Modular forms and their applications II

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The group $\mathrm{PSL}_2(\mathbb{Z})$

Let us look again at the transformation law for modular forms

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,.$$

Since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, by above definition any modular form of odd weight must vanish identically, so we may concentrate on the case when k is even.

In this case $(c\tau+d)^k$ is invariant under $(c,d) \mapsto (-c,-d)$ and hence we may work with the group $\Gamma_1 := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/(\pm I)$ instead.

The slash operator

Recall that the group $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}$ on the left:

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

It is convenient to define the right action of $\mathrm{PSL}_2(\mathbb{R})$ on functions $f\colon \mathbb{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(z) := (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Exercise

Show that $|_k$ indeed defines a right group action, i.e, $f|_k\gamma_1\gamma_2 = f|_k\gamma_1|_k\gamma_2$.

The generators of $\mathrm{PSL}_2(\mathbb{Z})$

The transformation law for modular forms then becomes

$$f|_k\gamma = f\,, \qquad \gamma \in \mathrm{PSL}_2(\mathbb{Z})$$

Let us denote

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Exercise

Show that the group $\mathrm{PSL}_2(\mathbb{Z})$ *is generated by* S *and* T *with* $S^2 = (ST)^3 = 1$.

This implies that it is enough to verify $f|_kS = f|_kT = f$.

Periodicity

The condition $f|_k T = f$ is simply saying that f is 1-periodic.
Note that any 1-periodic analytic function $f\colon \mathbb{H} \to \mathbb{C}$ can be written as a Laurent series

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n\,, \qquad q := e^{2\pi i\tau} \tag{*}$$

Definition

A **modular form** of weight k for Γ_1 is an analytic function $f\colon \mathbb{H} \to \mathbb{C}$ that satisfies $f|_k\gamma = f$ for all $\gamma \in \Gamma_1$ and such that (*) contains only nonnegative powers of q.

A modular form f is called a **cusp form** if $a_0 = 0$. We denote by $M_k(\Gamma_1)$ and $S_k(\Gamma_1)$ the spaces of modular forms and cusp forms respectively.

Other subgroups of finite index

If $\Gamma \subset \Gamma_1$ is a subgroup of finite index, then one needs to modify the definition a bit.

Definition

A **modular form** of weight k for Γ is an analytic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and such that $(f|_k\gamma)(\tau)$ is bounded as $\tau \to i\infty$ for all $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$.

A modular form f is called a **cusp form** if instead $(f|_k\gamma)(\tau)$ goes to 0 as $\tau \to i\infty$ for all $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$.

Again, the spaces are denoted by $M_k(\Gamma)$ and $S_k(\Gamma)$. Note that a product of modular forms of weight k and l is a modular form of weight $k + l$, so that

$$M_*(\Gamma) = \bigoplus_{k \geq 0} M_k(\Gamma)$$

is a graded ring.

The fundamental domain

Since the values of a modular form f at any two points in the same Γ_1-orbit are related by a transformation law, it is useful to have a set of representatives of these orbits.

Definition

A fundamental domain for Γ_1 is an open set $\mathcal{F}$ such that no two points of $\mathcal{F}$ lie in the same Γ_1-orbit, and every point is equivalent to a point of $\overline{\mathcal{F}}$

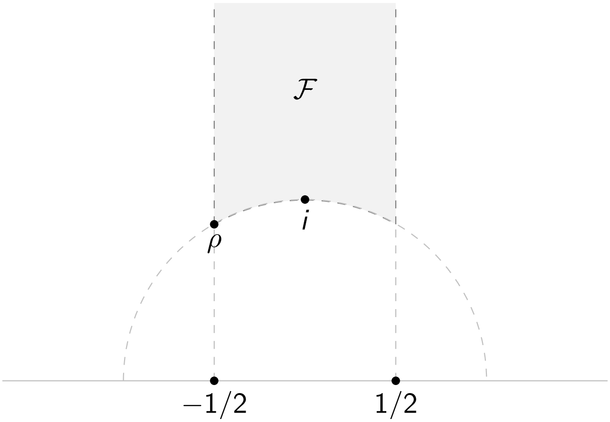
Proposition

The set

$$\mathcal{F} = \{\tau \in \mathbb{H} \colon |\operatorname{Re}\tau| < 1/2\,, |\tau| > 1\}$$

is a fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$.

The fundamental domain



The fundamental domain

Proof.

Use the identity

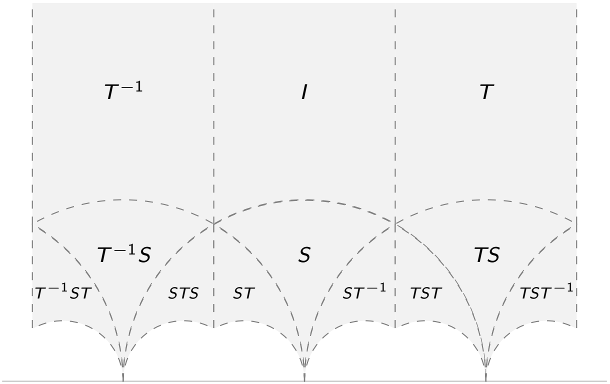
$$\operatorname{Im} \gamma\tau = \frac{\operatorname{Im} \tau}{|c\tau + d|^2} \, .$$

and consider $\tau' \in \Gamma_1 \tau$ with the largest imaginary part.
Since $\{c\tau + d \mid c, d \in \mathbb{Z}\}$ is a lattice, it has only finitely many vectors shorter than 1, and thus τ' is well-defined.
By applying some power of T if necessary, we may also assume that $|\operatorname{Re} \tau'| \leq 1/2$.
Since $\operatorname{Im} \tau'$ is maximal, we must have $|\tau'| \geq 1$, since otherwise $\operatorname{Im} S\tau' > \operatorname{Im} \tau'$.
This shows that every point in $\mathbb{H}$ is equivalent to a point in closure of $\mathcal{F}$. □

Exercise

Show that no two points of $\mathcal{F}$ are Γ_1-equivalent.

Some translates of the fundamental domain



Elliptic points

Note that some elements of $\mathrm{PSL}_2(\mathbb{Z})$ have fixed points in $\mathbb{H}$. For instance, S fixes i and $U = ST$ fixes the 3rd root of unity ρ.

We will call a point $z \in \mathbb{H}$ **elliptic**, if it has a nontrivial stabilizer in $\mathrm{PSL}_2(\mathbb{Z})$.

Exercise

Show that any elliptic point for $\mathrm{PSL}_2(\mathbb{Z})$ is equivalent to either i or ρ.

Define

$$n_z = \begin{cases} 3, & z \in \Gamma_1 \rho \\ 2, & z \in \Gamma_1 i \\ 1, & \text{otherwise} \end{cases}$$

Orders of zeros

Let $\mathrm{ord}_z(f)$ be the order of a zero of f at the point z, and let $\mathrm{ord}_\infty(f)$ be the smallest n such that $a_n \neq 0$ in the q-expansion.

Note that if f is a modular form, $\mathrm{ord}_z(f) = \mathrm{ord}_w(f)$ whenever $z = \gamma w$, $\gamma \in \Gamma_1$.

Proposition

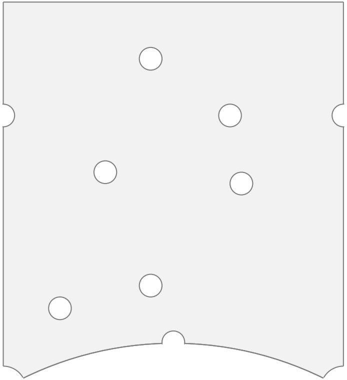
Let f be a nonzero modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Then

$$\sum_{z \in \Gamma_1 \backslash \mathbb{H}} \frac{\mathrm{ord}_z(f)}{n_z} + \mathrm{ord}_\infty(f) = \frac{k}{12}.$$

Proof.

Compute the integral of $\frac{1}{2\pi i}\frac{f'(z)}{f(z)}$ over the "Swiss cheese contour" illustrated on the next slide. We leave details as an exercise. □

Contour of integration



Proof of finite-dimensionality

Proposition

Let $k \geq 0$ be even. Then

$$\dim M_k(\mathrm{SL}_2(\mathbb{Z})) \leq \begin{cases} \lfloor k/12 \rfloor + 1\,, & k \not\equiv 2 \ (mod\ 12)\,, \\ \lfloor k/12 \rfloor\,, & k \equiv 2 \ (mod\ 12)\,. \end{cases}$$

Proof.

Given $m+1$ forms $f_1, \ldots, f_{m+1}$, where $m = \lfloor k/12 \rfloor + 1$, one can find a linear combination that vanishes at non-elliptic points $z_1, \ldots, z_m$, which must vanish.

If $k \equiv 2 \pmod{12}$, then $k/12 = a/3 + b/2 + c$ implies $a \equiv 2 \pmod 3$, $b \equiv 1 \pmod 2$, so that $a \geq 2$ and $b \geq 1$. Then the same argument shows $\dim M_k(\Gamma_1) \leq m - 1$. □

A different proof of finite-dimensionality

Exercise

(i) *Define the following norms on* $S_k(\Gamma_1)$*:*

$$\|f\|_2^2 := \int_{\mathcal{F}} |f(z)|^2 y^k \, d\mu(z)\,,$$

$$\|f\|_\infty := \sup_{\mathcal{F}} |f(z)| y^{k/2}\,.$$

where $d\mu(z) = y^{-2} dxdy$ *is the hyperbolic area measure. Show that there exists a constant* C_k *that depends only on* k*, such that*

$$\|f\|_\infty \leq C_k \|f\|_2\,, \qquad f \in S_k(\Gamma_1)\,.$$

(ii) *Let* $f_1, \ldots, f_m$ *be an orthonormal system with respect to the inner product associated to* $\|\cdot\|_2$*. Show that* $m \leq C_k^2 \, \mathrm{vol}(\mathcal{F})$*.*

Eisenstein series

Our next step is to show that the above upper bounds are exact.
For even $k > 2$ the Eisenstein series of weight k by is defined by

$$G_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n)\neq 0} \frac{1}{(m\tau+n)^k}$$

Since

$$\begin{aligned} \sum_{(m,n)\neq 0} \frac{1}{(m\frac{a\tau+b}{c\tau+d}+n)^k} &= (c\tau+d)^k \sum_{(m,n)\neq 0} \frac{1}{((ma+nc)\tau+(mb+nd))^k} \\ &= (c\tau+d)^k \sum_{(m',n')\neq 0} \frac{1}{(m'\tau+n')^k} \end{aligned}$$

(where $(m',n') = (m,n)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$) we get that $G_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Eisenstein series

To compute the q-expansion of Eisenstein series we use the Lipschitz formula

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{l\geq 1} l^{k-1} e^{2\pi i l z}, \qquad k \geq 2$$

Applying it to the sum over n gives

$$G_k(\tau) = \frac{(k-1)!}{(2\pi i)^k}\zeta(k) + \sum_{m\geq 1}\sum_{l\geq 1} l^{k-1} e^{2\pi i l m \tau} = -\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n) e^{2\pi i n \tau}$$

Here B_k is the k-th Bernoulli number, defined by the generating series

$$\sum_{k\geq 0} \frac{B_k x^k}{k!} = \frac{x}{e^x - 1}$$

Structure of the ring of modular forms

Let us denote by $E_k(\tau)$ the Eisenstein series normalized to have constant term 1.

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \dots$$
$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 + \dots$$
$$E_8(\tau) = 1 + 480q + 61920q^2 + 1050240q^3 + \dots$$

Note that E_4^3 and E_6^2 are linearly independent.

Proposition

The functions $E_4(\tau)$ and $E_6(\tau)$ are algebraically independent.

Structure of the ring of modular forms

Proof.

It is enough to show that E_4^3 and E_6^2 are algebraically independent. Let $P(E_4^3(\tau), E_6^2(\tau)) = 0$ for some polynomial $P \in \mathbb{C}[X, Y]$. We may assume that P is homogeneous since any homogeneous component P_d also satisfies $P_d(E_4^3(\tau), E_6^2(\tau)) = 0$. But if P is homogeneous, then $P(x, y) = 0$ has only finitely many solutions x/y, which implies that $E_4^3(\tau)/E_6^2(\tau)$ is constant, a contradiction. □

Corollary

We have

$$M_*(\Gamma_1) = \mathbb{C}[E_4, E_6]$$

Proof.

One can compute that $\#\{(a, b) \in \mathbb{Z}^2_{\geq 0} : 4a + 6b = k\} = \lfloor k/12 \rfloor + 1$ if $k \not\equiv 2 \pmod{12}$ and $\#\{(a, b) \in \mathbb{Z}^2_{\geq 0} : 4a + 6b = k\} = \lfloor k/12 \rfloor$ otherwise. Since these are also the upper bounds for $\dim M_k(\Gamma_1)$, they must match. □

Eisenstein series of weight 2

As we have mentioned in the first lecture,

$$G_2(\tau) = -\frac{1}{24} + \sum_{n\geq 1} \sigma(n) q^n$$

is not a modular form, but it does satisfy a similar transformation law

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - \frac{c(c\tau + d)}{4\pi i}\,. \qquad (1)$$

Exercise

Show that if (1) holds for γ_1 and γ_2, then it also holds for $\gamma_1\gamma_2$.

Eisenstein series of weight 2

There are several ways of proving the transformation law for G_2.
One way is to use Hecke's trick: define

$$G_{2,\varepsilon}(\tau) = -\frac{1}{8\pi^2} \sum_{(m,n)\neq 0} \frac{1}{(m\tau+n)^2|m\tau+n|^{2\varepsilon}}$$

and show that $\lim_{\varepsilon\to 0} G_{2,\varepsilon}(\tau) = G_2(\tau) + \frac{1}{8\pi y}$. Since $G_{2,\varepsilon}(\tau)$ does satisfy a modular transformation, one can show that it implies (1).

Eisenstein series of weight 2

A different way: it is enough to show that

$$G_2(i/y) = -y^2 G_2(iy) - \frac{y}{4\pi} \tag{2}$$

Consider the L-series associated to G_2:

$$\begin{aligned} L(s) = \int_0^\infty (G_2(iy) + 1/24) y^{s-1} dy &= (2\pi)^{-s}\Gamma(s) \sum_{n\geq 1} \frac{\sigma(n)}{n^s} \\ &= (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1) \end{aligned}$$

The functional equation for $\zeta(s)$ then implies $L(2-s) = -L(s)$, and knowing that L has poles only at $s = 0, 1, 2$ with known residues gives (2) by inverse Mellin transform.

Modular discriminant function

An immediate application of G_2 is the proof of the fact that $\Delta(\tau)$ is modular

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\Gamma_1)$$

Proof.

The product expansion shows that $\Delta(\tau) \neq 0$ for $\tau \in \mathbb{H}$. Note that

$$\frac{1}{2\pi i} \frac{\Delta'(\tau)}{\Delta(\tau)} = 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} = E_2(\tau)$$

Then the functional equation $\tau^{-2} E_2(-1/\tau) = E_2(\tau) + \frac{12}{2\pi i \tau}$ implies that $\Delta(-1/\tau) = c\tau^{12}\Delta(\tau)$, and since $\Delta(i) \neq 0$ we must have $c = 1$. □

Modular discriminant function

Since the space $S_{12}(\Gamma_1)$ is 1-dimensional, we obtain

$$\Delta(\tau) = \frac{E_4^3(\tau) - E_6^2(\tau)}{1728}$$

Since $\Delta(\tau)$ does not vanish anywhere in $\mathbb{H}$, $j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)}$ is a modular function.
Ramanujan has observed experimentally that the coefficients of the q-expansion

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

are multiplicative and satisfy $|\tau(p)| < 2p^{11/2}$ for primes p.
The first statement was proved one year later by Mordell, but the second was only proved in 1974 by Deligne as a corollary of the Weil conjectures.