

Optimization for lattices, packings, and coverings

Lecture 1

Frank Vallentin (Universität zu Köln)



Online summer school on optimization, interpolation and modular forms
August 24 to 28, 2020
EPF Lausanne

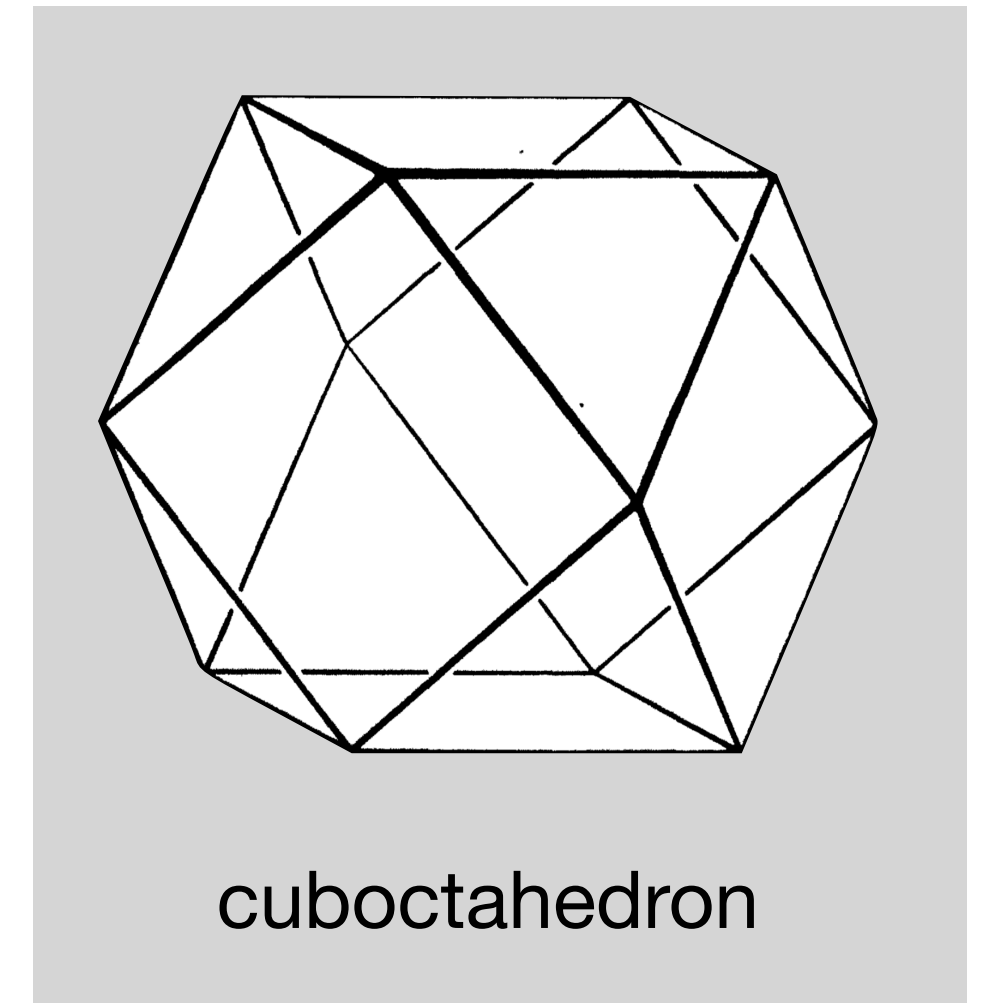
1. Introduction to conic optimization

WHAT IS... LP and SDP?

LP (linear programming) Maximizing/minimizing a linear functional over a *polyhedron*

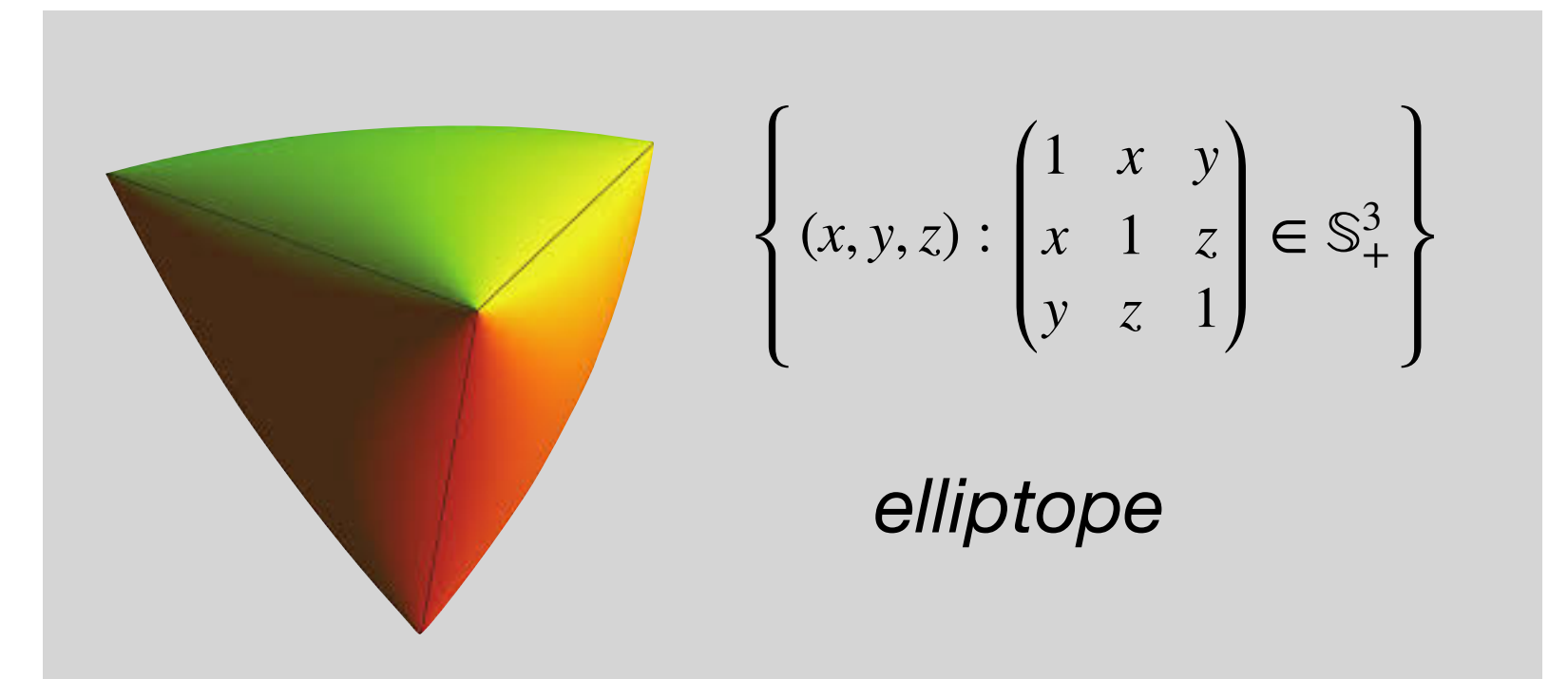
polyhedron intersection of finitely many linear half spaces

$$= \mathbb{R}_+^n \cap \text{affine subspace}$$



SDP (semidefinite programming) Maximizing/minimizing a linear functional over a *spectrahedron*

spectrahedron = $\mathbb{S}_+^n \cap \text{affine subspace}$



Why are LP and SDP interesting?

1. Describe a **wide class** of convex optimization problems
2. LP and SDP can be solved **efficiently** (in theory and practice)
3. Duality theory gives **optimality criteria** and a **systematic** way to prove rigorous upper/lower bounds
4. LP and SDP can be used to prove that a **point configuration** is optimal or near optimal
5. Lots of other **applications**... combinatorial optimization, global polynomial optimization, engineering, machine learning, quantum information, game theory, ...

General framework: Conic optimization

E finite-dimensional Euclidean space with inner product $\langle x, y \rangle$

$K \subseteq E$ **proper convex cone**

$\alpha K + \beta K \subseteq K$ for $\alpha, \beta \in \mathbb{R}_+$, K full-dimensional, K closed, $K \cap (-K) = \{0\}$

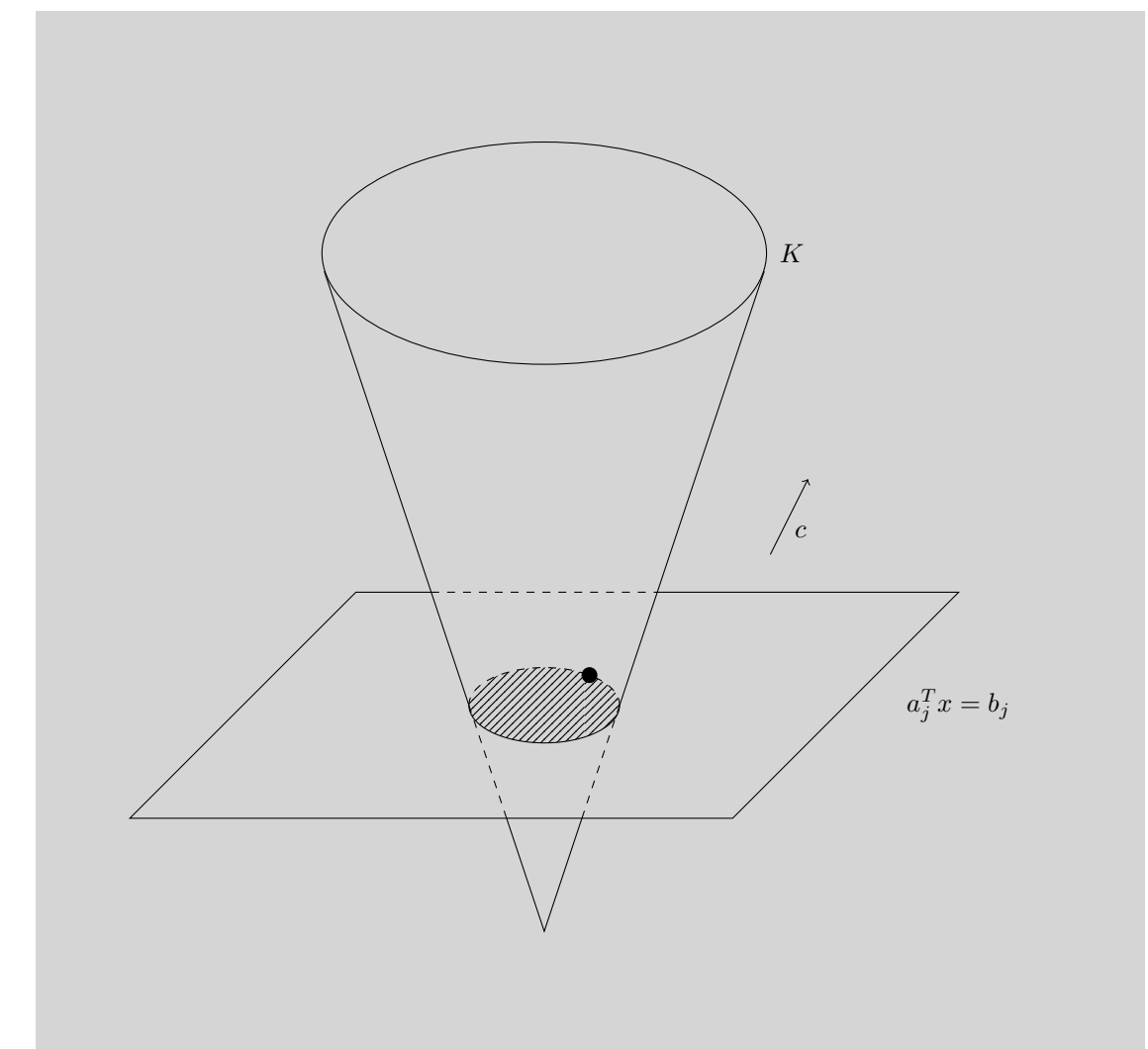
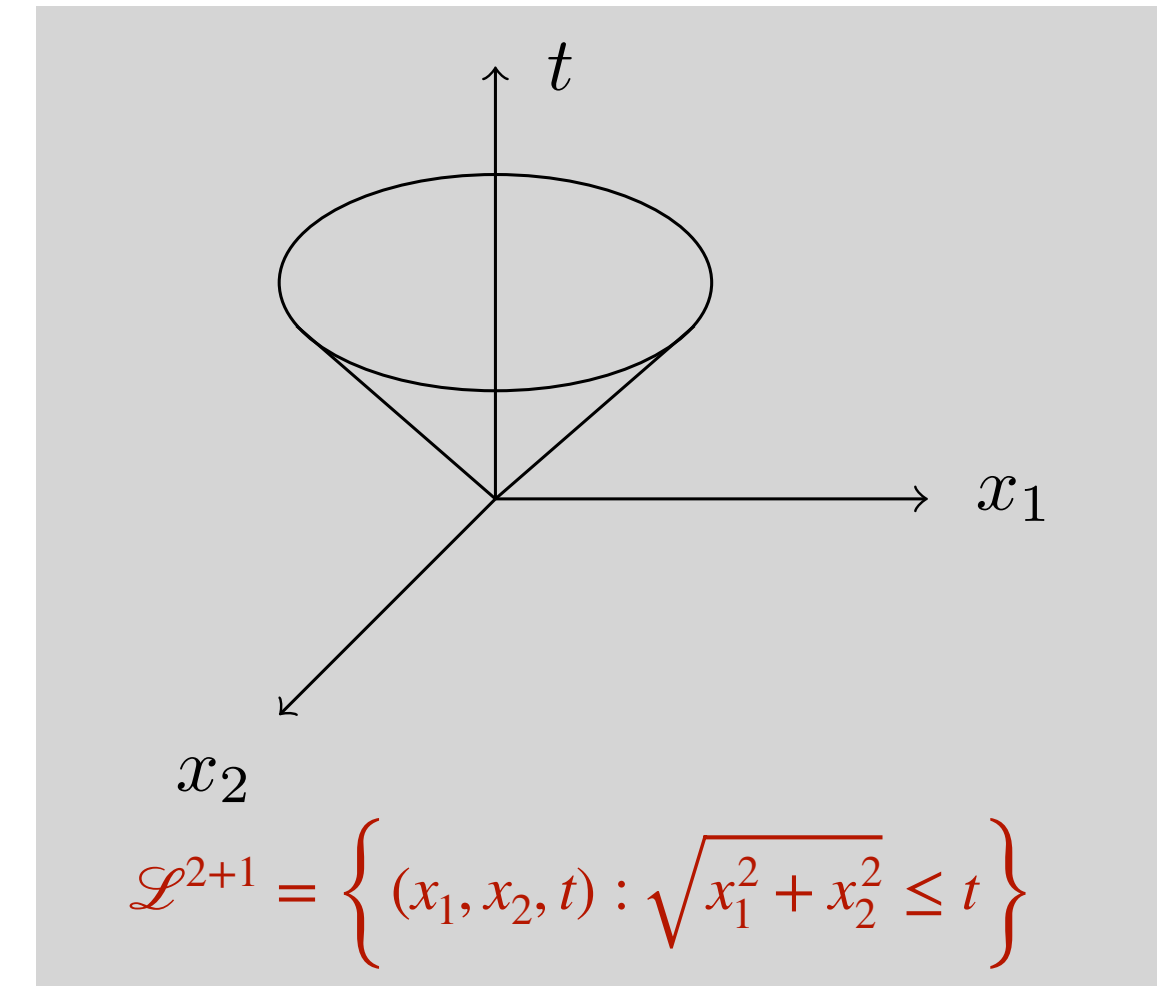
K gives **partial ordering** on E by $x \succeq_K y \iff x - y \in K$

K is the domain of nonnegative elements

primal standard form of conic program

$$p^* = \sup \left\{ \langle c, x \rangle : x \in E, x \succeq_K 0, \langle a_j, x \rangle = b_j \ (j \in [m]) \right\}$$

x is the optimization variable



Important examples

Linear programming (LP)

$$E = \mathbb{R}^n, \quad \langle x, y \rangle = x^\top y, \quad K = \mathbb{R}_+^n$$

Second order cone programming (SOCP)

$$E = \mathbb{R}^{n+1}, \quad \langle (x, s), (y, t) \rangle = x^\top y + st, \quad K = \mathcal{L}^{n+1} = \{(x, s) : \|x\|_2 \leq s\}$$

Semidefinite programming (SDP)

$$E = \mathcal{S}^n, \quad \langle X, Y \rangle = \text{Tr } XY, \quad K = \mathcal{S}_+^n$$

Determinant maximization (MAXDET)

$$E = \mathcal{S}^n \times \mathbb{R}, \quad \langle (X, s), (Y, t) \rangle = \text{Tr } XY + st, \quad K = \mathcal{D}^{n+1} = \{(X, s) : X \in \mathcal{S}_+^n, s \geq 0, (\det X)^{1/n} \geq s\}$$

Polynomial optimization (POP)

$$E = \mathbb{R}[x_1, \dots, x_n]_d, \quad \langle f, g \rangle = \frac{1}{d!} f(\nabla)g, \quad K = P_{n,d} = \{f : f(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\}, \quad d \text{ even}$$



Dualization

primal standard form of conic program

$$p^* = \sup \left\{ \langle c, x \rangle : x \in E, x \succeq_K 0, \langle a_j, x \rangle = b_j (j \in [m]) \right\}$$

dual standard form of conic program

$$d^* = \inf \left\{ b_1 y_1 + \dots + b_m y_m : y_1, \dots, y_m \in \mathbb{R}, y_1 a_1 + \dots + y_m a_m \succeq_{K^*} c \right\}$$

where $K^* = \{y \in E : \langle x, y \rangle \geq 0 \text{ for all } x \in K\}$ is the **dual cone** of K Bipolar theorem: $(K^*)^* = K$

Important examples

self dual cones: $(\mathbb{R}_+^n)^* = \mathbb{R}_+^n$ $(\mathcal{L}^{n+1})^* = \mathcal{L}^{n+1}$ $(\mathbb{S}_+^n)^* = \mathbb{S}_+^n$

Koecher-Vinberg classification of symmetric cones (real Euclidean Jordan algebras)

non self dual cones:

$$(\mathcal{D}^{n+1})^* = \left\{ (Y, t) \in \mathbb{S}_+^n \times \mathbb{R} : (\det Y)^{1/n} \geq -\frac{t}{n} \right\}$$

$$(P_{n,d})^* = Q_{n,d} = \left\{ (\alpha_1^\top x)^d + \dots + (\alpha_r^\top x)^d : \alpha_1, \dots, \alpha_r \in \mathbb{R}^n, r \in \mathbb{N} \right\} \quad \text{sums of even powers of linear forms}$$

The dualization cheat sheet

maximize	minimize
variable $\succeq_K 0$	\succeq_{K^*} constraint
variable $\preceq_K 0$	\preceq_{K^*} constraint
unconstrained variable	= constraint
= constraint	unconstrained variable
\leq constraint	variable ≥ 0
\geq constraint	variable ≤ 0
right-hand side	objective function
objective function	right-hand side

ganze Version von Eger vary: $G=(V,E)$ bip.

$y^T b \cdot y \geq 0$
 $1 u(\epsilon) P =$
 Charak
 $E : x \geq 0$

$(u) = \{(v,w) \in \dots\}$
 $(v_1, a_1, v_2, \dots, a_n)$
 $\forall a = (a_1, \dots, a_n)$
 (v, T) Spann
 $T = \min \{ \dots \}$
 $x_1, \dots, x_n \in \mathbb{R}^n$. Dann ist $y \in \mathbb{R}^n$ eine **Konvexkombination** von x_1, \dots, x_n , falls $y = \sum d_i x_i$ mit $d_i \geq 0, \sum d_i = 1$
 \mathbb{R}^n **Konvex**, wenn: $\forall N \in \mathbb{N} \forall x_1, \dots, x_N \in C \forall d_1, \dots, d_N \geq 0, \sum d_i = 1$
 $x_1, \dots, x_N \in \mathbb{R}^n$ **affin unabh.**, wenn $\forall d_1, \dots, d_N \in \mathbb{R} : \sum d_i = 0, \sum d_i x_i = 0$
Hyperebene, falls $\exists c \in \mathbb{R}^n \setminus \{0\}, s \in \mathbb{R} : H = \{x \in \mathbb{R}^n : c^T x = s\}$. Die
 $\{x \in \mathbb{R}^n : c^T x \leq s\}$ heißen **Halbräume** $K, D \in \mathbb{R}^n$, Hyperebene
 $D \subseteq H+1$ Hyperebene H heißt **Stützhyperebene** von C , falls
 $\mathbb{R}^m : P = \{x \in \mathbb{R}^n : Ax \leq b\} / P = \text{conv}\{x_1, \dots, x_k\}$ heißt **Polyto**
 $\in \mathbb{R}^n : x^T y \leq 1 \forall x \in A\} / C \subseteq \mathbb{R}^n$ heißt **Veget**, falls $\forall x, y$
 $\text{cone}\{y_1, \dots, y_s\} = \{y \mid y = \sum d_i y_i, d_i \geq 0\}$ / **LP in Standard**
proid: $A \in \mathbb{R}^{m \times n}$ pos. def., $x \in \mathbb{R}^n, E(A, x) = \{y \in \mathbb{R}^m : (y$
 pen **gemischte Strategien**, wobei u_i Wkheit, dass Zeile
 j wählt / Die Standardbasisvektoren $e_i, i = 1, \dots, n$
Wert der Bezahlung von Z an S ist $u \cdot A \cdot v$
 $A \in \mathbb{R}^{m \times n}$, geg. als Optimum des lin. Prog: $\max \mu$
 e Prog. zu: $\min \tau, u \geq 0, u e = \tau, u A \leq \tau e / G=(V,E)$ unger.
 . Werten, d.h. $\forall e, f \in M : e \cap f = \emptyset$ falls $e \neq f / M \subseteq E$ Mak
 a) P hat unger. Länge, b) Enden nicht von M überdeckt,
 r. heißt **bipartit**, falls $\exists U, W \subseteq V : V = U \cup W$ und $\forall e \in E : |e \cap U| = |e \cap W| = 1$ / **Matching M extrem**,
 $\forall M' \subseteq E$ Matching mit $|M'| = |M| : \omega(M') \leq \omega(M) / M \subseteq E$ Matching. **Längenfrist L** def. durch:
 $\forall e \in M : \exists e' \in M : e \cap e' \neq \emptyset$ / **Incidenzvektor χ_M** von M def. als: $\chi_M(e) = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$

Duality theory

primal $p^* = \sup \left\{ \langle c, x \rangle : x \in E, x \succeq_K 0, \langle a_j, x \rangle = b_j (j \in [m]) \right\}$

dual $d^* = \inf \left\{ b_1 y_1 + \dots + b_m y_m : y_1, \dots, y_m \in \mathbb{R}, y_1 a_1 + \dots + y_m a_m \succeq_{K^*} c \right\}$

weak duality $p^* \leq d^*$

strong duality if $d^* > -\infty$ and \exists strictly feasible dual solution y (i.e. $y_1 a_1 + \dots + y_m a_m - c \in \text{int } K^*$)

then $\exists x^*$ feasible for primal with $p^* = \langle c, x^* \rangle$ and $p^* = d^*$

(similarly with primal and dual interchanged)

optimality condition / complementary slackness Suppose primal and dual are both strictly feasible, suppose x primal feasible and y dual feasible, then (x, y) is optimal iff $\langle x, y_1 a_1 + \dots + y_m a_m - c \rangle = 0$