Problem 1.1 Use the spectral theorem to show that the cone of positive semidefinite matrices is self dual.

Problem 1.2
a) Show that the set
\[ D^{n+1} = \{(X, s) \in S^n_\geq \times \mathbb{R}_\geq : (\det X)^{1/n} \geq s\} \]
is a proper convex cone.

b) Use the inequality of arithmetic and geometric means to determine its dual cone \((D^{n+1})^*\).

Problem 1.3 Let \( f \in \mathbb{R}[x_1, \ldots, x_n]_d \) be a homogeneous polynomial of degree \( d \) and let \( \alpha \in \mathbb{R}^n \) be a vector. Compute \( \langle f, (\alpha^T x)^d \rangle \) with \( x = (x_1, \ldots, x_n) \).

Problem 1.4 Show that the PARTITION problem has a solution if and only if
\[ \sup \left\{ t : (c^T x)^4 + n \sum_{i=1}^{n} x_i^4 = \left( \sum_{i=1}^{n} x_i^2 \right)^2 - t \in P_{n,4} \right\} \]
has optimal solution \( t = 0 \).