Let us start by looking at unary theta functions.

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \ldots$$

Clearly, this function is 1-periodic.

**Proposition**

*The function $\theta(\tau)$ satisfies*

$$\theta\left(-\frac{1}{4\tau}\right) = \sqrt{2\tau/i} \theta(\tau), \quad \tau \in \mathbb{H}$$
To prove this we recall the Poisson summation formula. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a Schwartz function, and let $\hat{f}(\xi)$ be its Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$$

Then

$$\sum_{n \in \mathbb{Z}} f(n + x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

The standard way to prove this is to consider the left-hand side as a function on $\mathbb{R}/\mathbb{Z}$ and look at its Fourier series.
Proposition

The function $\theta(\tau)$ satisfies

$$\theta\left(-\frac{1}{4\tau}\right) = \sqrt{2\tau/i} \theta(\tau), \quad \tau \in \mathbb{H}$$

Proof.

Let $f(x) = e^{-\pi tx^2}$, so that $\hat{f}(\xi) = t^{-1/2} e^{-\pi t^{-1}x^2}$. Then by the Poisson summation formula for $x = 0$

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = t^{-1/2} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / t}.$$  

This is equivalent to

$$\theta\left(-\frac{1}{4\tau}\right) = \sqrt{2\tau/i} \theta(\tau)$$

for $\tau$ on the imaginary axis, and by the identity theorem we get it for all $\tau \in \mathbb{H}$. 

\qed
Application: Jacobi’s identities for sums of squares

We have proved that $\theta$ is modular with respect to $T$ and $W_4 = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$.
Note that $W_4TW_4 = \begin{pmatrix} -4 & 0 \\ 16 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$.

One can show what $T$ and $\begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}$ generate the subgroup $\Gamma_0(4)$.

Exercise

Show that $\theta^4(\tau) \in M_2(\Gamma_0(4))$.

Note that

$$\theta^k(\tau) = 1 + \sum_{n \geq 1} r_k(n)q^n,$$

where $r_k(n)$ is the number of representations of $n$ as a sum of $k$ squares.
Application: Jacobi’s identities for sums of squares

Since $\theta^4(\tau) \in M_2(\Gamma_0(4))$ and the latter space is 2-dimensional, spanned by $G_2(\tau) - 2G_2(2\tau)$ and $G_2(\tau) - 4G_2(4\tau)$, one gets

$$\theta^4(\tau) = 8(G_2(\tau) - 4G_2(4\tau))$$

From this we get Jacobi’s identity

$$r_4(n) = 8 \sum_{4|d|n} d, \quad n \geq 1$$

and also a proof of Lagrange’s four-square theorem.

Similarly, the space $M_4(\Gamma_0(4))$ is spanned by $G_4(\tau)$, $G_4(2\tau)$, and $G_4(4\tau)$, which implies

$$r_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3$$
Exercise: multiplicativity of $r_k(n)$

The explicit formulas for $k = 4$ and $k = 8$ show that the sequences $\left\{ \frac{r_4(n)}{8} \right\}$ and $\left\{ \frac{r_8(n)}{16} \right\}$ are both multiplicative, i.e., they satisfy $a(mn) = a(m)a(n)$ for $(m, n) = 1$.

Exercise

Let $r_k(n)$ be the number of representations of $n$ as a sum of squares of $k$ integers. Show that the sequence $\left\{ \frac{r_k(n)}{2^k} \right\}_{n \geq 1}$ is multiplicative if and only if $k \in \{1, 2, 4, 8\}$.
Recall that a lattice $\Lambda \subset \mathbb{R}^d$ is a discrete subgroup of rank $d$.

$\Lambda$ is called integral if

$$\langle x, y \rangle \in \mathbb{Z} \quad \text{for all} \quad x, y \in \Lambda.$$  

$\Lambda$ is called even if

$$|x|^2 \in 2\mathbb{Z} \quad \text{for all} \quad x \in \Lambda.$$  

We define the dual lattice by

$$\Lambda^* = \{ \xi \in \mathbb{R}^d \mid \langle x, \xi \rangle \in \mathbb{Z} \quad \text{for all} \quad x \in \Lambda \}.$$  

A lattice is called unimodular if $\Lambda = \Lambda^*$.

For integral lattices this is equivalent to $\text{vol}(\mathbb{R}^d/\Lambda) = 1$.  

For any lattice $\Lambda \subset \mathbb{R}^d$ and any Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$ we have

$$\sum_{v \in \Lambda} f(x + v) = \frac{1}{|\Lambda|} \sum_{\xi \in \Lambda^*} \hat{f}(\xi)e^{2\pi i \langle x, \xi \rangle}$$

Here we define

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i \langle x, \xi \rangle} dx$$

The proof is more or less the same as in one dimension: consider the Fourier series of the left hand side as a function on the torus $\mathbb{R}^d / \Lambda$. 
Proposition

Let $\Lambda \subseteq \mathbb{R}^d$ be an even unimodular lattice. Then $8|d$ and the function

$$\Theta_\Lambda(\tau) = \sum_{v \in \Lambda} q^{|v|^2/2}$$

is a modular form of weight $d/2$ for $\text{PSL}_2(\mathbb{Z})$.

Proof

The Poisson summation formula applied to $e^{\pi i \tau |x|^2}$ shows that

$$\sum_{v \in \Lambda} e^{\pi i \tau |v|^2} = \frac{1}{|\Lambda|} (\tau/i)^{-d/2} \sum_{v \in \Lambda^*} e^{\pi i (-1/\tau)|v|^2}$$
Equivalently,
\[ \Theta_\Lambda(\tau) = \frac{1}{|\Lambda|}(\tau/i)^{-d/2}\Theta_{\Lambda^*}(-1/\tau) \]

Since \( \Lambda = \Lambda^* \) and \(|\Lambda| = 1\), and using periodicity we get that
\[ \Theta_\Lambda(1 - 1/\tau)(\tau/i)^{-d/2} = \Theta_\Lambda(\tau) \]

Since \( \tau \mapsto 1 - 1/\tau \) is cyclic of order 3, this implies
\[ (\tau/i)^{-d/2}((1 - 1/\tau)/i)^{-d/2}(1/i(1 - \tau))^{-d/2} = 1. \]

On the other hand one can directly check that the left hand side equals \( e^{-\frac{2\pi id}{8}} \). This implies \( 8|d \) and hence also that \( \Theta_\Lambda \in M_{d/2}(\Gamma_1). \)
Example: theta series of the $E_8$ lattice

Since the $E_8$-lattice

$$\Lambda_8 = \{ (x_1, \ldots, x_8) \in \mathbb{Z}^8 \cup (1/2 + \mathbb{Z})^8 \mid x_1 + \cdots + x_8 = 0 \pmod{2} \}$$

is even and unimodular, we have $\Theta_{\Lambda_8}(\tau) \in M_4(SL_2(\mathbb{Z}))$, and therefore

$$\Theta_{\Lambda_8}(\tau) = E_4(\tau)$$

In particular, the number of vectors of length $\sqrt{2n}$ in $\Lambda_8$ is equal to

$$r_{\Lambda_8}(n) = 240\sigma_3(n)$$
For any even unimodular lattice we can get an approximation to $r_\Lambda(n)$, the number of vectors of square length $2n$, as $n$ goes to infinity. For this we need the following estimate for coefficients of cusp forms due to Hecke.

**Proposition**

Let $f(\tau) = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma_1)$. Then $|a_n| \ll n^{k/2}$.

**Proof.**

Consider the function $F(\tau) = |f(\tau)|y^{k/2}$. It is $\Gamma_1$ invariant, and goes to 0 as $\tau \to i\infty$, and therefore it is bounded by some constant $C$. Then

$$|a_n| = \left| \int_{-1/2+i/n}^{1/2+i/n} f(\tau)q^{-n}d\tau \right| \leq Ce^{2\pi n^{k/2}}$$
Application: even unimodular lattices

Proposition

If $\Lambda \subset \mathbb{R}^{8l}$ is an even unimodular lattice, then

$$r_\Lambda(n) \sim -\frac{8l}{B_{4l}} \sigma_{4l-1}(n), \quad n \to \infty$$

Proof.

Since $\Theta_\Lambda \in M_{4l}(\Gamma_1)$ and $\Theta_\Lambda(\tau) = 1 + O(q)$, we have

$$\Theta_\Lambda(\tau) = E_{4l}(\tau) + f(\tau),$$

where $f = \sum_{n \geq 1} a(n)q^n$ is a cusp form. Since $\sigma_{4l-1}(n) \geq n^{4l-1}$ and $a_n = O(n^{2l})$, we get the claim.
Note that since $\Theta_\Lambda$ belongs to $M_{4l}(\Gamma_1)$, it is uniquely determined by $m = \dim M_{4l}(\Gamma_1)$ first coefficients. An even unimodular lattice $\Lambda$ is called extremal if

$$\Theta_\Lambda(\tau) = 1 + O(q^m)$$

In this case we define $a_l$ and $b_l$ by $\Theta_\Lambda(\tau) = 1 + a_l q^m + b_l q^{m+1} + O(q^{m+2})$.

**Theorem (Siegel)**

For all $l \geq 1$ the coefficient $a_l$ is positive. In particular, any even unimodular lattice has a nonzero vector of length $\leq \sqrt{2m}$.

**Theorem (Mallows–Odlyzko–Sloane)**

For all sufficiently large $l$ the coefficient $b_l$ is negative. In particular, there exists $C > 0$ such that there are no extremal lattices in $\mathbb{R}^d$ for $d > C$. 
The idea is to calculate $a_l, b_l$ using Lagrange inversion formula, and get asymptotic formulas.

For example, one can show that

$$a_{3k} = \frac{3k}{k+1} [q^k] \left( E_4^2 \frac{dE_4}{dq} \prod_{n \geq 1} (1 - q^n)^{-24(k+1)} \right)$$

which immediately shows that $a_{3k} > 0$ is positive.

The proof of the claim for $b_l$ is more involved but is based on a similar computation.
Let $\Lambda$ be an even unimodular lattice in $\mathbb{R}^d$, and let $P(x)$ be a homogeneous harmonic polynomial in $d$ variables of degree $m > 0$. We define

$$\Theta_{\Lambda, P}(\tau) = \sum_{v \in \Lambda} P(v) q^{|v|^2/2}$$

**Proposition**

*Under the above conditions* $\Theta_{\Lambda, P} \in S_{d/2+m}(\Gamma_1)$.

**Proof.**

The claim follows by applying the Poisson summation formula to $f(x) = P(x)e^{\pi i \tau |x|^2}$, using the fact that $\hat{f}(x) = im(\tau/i)^{-d/2-m} P(x)e^{\pi i(-1/\tau)|x|^2}$.
The fact that we can consider theta functions weighted by harmonic polynomials can be used to analyze the strength of a lattice shell as a spherical design.

Recall that a spherical $t$-design is a configuration of $N$ points $x_1, \ldots, x_N \in S^{d-1}$ such that for any polynomial $P \in \mathbb{R}[t_1, \ldots, t_d]$ of degree $\leq t$ one has

$$\int_{S^{d-1}} P(x) d\mu(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

One can show that it is enough to verify (*) for homogeneous harmonic polynomials.
**Proposition**

The set of vectors of length $\sqrt{2n}$ in the $E_8$ lattice forms a spherical 7-design.

**Proof.**

By the above theorem, if $P$ is harmonic of degree $d$, then $\Theta_{\Lambda, P} \in S_{4+d}(\Gamma_1)$, and hence it vanishes for $d < 8$.

Note that since $S_{14}(\Gamma_1) = 0$ the average of any harmonic polynomial of degree 10 over the set of vectors in $E_8$ of length $\sqrt{2n}$ is also zero.
Lehmer has conjectured that the Fourier coefficients of $\Delta(\tau)$ are non-zero.

One can reformulate this conjecture in more geometric terms as follows.

**Proposition**

*Lehmer’s conjecture is equivalent to the following statement: for all $n \geq 1$ the set of vectors of length $\sqrt{2n}$ in the $E_8$ lattice does not form an 8-design.*

To see this, note that the theta function of $\Lambda_8$ weighted by a harmonic polynomial of degree 8 lies in $S_{12}(\Gamma_1)$, which is spanned by $\Delta(\tau)$.

By the previous remark, we can also replace “8-design” by “9-design”, “10-design”, or even “11-design”.
A spherical \( t \)-design \( X \) on \( S^{n-1} \) is called tight if

\[
|X| = \left( n - 1 + \left\lfloor \frac{t}{2} \right\rfloor \right) + \left( n - 1 + \left\lfloor \frac{t-1}{2} \right\rfloor \right)
\]

Bannai and Damerell have proved that for \( n > 2 \) tight designs can exist only for \( t \in \{1, 2, 3, 4, 5, 7, 11\} \).

For \( t = 1, 2, 3 \) there is a simple classification, and for \( t = 11 \) there is only one such design, namely the 196560 shortest vectors of the Leech lattice.

For \( t = 4, 5, 7 \) there are only partial results: the only known examples are

\[
\begin{align*}
    t = 4 & : \quad n = 6, 22 \\
    t = 5 & : \quad n = 3, 7, 23 \\
    t = 7 & : \quad n = 8, 23
\end{align*}
\]
Application: tight spherical designs

It is known that for a tight 5-design, if \( n > 3 \), then \( n = (2m + 1)^2 - 2 \) for some \( m \geq 1 \). The two known examples correspond to \( m = 1, 2 \).

Bannai, Munemasa, and Venkov have proved that tight 5-designs do not exist for \( m = 3, 4 \) by analyzing the lattice generated by \( X \).

In particular, to prove that a tight 5-design in \( \mathbb{R}^{47} \) cannot exist they have constructed from it an even unimodular lattice \( \Lambda \subset \mathbb{R}^{48} \) with

\[
\Theta_\Lambda(\tau) = 1 + 2q + 4512q^2 + 1271256q^3 + \ldots
\]

Exercise

*Show that such a lattice does not exist.*