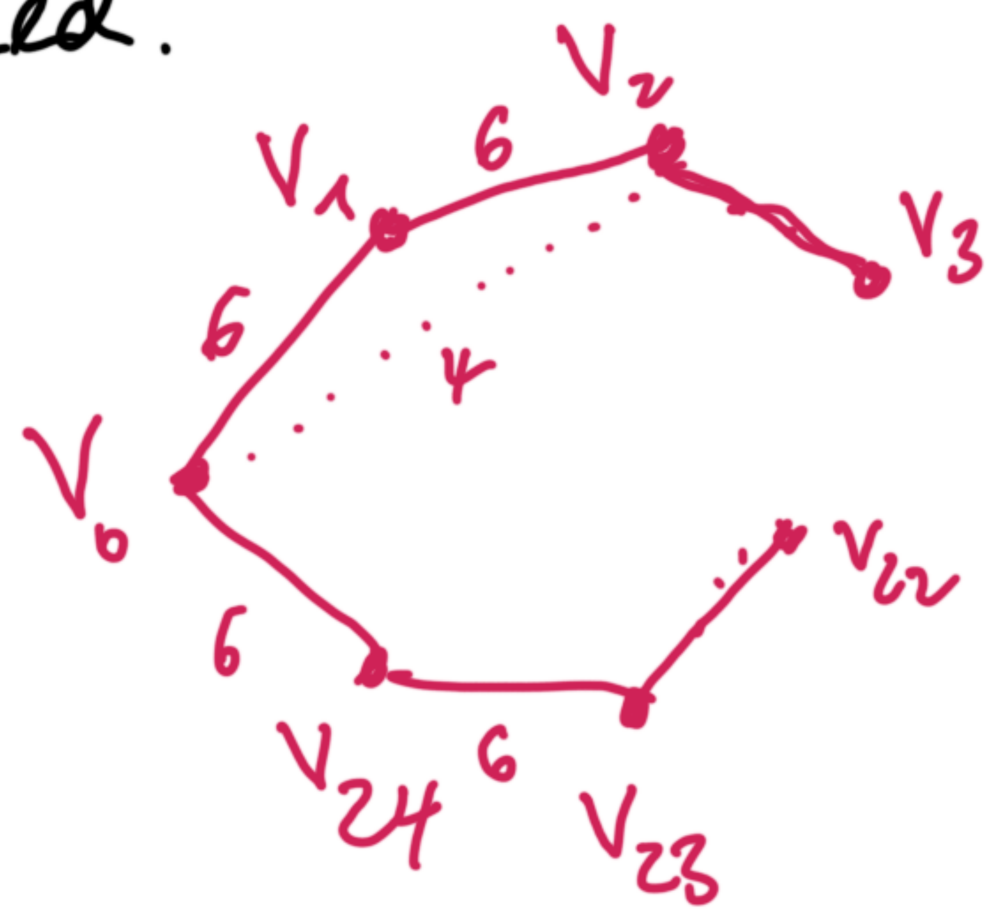
Currently, we don't have a good method (like LP bounds) to come anywhere close to a proof.

# $\Lambda_{24}$ *is locally optimal for covering*

Delaunay polytopes of $\Lambda_{24}$ were classified by
Conway, Parker, Sloane (1982)
and Borcherds, Conway, Queen (1984)

Important class of Delaunay simplices
where covering radius is attained.

$$A_{25} = \mathrm{Conv}\{v_0, v_1, \dots, v_{24}\}$$




center of circumsphere = barycenter

# Proof by MAXDET

- first proof (but more complicated) by Schürmann, Vallentin (2005)

- here: new proof using MAXDET framework (with support from Krupp, Rolfes and Zimmermann)

# Relaxation

$$\frac{1}{2} \leq \max \, (\det Q)^{1/n}$$
$$Q \in \mathcal{S}^{24}_{+}$$
$$\mu_{TA_{25}}(Q) \leq 1 \quad \forall T \in Co_0$$

because $\Lambda_{24}$ is feasible solution

$$= \max \; s$$
$$(Q, s) \in \mathcal{D}^n$$
$$V_T \in \mathcal{S}^{n+1}_{+}, \quad T \in Co_0$$

$$\langle V_T, E_{00} \rangle = 4$$
$$\langle V_T, E_{0i} \rangle = \langle Q, v_{T,i} v_{T,i}^T \rangle \qquad i = 1, \ldots, n$$
$$\langle V_T, E_{ij} \rangle = \langle Q, \tfrac{1}{2}(v_{T,i} v_{T,j}^T + v_{T,j} v_{T,i}^T) \rangle \qquad \begin{array}{l} 1 \leq i \leq j \leq n \\ T \in Co_0 \end{array}$$

# Dualization

$$= \max s$$

$$(Q,s) \in \mathcal{D}^n$$

$$V_T \in S_+^{n+1}, \quad T \in C_0$$

$$\lambda_{T,0} \quad \langle V_T, E_{00} \rangle = 4$$

$$\lambda_{T,i} \quad \langle V_T, E_{0i} \rangle = \langle Q, v_{T,i} v_{T,i}^T \rangle \qquad i = 1, \dots, n$$

$$\lambda_{T,i,j} \quad \langle V_T, E_{ij} \rangle = \langle Q, \tfrac{1}{2}(v_{T,i} v_{T,j}^T + v_{T,j} v_{T,i}^T) \rangle \qquad 1 \le i \le j \le n, \; T \in C_0$$

$$\le \min 4 \sum_T \lambda_{T,0}$$

$$\lambda_{T,0} E_{00} - \sum_i \lambda_{T,i} E_{0i} - \sum_{i,j} \lambda_{T,i,j} E_{ij} \succeq 0 \quad \forall T$$

$$\Big( \underbrace{\sum \lambda_{T,i} v_{T,i} v_{T,i}^T + \sum \lambda_{T,i,j} \tfrac{1}{2} (v_{T,i} v_{T,j}^T + v_{T,j} v_{T,i}^T)}_{Q(\lambda)}, 0 \Big) - (0,1) \in (\mathcal{D}^n)^*$$

$$Q(\lambda) \quad \rightarrow \quad (\det Q(\lambda))^{1/n} \ge \frac{1}{n}$$

# Ansatz

$$\leq \min 4 \sum_T \lambda_{T,0}$$

$$\lambda_{T,0} E_{00} - \sum_i \lambda_{T,i} E_{0i} - \sum_{i,j} \lambda_{T,i,j} E_{ij} \geq 0 \quad \forall T$$

$$\Big(\underbrace{\sum \lambda_{T,i} v_{T,i} v_{T,i}^T + \sum \lambda_{T,i,j} \tfrac{1}{2} \left(v_{T,i} v_{T,j}^T + v_{T,j} v_{T,i}^T\right)}_{Q(\lambda)}, 0\Big) - (0,1) \in (\mathbb{Q}^n)^*$$

$$\to \left(\det Q(\lambda)\right)^{1/n} \geq \frac{1}{n}$$

Ansatz : Set

$$\begin{aligned} \alpha &= \lambda_{T,0} && \forall T \\ 2\beta &= \lambda_{T,i} && \forall T, i \\ \gamma &= \lambda_{T,i,i} && \forall T, i \\ 2\gamma &= \lambda_{T,i,j} && \forall T, i, j,\ i \neq j \end{aligned}$$

Then

$$\lambda_{T,0} E_{00} - \sum_i \lambda_{T,i} E_{0i} - \sum_{i,j} \lambda_{T,i,j} E_{ij} = \begin{bmatrix} \alpha & -\beta & \cdots & -\beta \\ -\beta & & & \\ \vdots & & \boxed{-\gamma} & \\ -\beta & & & \end{bmatrix} \geq 0$$

# *First condition*

$$\begin{bmatrix} \alpha & -\beta & \cdots & -\beta \\ -\beta & & & \\ \vdots & & \boxed{\quad -\gamma \quad} & \\ -\beta & & & \end{bmatrix} \succ 0$$

$$\underset{\text{Schur complement}}{\overset{\text{by}}{\iff}}$$

$\alpha > 0$ and

$$-\gamma J - (-\beta e)\frac{1}{\alpha}(-\beta e)^T \succeq 0$$

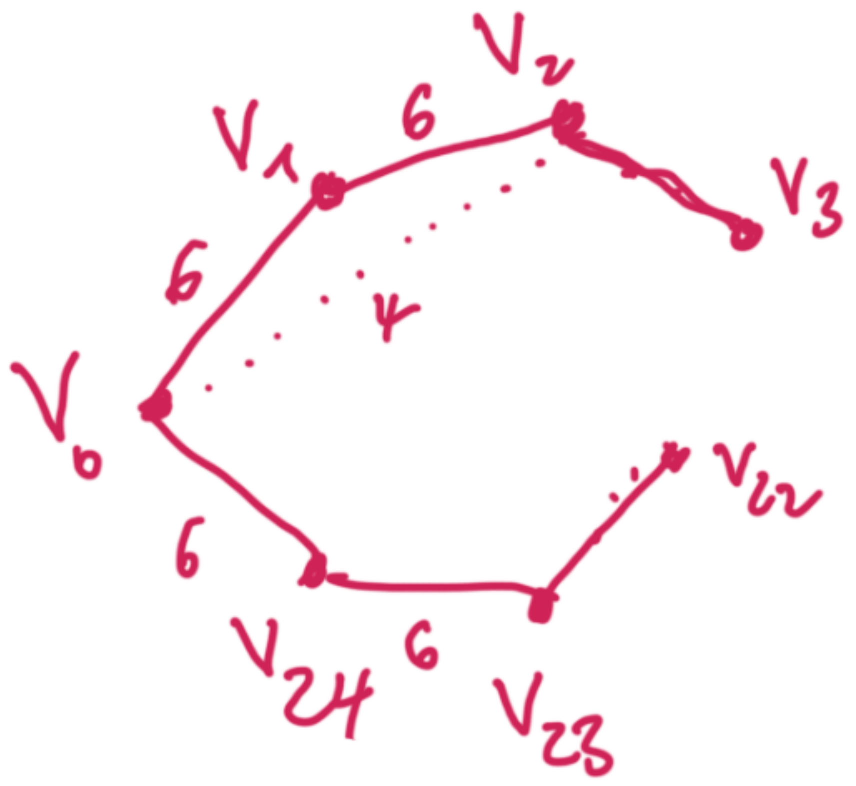
$$\iff -\gamma J - \frac{\beta^2}{\alpha} J \succeq 0$$

$$\iff -\gamma - \frac{\beta^2}{\alpha} \geq 0$$

# Second condition

$$\Big(\underbrace{\sum \lambda_{T,i}\, v_{T,i} v_{T,i}^T + \sum \lambda_{T,i,j} \tfrac{1}{2}\left(v_{T,i} v_{T,j}^T + v_{T,j} v_{T,i}^T\right)}_{Q(\lambda)}, 0\Big) - (0,1) \in (\mathcal{Q}^n)^*$$

$$Q(\lambda) \to \left(\det Q(\lambda)\right)^{1/n} \geq \frac{1}{n}$$




<u>$1^{st}$ term</u>

$$\sum_{T,i} \lambda_{T,i}\, v_{T,i} v_{T,i}^T = 2\beta \sum_T \sum_i T v_i (T v_i)^T$$

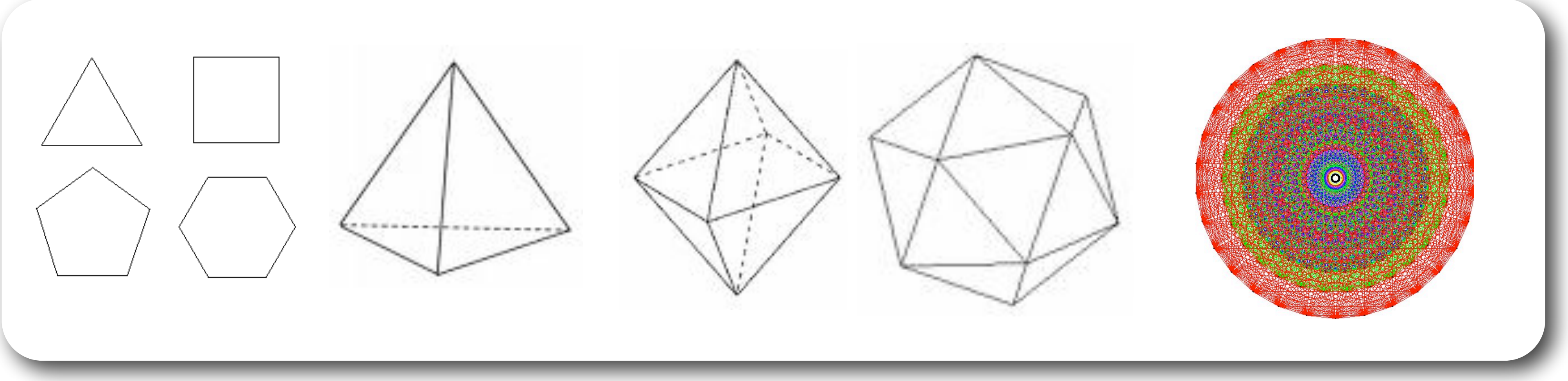
$$= 2\beta\left(\frac{22\cdot|Co_0|}{|S_4|}\sum_{e\in S_4} ee^T + \frac{2\,|Co_0|}{|S_6|}\sum_{e\in S_6} ee^T\right)$$

$$S_k = \{v \in \Lambda_{24} : v\cdot v = k\}$$

# *Spherical $t$—designs*

$X \subseteq S^{n-1}$ spherical $t$-design

$$\int_{S^{n-1}} f(x)d\omega(x) = \frac{1}{|X|}\sum_{x\in X} f(x) \quad \text{for all polynomials of degree} \leq t.$$



| **configuration** | **strength** |
|---|---|
| $n$-gon | $n-1$ |
| simplex | 2 |
| cross polytope | 3 |
| icosahedron | 5 |
| 240 | 7 |
| 196560 | 11 |

Useful for optimality conditions: $X$ spherical 2-design $\iff \sum_{x\in X} xx^{\mathsf{T}} = \frac{|X|}{n} I$

# Putting it together

know : $\frac{1}{\sqrt{k}} S_k$ is spherical 11-design

So $\sum_{e \in S_k} e e^T = \frac{k\,|S_k|}{24} I$

First term of $Q(\lambda)$ equals

$$\frac{50}{6} |C_0| \beta I$$

Similarly, second term of $Q(\lambda)$ equals

$$\frac{1250}{24} |C_0| \gamma I .$$

Together

$$\alpha = \frac{1}{|C_0|} \cdot \frac{1}{8}$$

$$\beta = \frac{1}{|C_0|} \cdot \frac{1}{100}$$

$$\gamma = \frac{1}{|C_0|} \cdot \frac{1}{1250}$$

gives optimal solution of dual problem.
This shows that $\Lambda_{24}$ is locally optimal for lattice sphere coverings. $\boxtimes$