Semidefinite programming hierarchies for packing and energy minimization (2/4)

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Topics for the four talks

1. Sums-of-squares hierarchies for polynomial optimization
2. **Moment hierarchies for polynomial optimization**
3. Packing problems
4. Energy minimization problems

Yesterday

$$\begin{aligned} P &= \inf \left\{ p(x) : x \in S(Q) \right\} \\ &= \sup \left\{ M \in \mathbb{R} : p(x) - M \geq 0 \text{ for } x \in S(Q) \right\} \\ &\geq \sup \left\{ M \in \mathbb{R} : p - M \in \mathcal{M}_t(Q) \right\} = P_t \end{aligned}$$

Semialgebraic set:

$$S(Q) = \{x \in \mathbb{R}^n : q(x) \geq 0 \text{ for } q \in Q\}$$

Quadratic module:

$$\mathcal{M}(Q) = \mathrm{cone}\left\{ qs^2 : q \in \{1\} \cup Q,\, s \in \mathbb{R}[x_1, \ldots, x_n] \right\}$$

Truncated quadratic module:

$$\mathcal{M}_t(Q) = \mathrm{cone}\left\{ qs^2 : q \in \{1\} \cup Q,\, s \in \mathbb{R}[x_1, \ldots, x_n],\, \deg(qs^2) \leq 2t \right\}$$

Convexification

Let $\mathcal{P}(S(Q))$ be the set of probability measures supported on $S(Q)$.

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Write the polynomial optimization problem

$$P = \inf \left\{ p(x) : x \in S(Q) \right\}$$

as

$$\inf \left\{ \int_{S(Q)} p(x)\, d\mu(x) : \mu \in \mathcal{P}(S(Q)) \right\}.$$

Moment and localizing matrices

Notation: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$

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Lemma If $\mu \in \mathcal{P}(S(Q))$, then $M^q(\mu) \succeq 0$ for every $q \in \{1\} \cup Q$.

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Lemma If $\mu \in \mathcal{P}(S(Q))$, then $M^q(\mu) \succeq 0$ for every $q \in \{1\} \cup Q$.
Proof For C a finite set of exponend vectors and $c \in \mathbb{R}^C$,

$$\sum_{\alpha,\beta \in C} c_\alpha c_\beta M^q(\mu)_{\alpha,\beta} = \sum_{\alpha,\beta \in C} c_\alpha c_\beta \int_{S(Q)} x^{\alpha+\beta}\, q(x) d\mu(x)$$

$$= \int_{S(Q)} \left(\sum_\alpha c_\alpha x^\alpha \right)^2 q(x) d\mu(x) \geq 0. \qquad \square$$

Relaxations

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for $|\alpha|, |\beta| \leq t - \lfloor \deg(q)/2 \rfloor$

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Exercise 5

Define y by $y_\alpha = \int_{S(Q)} x^\alpha \, d\mu(x)$ for $|\alpha| \leq 2t$. Show that

$$M_t^q(y)_{\alpha,\beta} = M^q(\mu)_{\alpha,\beta}$$

for $|\alpha|, |\beta| \leq t - \lfloor \deg(q)/2 \rfloor$

Relaxations

The Lasserre moment hierarchy:

$$P_t = \inf\left\{ \sum_{\alpha} p_\alpha y_\alpha : y_0 = 1,\ M_t^q(y) \succeq 0 \text{ for } q \in \{1\} \cup Q \right\}$$

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Lemma
If $t \geq \lceil \deg(h)/2 \rceil$ for $h \in \{p\} \cup Q$, then $P_t \leq P$.
Proof
For $x \in S(Q)$, define y by $y_\alpha = x^\alpha$ for $|\alpha| \leq 2t$.

Semidefinite programming duality

Primal and dual semidefinite programs:

$$P_t = \inf\Big\{ \sum_{\alpha} p_\alpha y_\alpha : y_0 = 1,\ M_t^q(y) \succeq 0 \text{ for } q \in \{1\} \cup Q \Big\}$$

$$P_t^{\mathrm{sos}} = \sup\big\{ c \in \mathbb{R} : p - c \in \mathcal{M}_t(Q) \big\}$$

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Weak duality:

$$\begin{aligned}
c &= \sum_\alpha (c-p)_\alpha y_\alpha + \sum_\alpha p_\alpha y_\alpha \\
&= \sum_\alpha - \left(\sum_q q(x) \langle A_q, m(x) m(x)^{\mathsf{T}} \rangle \right)_\alpha y_\alpha + \sum_\alpha p_\alpha y_\alpha \\
&= -\sum_q \left\langle A_q, \sum_\alpha \left(q(x) m(x) m(x)^{\mathsf{T}} \right)_\alpha y_\alpha \right\rangle + \sum_\alpha p_\alpha y_\alpha \\
&= -\sum_q \langle A_q, M_t^q(y) \rangle + \sum_\alpha p_\alpha y_\alpha \leq \sum_\alpha p_\alpha y_\alpha.
\end{aligned}$$

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\end{aligned}$$

If $S(Q)$ is full dimensional, then strong duality holds: $P_t = P_t^{\mathrm{sos}}$

Asymptotic convergence

Strong duality shows $P_t \to P$ holds whenever $\mathcal{M}(Q)$ is archimedean (since we know $P_t^{\mathrm{sos}} \to P$ by Putinar's theorem)

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Yes, we first show $P_t \to P_\infty$ as $t \to \infty$, and then we show $P_\infty = P$

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Note that $P_t \leq P_{t+1} \leq ... \leq P$

Asymptotic convergence

Theorem If $\mathcal{M}(Q)$ is archimedean, then $P_t \to P_\infty$.

Proof sketch

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Proof sketch

Fix $\epsilon > 0$ and let y^t be feasible for P_t with $\sum_\alpha p_\alpha y^t_\alpha \leq P_t + \epsilon$. Since $\mathcal{M}(Q)$ is archimedean, there is a C with $C - \sum_{i=1}^n x_i^2 \in \mathcal{M}(Q)$. Since $M_t^q(y) \succeq 0$ for all $q \in \{1\} \cup Q$, we get $\sum_\alpha h_\alpha y^t_\alpha \geq 0$ for all $h \in \mathcal{M}_t(Q)$. From this we get $y^t_\alpha \leq C^{|\alpha|/2} y_0 = C^{|\alpha|/2}$.

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Define the linear functional L^t on $\mathbb{R}[x_1, \ldots, x_n]$ by

$$L^t(x^\alpha) = \begin{cases} y^t_\alpha / C^{|\alpha|/2} & \text{if } |\alpha| \leq 2t, \\ 0 & \text{otherwise.} \end{cases}$$

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The functional L^t lies in the unit ball, which is closed in the weak* topology by the Banach-Alaoglu theorem. This shows L^t has a pointwise limit point L, and thus y^t has a limit point y which is feasible for P. This gives $P_\infty \leq \lim_{t\to\infty} P_t + \epsilon$.

Asymptotic convergence

Riesz-Haviland Theorem Let y be a vector indexed by all exponent vectors and let $K \subseteq \mathbb{R}^n$ be closed. There exists a finite Borel measure μ on K such that $y_\alpha = \int_K x^\alpha \, d\mu(x)$ for all α if and only if $\sum_\alpha h_\alpha y_\alpha \geq 0$ for all polynomials h that are nonnegative on K.

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Theorem If $\mathcal{M}(Q)$ is archimedean, then $P_\infty = P$

Proof

If $h \geq 0$ on $S(Q)$, then $h + \epsilon > 0$ on $S(Q)$, so by Putinar's theorem $h + \epsilon \in \mathcal{M}(Q)$. The conditions $y_0 = 1$ and $M_t^q(y) \succeq 0$ then imply $\sum_\alpha y_\alpha h_\alpha + \varepsilon \geq 0$. Since $\epsilon > 0$ was arbitrary we get $\sum_\alpha y_\alpha h_\alpha \geq 0$ so by the Riesz-Haviland theorem there is a probability measure μ supported on $S(Q)$ with $\mu(p) = \sum_\alpha p_\alpha y_\alpha$.

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Alternatively we can avoid using Putinar's theorem and instead work fully 'on the moment side' by a Gelfand–Naimark–Segal construction and the Gelfand and Riesz representation theorems.

Strengthening the moment hierarchy

We defined the localizing matrices by

$$M_t^q(y)_{\alpha,\beta} = \sum_\gamma q_\gamma y_{\gamma+\alpha+\beta}$$

for $|\alpha|, |\beta| \leq t - \lfloor \deg(q)/2 \rfloor$

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Instead we can define the partial matrix

$$M_t^q(y)_{\alpha,\beta} = \begin{cases} \sum_\gamma q_\gamma y_{\gamma+\alpha+\beta} & \text{if } |\alpha+\beta| \leq 2t - \deg(q), \\ \text{unspecified} & \text{otherwise} \end{cases}$$

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The constraint $M_t^q(y) \succeq 0$ now means: There is a positive semidefinite completion of the partial matrix $M_t^q(y)$

Specialization to $0/1$ polynomial optimization problems

Exercise 6

Show that if Q contains the polynomials $x_i(1-x_i)$ and $-x_i(1-x_i)$ for $i = 1, \ldots, n$, then we may assume the exponent vector of each term of all other polynomials in $\{p\} \cup Q$ lies in $\{0,1\}^n$, and if y is a feasible vector to P_t, then $y_\alpha = y_{\bar{\alpha}}$, where $\bar{\alpha}$ is obtained from α by replacing all nonzero entries by ones.

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This shows the moment hierarchy can be written as

$$P_t = \inf\Big\{ \sum_S p_S y_S : y_\emptyset = 1,\, M_t^q(y) \succeq 0 \text{ for } q \in \{1\} \cup Q \Big\}$$

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Here $p(x) = \sum_{S \subseteq \{1,\ldots,n\}} p_S \prod_{i \in S} x_i$ and

$$M_t^q(y)_{J,J'} = \begin{cases} \sum_S q_S \, y_{J \cup J' \cup S} & \text{if } |J \cup J'| \leq 2t - \deg(q), \\ \text{unspecified} & \text{otherwise} \end{cases}$$

for $|J|, |J'| \leq t$

Specialization to the independent set problem

$0/1$ polynomial optimization problem for the independent set problem:

$$P = \max\Big\{ \sum_{i=1}^{n} x_i : x_i(1-x_i) = 0 \text{ for } i = 1, \ldots, n,\, x_i + x_j \leq 1 \text{ if } i \sim j \Big\}$$

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The moment hierarchy thus reduces to

$$P_t = \sup\Big\{ \sum_{i=1}^{n} y_{\{i\}} : y_\emptyset = 1, \; M_t^1(y) \succeq 0, \; M_t^{1-x_i-x_j}(y) \succeq 0 \text{ for } i \sim j \Big\}$$

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Exercise 7 Assume we use the strengthened version of the moment hierarchy. Show that if y is feasible for P_t for $t \geq 1$, we have $y_S = 0$ for all dependent sets.

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(That these hierarchies are the same is shown by Laurent in 2001)

Tomorrow

Adapt this to infinite graphs.

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P_1 for the independent set problem is (almost) the Delsarte bound.