Optimization for lattices, packings, and coverings

Lecture 1

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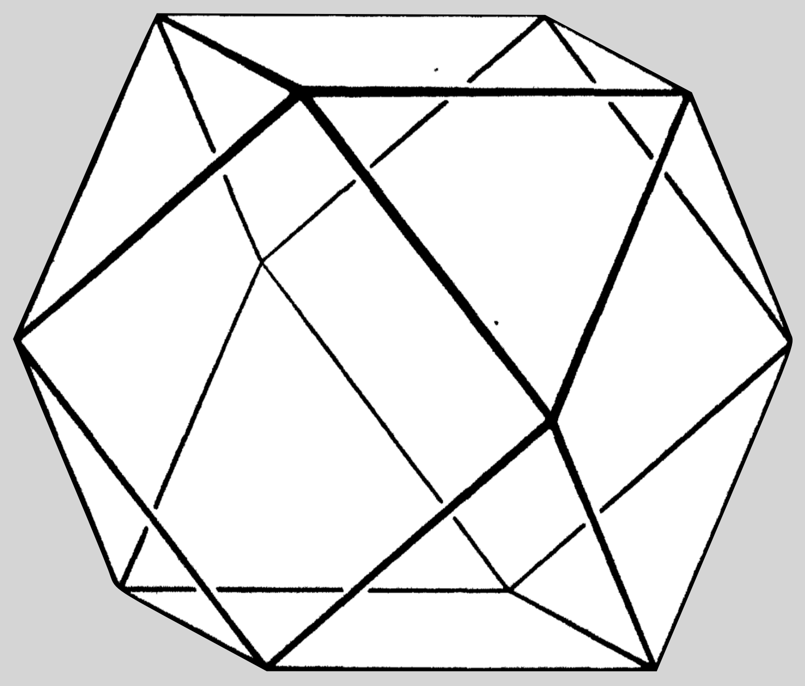
1. Introduction to conic optimization

WHAT IS… LP and SDP?

LP (linear programming) Maximizing/minimizing a linear functional over a *polyhedron*

polyhedron intersection of finitely many linear half spaces

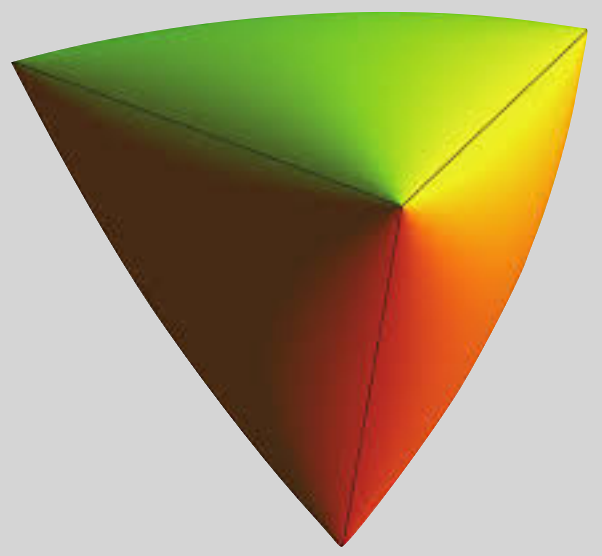
$$= \mathbb{R}^n_+ \cap \text{affine subspace}$$



cuboctahedron

SDP (semidefinite programming) Maximizing/minimizing a linear functional over a spectrahedron

spectrahedron $= \mathbb{S}^n_+ \cap$ *affine subspace*



$$\left\{ (x, y, z) : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \in \mathbb{S}^3_+ \right\}$$

elliptope

Why are LP and SDP interesting?

1. Describe a wide class of convex optimization problems
2. LP and SDP can be solved efficiently (in theory and practice)
3. Duality theory gives optimality criteria and a systematic way to prove rigorous upper/lower bounds
4. LP and SDP can be used to prove that a point configuration is optimal or near optimal
5. Lots of other applications… combinatorial optimization, global polynomial optimization, engineering, machine learning, quantum information, game theory, …

General framework: Conic optimization

E finite-dimensional Euclidean space with inner product $\langle x, y \rangle$

$K \subseteq E$ proper convex cone

$\alpha K + \beta K \subseteq K$ for $\alpha, \beta \in \mathbb{R}_+$, K full-dimensional, K closed, $K \cap (-K) = \{0\}$

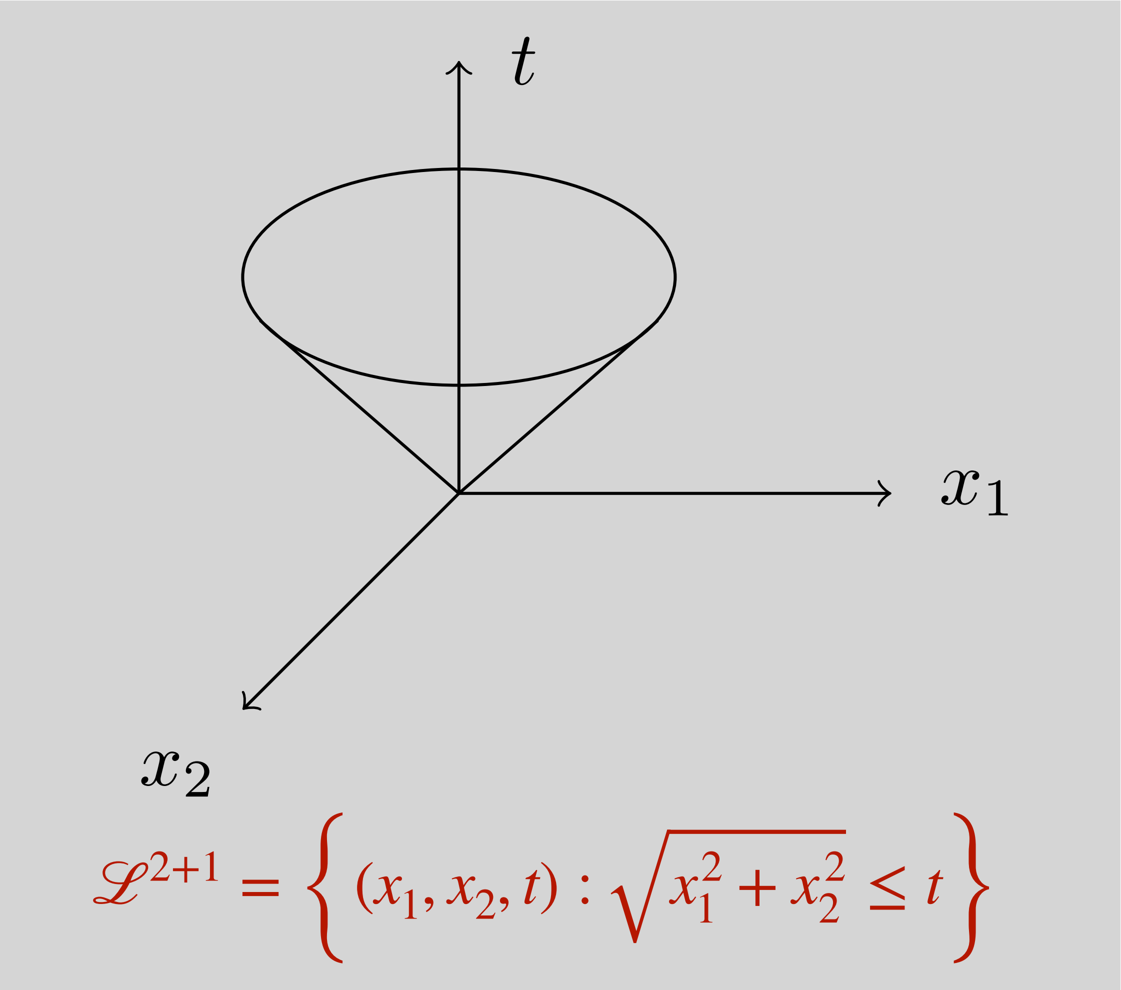
K gives partial ordering on E by $x \succeq_K y \iff x - y \in K$

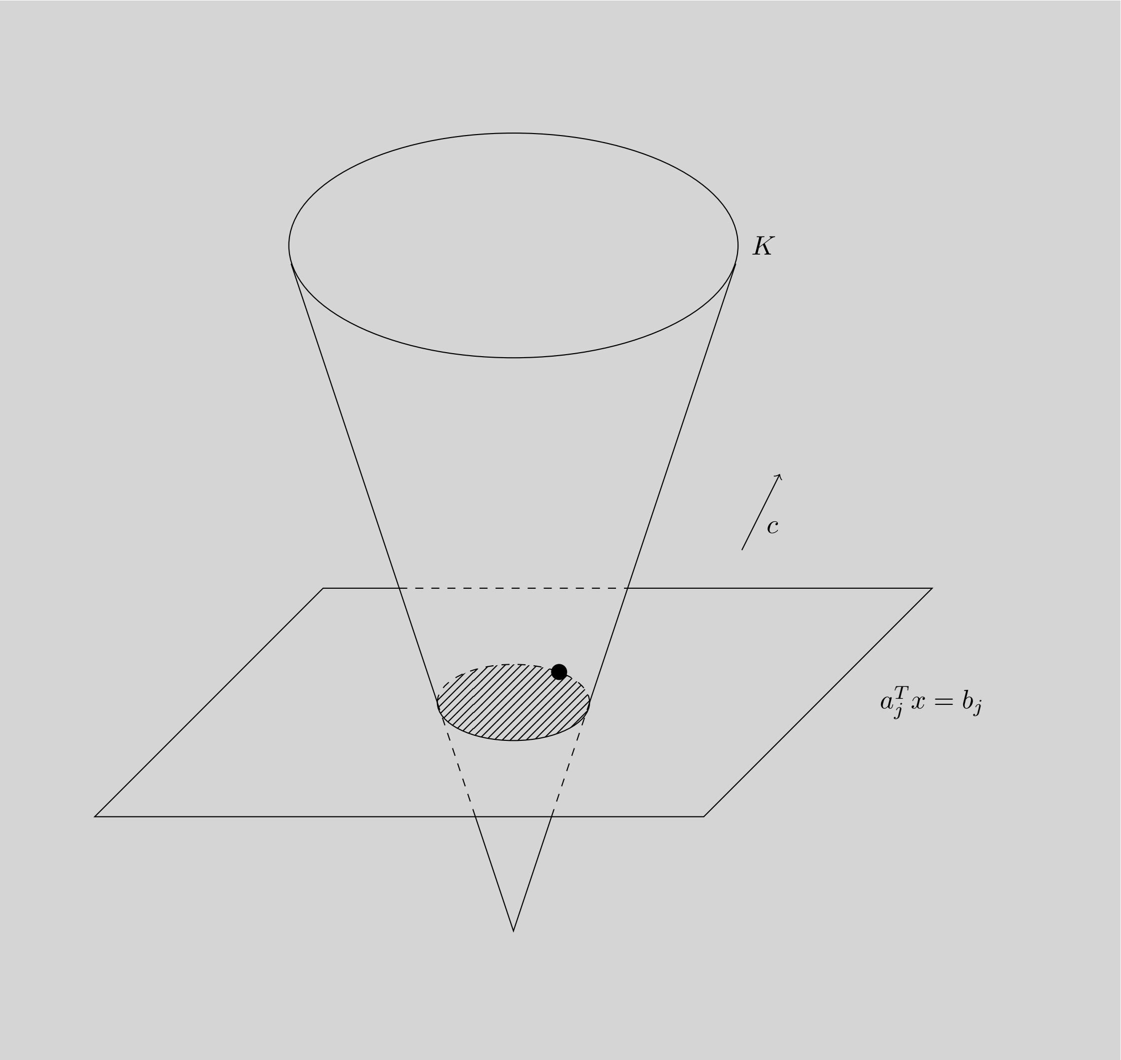
K is the domain of nonnegative elements

primal standard form of conic program

$$p^* = \sup \left\{ \langle c, x \rangle : x \in E,\, x \succeq_K 0,\, \langle a_j, x \rangle = b_j \; (j \in [m]) \right\}$$

x is the optimization variable





Important examples

Linear programming (LP)

$$E = \mathbb{R}^n, \quad \langle x, y\rangle = x^\top y, \quad K = \mathbb{R}^n_+$$

Second order cone programming (SOCP)

$$E = \mathbb{R}^{n+1}, \quad \langle (x,s),(y,t)\rangle = x^\top y + st, \quad K = \mathscr{L}^{n+1} = \{(x,s) : \|x\|_2 \le s\}$$

Semidefinite programming (SDP)

$$E = \mathbb{S}^n, \quad \langle X, Y\rangle = \mathrm{Tr}\, XY, \quad K = \mathbb{S}^n_+$$

Determinant maximization (MAXDET)

$$E = \mathbb{S}^n \times \mathbb{R}, \quad \langle (X,s),(Y,t)\rangle = \mathrm{Tr}\, XY + st, \quad K = \mathscr{D}^{n+1} = \{(X,s) : X \in \mathbb{S}^n_+, s \ge 0, (\det X)^{1/n} \ge s\}$$

Polynomial optimization (POP)

$$E = \mathbb{R}[x_1, \dots, x_n]_d, \quad \langle f, g\rangle = \frac{1}{d!} f(\nabla) g, \quad K = P_{n,d} = \{f : f(x) \ge 0 \text{ for all } x \in \mathbb{R}^n\}, \quad d \text{ even}$$

Dualization

primal standard form of conic program

$$p^* = \sup\left\{\langle c, x\rangle : x \in E,\ x \succeq_K 0,\ \langle a_j, x\rangle = b_j\ (j \in [m])\right\}$$

dual standard form of conic program

$$d^* = \inf\left\{b_1 y_1 + \cdots + b_m y_m : y_1, \ldots, y_m \in \mathbb{R},\ y_1 a_1 + \cdots + y_m a_m \succeq_{K^*} c\right\}$$

where $K^* = \{y \in E : \langle x, y\rangle \geq 0$ for all $x \in K\}$ is the dual cone of K Bipolar theorem: $(K^*)^* = K$

Important examples

self dual cones: $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$ $(\mathscr{L}^{n+1})^* = \mathscr{L}^{n+1}$ $(\mathbb{S}^n_+)^* = \mathbb{S}^n_+$

Koecher-Vinberg classification of symmetric cones (real Euclidean Jordan algebras)

non self dual cones:

$$(\mathscr{D}^{n+1})^* = \left\{(Y, t) \in \mathbb{S}^n_+ \times \mathbb{R} : (\det Y)^{1/n} \geq -\frac{t}{n}\right\}$$

$$(P_{n,d})^* = Q_{n,d} = \left\{(\alpha_1^\mathsf{T} x)^d + \cdots + (\alpha_r^\mathsf{T} x)^d : \alpha_1, \ldots, \alpha_r \in \mathbb{R}^n, r \in \mathbb{N}\right\}$$ sums of even powers of linear forms

The dualization cheat sheet

maximize	minimize
variable $\succeq_K 0$	$\succeq_{K^*}$ constraint
variable $\preceq_K 0$	$\preceq_{K^*}$ constraint
unconstrained variable	$=$ constraint
$=$ constraint	unconstrained variable
$\leq$ constraint	variable ≥ 0
$\geq$ constraint	variable ≤ 0
right-hand side	objective function
objective function	right-hand side

Duality theory

primal	$p^* = \sup \left\{ \langle c, x \rangle : x \in E,\ x \succeq_K 0,\ \langle a_j, x \rangle = b_j\ (j \in [m]) \right\}$
dual	$d^* = \inf \left\{ b_1 y_1 + \cdots + b_m y_m : y_1, \ldots, y_m \in \mathbb{R},\ y_1 a_1 + \cdots + y_m a_m \succeq_{K^*} c \right\}$

weak duality $p^* \le d^*$

strong duality if $d^* > -\infty$ and $\exists$ strictly feasible dual solution y (i.e. $y_1 a_1 + \cdots + y_m a_m - c \in \text{int } K^*$)

then $\exists x^*$ feasible for primal with $p^* = \langle c, x^* \rangle$ and $p^* = d^*$

(similarly with primal and dual interchanged)

optimality condition / complementary slackness Suppose primal and dual are both strictly feasible,
suppose x primal feasible and y dual feasible,
then (x, y) is optimal iff $\langle x, y_1 a_1 + \cdots + y_m a_m - c \rangle = 0$