MODULAR FORMS AND THEIR APPLICATIONS: EXERCISES 2

Exercise 1. For $k \in \{4, 6, 8, 10, 14\}$ define $f_k = E_k$ and let $f_0 = 1$. Prove that if $k = 12l + k_0$ for $k_0 \in \{0, 4, 6, 8, 10, 14\}$ and $l \geq 0$, then

$$M_k(\mathrm{SL}_2(\mathbb{Z})) \;=\; \Delta^l(z) f_{k_0}(z) \mathbb{C}[j(z)]_{\leq l}\,,$$

where $j(z) = E_4^3(z)/\Delta(z)$ and $\mathbb{C}[X]_{\leq k}$ denotes the space of polynomials of degree $\leq k$ in X.

Exercise 2. Verify, using the transformation $\theta(-1/4\tau) = \sqrt{2\tau/i}\theta(\tau)$, that $\theta^4 \in M_k(\Gamma_0(4))$. One may use the fact that $\Gamma_0(4)$ is generated by T and $\left(\begin{smallmatrix} 1 & 0 \\ -4 & 1 \end{smallmatrix}\right)$.

Exercise 3. Let $r_k(n)$ be the number of representations of n as a sum of squares of k integers. Show that the sequence $\{\frac{r_k(n)}{2k}\}_{n\geq 1}$ is multiplicative if and only if $k \in \{1, 2, 4, 8\}$. In each of these cases compute the corresponding Dirichlet series

$$L_k(s) \;=\; \frac{1}{2k} \sum_{n \geq 1} \frac{r_k(n)}{n^s}\,.$$

(For $k = 2$ one may use the identity $r_2(n) = 4 \sum_{2 \nmid d | n} (-1)^{(d-1)/2}$.)