Optimization for lattices, packings, and coverings

Lecture 1

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1. Introduction to conic optimization
WHAT IS... LP and SDP?

**LP (linear programming)** Maximizing/minimizing a linear functional over a polyhedron

polyhedron intersection of finitely many linear half spaces

\[ = \mathbb{R}^n_+ \cap \text{affine subspace} \]

**SDP (semidefinite programming)** Maximizing/minimizing a linear functional over a spectrahedron

spectrahedron \( = \mathbb{S}^n_+ \cap \text{affine subspace} \)

\[ \left\{ (x, y, z) : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \in \mathbb{S}^3_+ \right\} \]

elliptope
1. Describe a **wide class** of convex optimization problems

2. LP and SDP can be solved **efficiently** (in theory and practice)

3. Duality theory gives **optimality criteria** and a **systematic** way to prove rigorous upper/lower bounds

4. LP and SDP can be used to prove that a **point configuration** is optimal or near optimal

5. Lots of other **applications**… combinatorial optimization, global polynomial optimization, engineering, machine learning, quantum information, game theory, …
General framework: Conic optimization

$E$ finite-dimensional Euclidean space with inner product $\langle x, y \rangle$

$K \subseteq E$ proper convex cone

$$\alpha K + \beta K \subseteq K \text{ for } \alpha, \beta \in \mathbb{R}_+, K \text{ full-dimensional, } K \text{ closed, } K \cap (-K) = \{0\}$$

$K$ gives partial ordering on $E$ by $x \geq_K y \iff x - y \in K$

$K$ is the domain of nonnegative elements

primal standard form of conic program

$$p^* = \sup \left\{ \langle c, x \rangle : x \in E, x \geq_K 0, \langle a_j, x \rangle = b_j \ (j \in [m]) \right\}$$

$x$ is the optimization variable
Important examples

Linear programming (LP)
\[ E = \mathbb{R}^n, \quad \langle x, y \rangle = x^T y, \quad K = \mathbb{R}_+^n \]

Second order cone programming (SOCP)
\[ E = \mathbb{R}^{n+1}, \quad \langle (x, s), (y, t) \rangle = x^T y + st, \quad K = \mathcal{L}^{n+1} = \{(x, s) : \|x\|_2 \leq s\} \]

Semidefinite programming (SDP)
\[ E = \mathbb{S}^n, \quad \langle X, Y \rangle = \text{Tr } XY, \quad K = \mathbb{S}_+^n \]

Determinant maximization (MAXDET)
\[ E = \mathbb{S}^n \times \mathbb{R}, \quad \langle (X, s), (Y, t) \rangle = \text{Tr } XY + st, \quad K = \mathcal{D}^{n+1} = \{(X, s) : X \in \mathbb{S}_+^n, s \geq 0, (\det X)^{1/n} \geq s\} \]

Polynomial optimization (POP)
\[ E = \mathbb{R}[x_1, \ldots, x_n]_d, \quad \langle f, g \rangle = \frac{1}{d!} f(\nabla)g, \quad K = P_{n,d} = \{f : f(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\}, \quad d \text{ even} \]
Dualization

**Primal standard form of conic program**

\[ p^* = \sup \left\{ \langle c, x \rangle : x \in E, x \geq_K 0, \langle a_j, x \rangle = b_j \,(j \in [m]) \right\} \]

**Dual standard form of conic program**

\[ d^* = \inf \left\{ b_1 y_1 + \cdots + b_m y_m : y_1, \ldots, y_m \in \mathbb{R}, y_1 a_1 + \cdots + y_m a_m \geq_{K^*} c \right\} \]

where \( K^* = \{ y \in E : \langle x, y \rangle \geq 0 \text{ for all } x \in K \} \) is the dual cone of \( K \)

Bipolar theorem: \( (K^*)^* = K \)

**Important examples**

- **Self dual cones:** \( (\mathbb{R}^n_+)*) = \mathbb{R}^n_+ \quad (\mathcal{L}^{n+1})^* = \mathcal{L}^{n+1} \quad (\mathbb{S}_n^+)^* = \mathbb{S}_n^+ \)

  Koecher-Vinberg classification of symmetric cones (real Euclidean Jordan algebras)

- **Non self dual cones:**
  \[ (\mathcal{D}^{n+1})^* = \left\{ (Y, t) \in \mathbb{S}_n^+ \times \mathbb{R} : (\det Y)^{1/n} \geq -\frac{t}{n} \right\} \]
  \[ (P_{n,d})^* = Q_{n,d} = \left\{ (\alpha_1^T x)^d + \cdots + (\alpha_r^T x)^d : \alpha_1, \ldots, \alpha_r \in \mathbb{R}^n, r \in \mathbb{N} \right\} \]

  sums of even powers of linear forms
# The dualization cheat sheet

<table>
<thead>
<tr>
<th>maximize</th>
<th>minimize</th>
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<tbody>
<tr>
<td>variable ( \geq_K 0 )</td>
<td>( \leq_{K^*} ) constraint</td>
</tr>
<tr>
<td>variable ( \leq_K 0 )</td>
<td>( \leq_{K^*} ) constraint</td>
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<td>unconstrained variable</td>
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<td>( = ) constraint</td>
<td>( \leq ) constraint</td>
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<td>( \geq ) constraint</td>
<td>( \geq 0 ) variable</td>
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<td>right-hand side</td>
<td>objective function</td>
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<td>objective function</td>
<td>right-hand side</td>
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Duality theory

**primal** \( p^* = \sup \left\{ \langle c, x \rangle : x \in E, x \geq_K 0, \langle a_j, x \rangle = b_j \ (j \in [m]) \right\} \)

**dual** \( d^* = \inf \left\{ b_1y_1 + \cdots + b_my_m : y_1, \ldots, y_m \in \mathbb{R}, y_1a_1 + \cdots + y_mA_m \geq_{K^*} c \right\} \)

**weak duality** \( p^* \leq d^* \)

**strong duality** if \( d^* > -\infty \) and \( \exists \) strictly feasible dual solution \( y \) (i.e. \( y_1a_1 + \cdots + y_mA_m - c \in \text{int} \ K^* \))

then \( \exists x^* \) feasible for primal with \( p^* = \langle c, x^* \rangle \) and \( p^* = d^* \)

(similarly with primal and dual interchanged)

**optimality condition / complementary slackness** Suppose primal and dual are both strictly feasible, suppose \( x \) primal feasible and \( y \) dual feasible, then \( (x, y) \) is optimal iff \( \langle x, y_1a_1 + \cdots + y_mA_m - c \rangle = 0 \)