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Online summer school on optimization, interpolation and modular forms

— Problem Sheet 3: Application to lattice sphere coverings —

Problem 3.1 Define the *Cayley-Menger determinant* of n points $x_1, \dots, x_n$, where the pairwise distances $d(x_i, x_j) = \|x_i - x_j\|$ are given, by

$$\mathrm{CM}(x_1, \dots, x_n) = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & d(x_1, x_1)^2 & \dots & d(x_1, x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d(x_n, x_1)^2 & \dots & d(x_n, x_n)^2 \end{vmatrix}.$$

a) The Cayley-Menger determinant of $n+2$ points in $\mathbb{R}^n$ vanishes.

b) Let $L = \mathrm{conv}\{v_0, \dots, v_n\}$ be an n-dimensional simplex. Then the circumsphere of L has the squared radius

$$R^2 = -\frac{1}{2} \cdot \frac{\det \left(d(v_i, v_j)^2\right)_{0 \le i,j \le n}}{\mathrm{CM}(v_0, \dots, v_n)}.$$

Problem 3.2 Consider $Q[x] = n \sum_{i=1}^n x_i^2 - \sum_{i \neq j} x_i x_j$. Show that Q is locally optimal for the lattice sphere covering problem.