MODULAR FORMS AND THEIR APPLICATIONS: EXERCISES 1

Exercise 1. (i) Show that

$$\gamma\tau - \gamma z = \frac{\tau - z}{(c\tau + d)(cz + d)}\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\,.$$

(ii) Show that

$$\operatorname{Im}\gamma\tau = \frac{\operatorname{Im}\tau}{|c\tau + d|^2}\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\,.$$

Exercise 2. Show that the group $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and T with $S^2 = (ST)^3 = 1$. (it is not necessary to prove that these are the only relations).

Exercise 3. Show that any elliptic point for $\mathrm{PSL}_2(\mathbb{Z})$ is equivalent to either i or ρ.

Exercise 4. Prove Lipschitz's formula

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{l\geq 1} l^{k-1} e^{2\pi i l z}\,, \qquad z \in \mathbb{H}\,, \quad k \geq 2\,.$$

Exercise 5. In this exercise we give an alternative proof of finite-dimensionality of $M_k(\Gamma_1)$:

(i) Let f be a cusp form of weight k. Show that

$$\varphi(z) = |f(z)|y^{k/2}$$

is $\mathrm{PSL}_2(\mathbb{Z})$-invariant and bounded.

(ii) Define the following two norms on $S_k(\Gamma_1)$:

$$\|f\|_2^2 := \int_{\mathcal{F}} |f(z)|^2 y^k d\mu(z)\,,$$

where $d\mu(z) = y^{-2}dxdy$ is the hyperbolic area measure and

$$\|f\|_\infty := \sup_{\mathcal{F}} |f(z)|y^{k/2}\,.$$

Show that there exists a constant C_k that depends only on k, such that

$$\|f\|_\infty \leq C_k \|f\|_2\,, \qquad f \in S_k(\Gamma_1)\,.$$

(iii) Let $f_1, \ldots, f_m$ be an orthonormal system with respect to the inner product associated to $\|\cdot\|_2$. Using (a) show that $m \leq C_k^2 \operatorname{vol}(\mathcal{F})$.

(Hint: in (a) use the invariance of the measure μ and consider an integral of $|f(z)|^2 y^k d\mu(z)$ over a strip $[-1/2, 1/2] \times [\delta, \infty)$ for some $\delta < \sqrt{3}/2$.)