Modular forms and their applications III

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Jacobi's theta function

Let us start by looking at unary theta functions.

$$\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \dots$$

Clearly, this function is 1-periodic.

Proposition

The function $\theta(\tau)$ satisfies

$$\theta\Big(-\frac{1}{4\tau}\Big) = \sqrt{2\tau/i}\,\theta(\tau)\,, \quad \tau \in \mathbb{H}$$

Poisson summation formula

To prove this we recall the Poisson summation formula.
Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function, and let $\widehat{f}(\xi)$ be its Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$$

Then

$$\sum_{n \in \mathbb{Z}} f(n + x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$$

The standard way to prove this is to consider the left-hand side as a function on $\mathbb{R}/\mathbb{Z}$ and look at its Fourier series.

Jacobi's theta function

Proposition

The function $\theta(\tau)$ *satisfies*

$$\theta\Big(-\frac{1}{4\tau}\Big) = \sqrt{2\tau/i}\,\theta(\tau)\,, \quad \tau \in \mathbb{H}$$

Proof.

Let $f(x) = e^{-\pi t x^2}$, so that $\widehat{f}(\xi) = t^{-1/2} e^{-\pi t^{-1} x^2}$. Then by the Poisson summation formula for $x = 0$

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = t^{-1/2} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/t}\,.$$

This is equivalent to

$$\theta\Big(-\frac{1}{4\tau}\Big) = \sqrt{2\tau/i}\,\theta(\tau)$$

for τ on the imaginary axis, and by the identity theorem we get it for all $\tau \in \mathbb{H}$. □

Application: Jacobi's identities for sums of squares

We have proved that θ is modular with respect to T and $W_4 = \left(\begin{smallmatrix} 0 & 1 \\ -4 & 0 \end{smallmatrix}\right)$.
Note that $W_4 T W_4 = \left(\begin{smallmatrix} -4 & 0 \\ 16 & -4 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ -4 & 1 \end{smallmatrix}\right)$.
One can show what T and $\left(\begin{smallmatrix} -1 & 0 \\ 4 & -1 \end{smallmatrix}\right)$ generate the subgroup $\Gamma_0(4)$.

Exercise

Show that $\theta^4(\tau) \in M_2(\Gamma_0(4))$.

Note that

$$\theta^k(\tau) = 1 + \sum_{n \geq 1} r_k(n) q^n \,,$$

where $r_k(n)$ is the number of representations of n as a sum of k squares.

Application: Jacobi's identities for sums of squares

Since $\theta^4(\tau) \in M_2(\Gamma_0(4))$ and the latter space is 2-dimensional, spanned by $G_2(\tau) - 2G_2(2\tau)$ and $G_2(\tau) - 4G_2(4\tau)$, one gets

$$\theta^4(\tau) = 8(G_2(\tau) - 4G_2(4\tau))$$

From this we get Jacobi's identity

$$r_4(n) = 8 \sum_{4 \nmid d | n,} d \,, \quad n \geq 1$$

and also a proof of Lagrange's four-square theorem.

Similarly, the space $M_4(\Gamma_0(4))$ is spanned by $G_4(\tau)$, $G_4(2\tau)$, and $G_4(4\tau)$, which implies

$$r_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3$$

Exercise: multiplicativity of $r_k(n)$

The explicit formulas for $k = 4$ and $k = 8$ show that the sequences $\{\frac{r_4(n)}{8}\}$ and $\{\frac{r_8(n)}{16}\}$ are both multiplicative, i.e., they satisfy $a(mn) = a(m)a(n)$ for $(m, n) = 1$.

Exercise

Let $r_k(n)$ be the number of representations of n as a sum of squares of k integers. Show that the sequence $\{\frac{r_k(n)}{2k}\}_{n\geq 1}$ is multiplicative if and only if $k \in \{1, 2, 4, 8\}$.

Theta functions of lattices

Recall that a lattice $\Lambda \subset \mathbb{R}^d$ is a discrete subgroup of rank d.

Λ is called integral if

$$\langle x, y \rangle \in \mathbb{Z} \quad \text{for all} \quad x, y \in \Lambda\,.$$

Λ is called even if

$$|x|^2 \in 2\mathbb{Z} \quad \text{for all} \quad x \in \Lambda\,.$$

We define the dual lattice by

$$\Lambda^* = \{\xi \in \mathbb{R}^d \mid \langle x, \xi \rangle \in \mathbb{Z} \quad \text{for all } x \in \Lambda\}$$

A lattice is called unimodular if $\Lambda = \Lambda^*$.
For integral lattices this is equivalent to $\mathrm{vol}(\mathbb{R}^d/\Lambda) = 1$.

Poisson summation formula for lattices

For any lattice $\Lambda \subset \mathbb{R}^d$ and any Schwartz function $f\colon \mathbb{R}^d \to \mathbb{C}$ we have

$$\sum_{v\in\Lambda} f(x+v) = \frac{1}{|\Lambda|} \sum_{\xi\in\Lambda^*} \widehat{f}(\xi) e^{2\pi i \langle x,\xi\rangle}$$

Here we define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,\xi\rangle} dx$$

The proof is more or less the same as in one dimension: consider the Fourier series of the left hand side as a function on the torus $\mathbb{R}^d/\Lambda$.

Theta functions of even unimodular lattices

Proposition

Let $\Lambda \subseteq \mathbb{R}^d$ be an even unimodular lattice. Then $8|d$ and the function

$$\Theta_\Lambda(\tau) = \sum_{v \in \Lambda} q^{|v|^2/2}$$

is a modular form of weight $d/2$ for $\mathrm{PSL}_2(\mathbb{Z})$.

Proof

The Poisson summation formula applied to $e^{\pi i \tau |x|^2}$ shows that

$$\sum_{v \in \Lambda} e^{\pi i \tau |v|^2} = \frac{1}{|\Lambda|}(\tau/i)^{-d/2} \sum_{v \in \Lambda^*} e^{\pi i (-1/\tau)|v|^2}$$

Theta functions of even unimodular lattices

Proof (cont.)

Equivalently,

$$\Theta_\Lambda(\tau) = \frac{1}{|\Lambda|}(\tau/i)^{-d/2}\Theta_{\Lambda^*}(-1/\tau)$$

Since $\Lambda = \Lambda^*$ and $|\Lambda| = 1$, and using periodicity we get that

$$\Theta_\Lambda(1 - 1/\tau)(\tau/i)^{-d/2} = \Theta_\Lambda(\tau)$$

Since $\tau \mapsto 1 - 1/\tau$ is cyclic of order 3, this implies

$$(\tau/i)^{-d/2}((1 - 1/\tau)/i)^{-d/2}(1/i(1-\tau))^{-d/2} = 1\,.$$

On the other hand one can directly check that the left hand side equals $e^{-\frac{2\pi i d}{8}}$. This implies $8|d$ and hence also that $\Theta_\Lambda \in M_{d/2}(\Gamma_1)$. □

Example: theta series of the E_8 lattice

Since the E_8-lattice

$$\Lambda_8 = \{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \cup (1/2 + \mathbb{Z})^8 \mid x_1 + \cdots + x_8 = 0 \pmod 2\}$$

is even and unimodular, we have $\Theta_{\Lambda_8}(\tau) \in M_4(\mathrm{SL}_2(\mathbb{Z}))$, and therefore

$$\Theta_{\Lambda_8}(\tau) = E_4(\tau)$$

In particular, the number of vectors of length $\sqrt{2n}$ in Λ_8 is equal to

$$r_{\Lambda_8}(n) = 240\sigma_3(n)$$

Application: even unimodular lattices

For any even unimodular lattice we can get an approximation to $r_\Lambda(n)$, the number of vectors of square length $2n$, as n goes to infinity.
For this we need the following estimate for coefficients of cusp forms due to Hecke.

Proposition

Let $f(\tau) = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_1)$. Then $|a_n| \ll n^{k/2}$.

Proof.

Consider the function $F(\tau) = |f(\tau)|y^{k/2}$. It is Γ_1 invariant, and goes to 0 as $\tau \to i\infty$, and therefore it is bounded by some constant C. Then

$$|a_n| = \left| \int_{-1/2+i/n}^{1/2+i/n} f(\tau) q^{-n} d\tau \right| \leq Ce^{2\pi} n^{k/2}$$

□

Application: even unimodular lattices

Proposition

If $\Lambda \subset \mathbb{R}^{8l}$ is an even unimodular lattice, then

$$r_\Lambda(n) \sim -\frac{8l}{B_{4l}}\sigma_{4l-1}(n)\,, \quad n \to \infty$$

Proof.

Since $\Theta_\Lambda \in M_{4l}(\Gamma_1)$ and $\Theta_\Lambda(\tau) = 1 + O(q)$, we have

$$\Theta_\Lambda(\tau) = E_{4l}(\tau) + f(\tau)\,,$$

where $f = \sum_{n\geq 1} a(n)q^n$ is a cusp form. Since $\sigma_{4l-1}(n) \geq n^{4l-1}$ and $a_n = O(n^{2l})$, we get the claim. □

Application: extremal lattices

Note that since Θ_Λ belongs to $M_{4l}(\Gamma_1)$, it is uniquely determined by $m = \dim M_{4l}(\Gamma_1)$ first coefficients. An even unimodular lattice Λ is called extremal if

$$\Theta_\Lambda(\tau) = 1 + O(q^m)$$

In this case we define a_l and b_l by $\Theta_\Lambda(\tau) = 1 + a_l q^m + b_l q^{m+1} + O(q^{m+2})$.

Theorem (Siegel)

For all $l \geq 1$ the coefficient a_l is positive. In particular, any even unimodular lattice has a nonzero vector of length $\leq \sqrt{2m}$.

Theorem (Mallows–Odlyzko–Sloane)

For all sufficiently large l the coefficient b_l is negative. In particular, there exists $C > 0$ such that there are no extremal lattices in $\mathbb{R}^d$ for $d > C$.

The idea is to calculate a_l, b_l using Lagrange inversion formula, and get asymptotic formulas.
For example, one can show that

$$a_{3k} = \frac{3k}{k+1}[q^k]\Big(E_4^2\frac{dE_4}{dq}\prod_{n\geq 1}(1-q^n)^{-24(k+1)}\Big)$$

which immediately shows that $a_{3k} > 0$ is positive.

The proof of the claim for b_l is more involved but is based on a similar computation.

Theta functions with polynomial weights

Let Λ be an even unimodular lattice in $\mathbb{R}^d$, and let $P(x)$ be a homogeneous harmonic polynomial in d variables of degree $m > 0$. We define

$$\Theta_{\Lambda,P}(\tau) = \sum_{v\in\Lambda} P(v)q^{|v|^2/2}$$

Proposition

Under the above conditions $\Theta_{\Lambda,P} \in S_{d/2+m}(\Gamma_1)$.

Proof.

The claim follows by applying the Poisson summation formula to $f(x) = P(x)e^{\pi i\tau|x|^2}$, using the fact that $\widehat{f}(x) = i^m(\tau/i)^{-d/2-m}P(x)e^{\pi i(-1/\tau)|x|^2}$. □

Application: spherical designs

The fact that we can consider theta functions weighted by harmonic polynomials can be used to analyze the strength of a lattice shell as a spherical design.

Recall that a spherical t-design is a configuration of N points $x_1, \ldots, x_N \in S^{d-1}$ such that for any polynomial $P \in \mathbb{R}[t_1, \ldots, t_d]$ of degree $\leq t$ one has

$$\int_{S^{d-1}} P(x) d\mu(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i) \qquad (*)$$

One can show that it is enough to verify (*) for homogeneous harmonic polynomials.

Application: spherical designs

Proposition

The set of vectors of length $\sqrt{2n}$ in the E_8 lattice forms a spherical 7-design.

Proof.

By the above theorem, if P is harmonic of degree d, then $\Theta_{\Lambda,P} \in S_{4+d}(\Gamma_1)$, and hence it vanishes for $d < 8$. □

Note that since $S_{14}(\Gamma_1) = 0$ the average of any harmonic polynomial of degree 10 over the set of vectors in E_8 of length $\sqrt{2n}$ is also zero.

Application (?): Lehmer's conjecture

Lehmer has conjectured that the Fourier coefficients of $\Delta(\tau)$ are non-zero.

One can reformulate this conjecture in more geometric terms as follows.

Proposition

Lehmer's conjecture is equivalent to the following statement: for all $n \geq 1$ the set of vectors of length $\sqrt{2n}$ in the E_8 lattice does not form an 8-design.

To see this, note that the theta function of Λ_8 weighted by a harmonic polynomial of degree 8 lies in $S_{12}(\Gamma_1)$, which is spanned by $\Delta(\tau)$.

By the previous remark, we can also replace "8-design" by "9-design", "10-design", or even "11-design".

Application: tight spherical designs

A spherical t-design X on S^{n-1} is called tight if

$$|X| = \binom{n-1+\lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} + \binom{n-1+\lfloor \frac{t-1}{2} \rfloor}{\lfloor \frac{t-1}{2} \rfloor}$$

Bannai and Damerell have proved that for $n > 2$ tight designs can exist only for $t \in \{1, 2, 3, 4, 5, 7, 11\}$.

For $t = 1, 2, 3$ there is a simple classification, and for $t = 11$ there is only one such design, namely the 196560 shortest vectors of the Leech lattice.

For $t = 4, 5, 7$ there are only partial results: the only known examples are

$$\begin{aligned} t = 4: &\qquad n = 6, 22 \\ t = 5: &\qquad n = 3, 7, 23 \\ t = 7: &\qquad n = 8, 23 \end{aligned}$$

Application: tight spherical designs

It is known that for a tight 5-design, if $n > 3$, then $n = (2m+1)^2 - 2$ for some $m \geq 1$.
The two known examples correspond to $m = 1, 2$.
Bannai, Munemasa, and Venkov have proved that tight 5-designs do not exist for $m = 3, 4$ by analyzing the lattice generated by X.

In particular, to prove that a tight 5-design in $\mathbb{R}^{47}$ cannot exist they have constructed from it an even unimodular lattice $\Lambda \subset \mathbb{R}^{48}$ with

$$\Theta_\Lambda(\tau) = 1 + 2q + 4512q^2 + 1271256q^3 + \dots$$

Exercise

Show that such a lattice does not exist.