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Online summer school on optimization, interpolation and modular forms

— Problem Sheet 2: Voronoi's lattice reduction theory —

Problem 2.1 Given $Q \in \mathbb{S}^n$ and $V = \{v_1, \dots, v_{n+1}\}$ be affinely independent.
For $w \in \mathbb{R}^n$ there exist $\alpha_1, \dots, \alpha_{n+1}$ so that

$$w = \sum_{i=1}^{n+1} \alpha_i v_i, \quad \sum_{i=1}^{n+1} \alpha_i = 1.$$

Define

$$N_{V,w} = ww^{\mathsf{T}} - \sum_{i=1}^{n+1} \alpha_i v_i v_i^{\mathsf{T}}.$$

Let $c \in \mathbb{R}^n$ and $r > 0$ be such that $Q[c - v_i] = r^2$ for all $i = 1, \dots, n+1$.
Show that

$$\langle Q, N_{V,w} \rangle = Q[w - c] - r^2.$$

Think about how this could help to show that $\Delta(\mathrm{Del}(Q))$ is an open polyhedral cone.

Problem 2.2 Consider $Q[x] = n \sum_{i=1}^n x_i^2 - \sum_{i \neq j} x_i x_j$. For a permutation $\pi \in S_{n+1}$ define the n-dimensional simplex

$$L_\pi = \mathrm{conv} \left\{ \sum_{i=1}^{k} e_{\pi(i)} : k = 1, \dots, n+1 \right\},$$

where $e_{n+1} = -(e_1 + \cdots + e_n)$.

a) Then, $L_\pi \in \mathrm{Del}(Q)$ holds for any $\pi \in S_{n+1}$.

b) What is $\Delta(\mathrm{Del}(Q))$?

Problem 2.3 Let $V = \{v_1, \dots, v_{n+2}\} \subseteq \mathbb{R}^n$ be points which affinely span $\mathbb{R}^n$. Show that the polytope $P = \mathrm{conv}\, V$ has exactly two different triangulations (i.e. it can be decomposed into simplices where all vertices lie in V).