Packing, coding, and ground states IV

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(today: joint work with Afkhami-Jeddi, Hartman, de Laat, Tajdini)

Today: applications to physics

Conformal field theory in two dimensions

- many different CFTs
- govern limiting behavior of various stat. mech. models at critical points
- think scale invariance + ε (where ε is conceptually significant but automatic in some cases)

Hierarchy of exceptional structures

binary codes	Mathieu groups
$\downarrow$	
lattices	Conway groups, etc.
$\downarrow$	
vertex algebras / CFTs	Monster group

Each level used to construct the next.

conformal bootstrap

=

What can we learn about CFTs from self-consistency?

modular bootstrap

=

specifically conformal invariance of partition fns.

Much of quantum field theory is not rigorous, but here we can formulate clean mathematics (even if some consequences are heuristic).

Quantum Gravity

AdS/CFT correspondence (conjectural)

very mysterious

3d quantum gravity
in anti-de Sitter space

$\longleftrightarrow$

fairly mysterious
(but less so)

2d CFT on boundary

What can we learn about quantum gravity by studying the space of CFTs (theory space)?

Toy model, since AdS = negative cosmological constant (doesn't match data).

Rigorous understanding of CFTs is way behind what physics needs.

Let's treat it as a black box, via (torus) partition functions

What does conformal invariance mean?

Every 2d torus is conformally equivalent to a flat torus (uniformization).

$\mathbb{C}/\Lambda$ for some lattice Λ.

Can rescale size of Λ rotate.

Pick basis $\tau, 1$ of Λ with τ in upper half-plane.

Change of basis

$$\tau, 1 \longleftrightarrow a\tau+b, \quad c\tau+d$$

w/ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$\longleftrightarrow \frac{a\tau+b}{c\tau+d}, 1$$

after rotation, rescaling

<u>Conclusion:</u>

$$\text{2d tori} \Big/ \begin{matrix}\text{conformal} \\ \text{equiv.}\end{matrix} \cong \text{upper half-plane} \big/ SL_2(\mathbb{Z})$$

2d unitary, compact CFT

partition function $Z(\tau, \bar{\tau})$

$\tau, -\bar{\tau} \in$ upper half-plane

Note: $\tau^* =$ conjugate of τ

$\bar{\tau} =$ just some variable (but often set equal to τ^*)

Blame physicists, not me!

$Z(\tau, \bar{\tau})$ = sum of

$$q^{h - c/24} \, \bar{q}^{\bar{h} - \bar{c}/24}$$

over all states

$$q = e^{2\pi i \tau}, \quad \bar{q} = e^{-2\pi i \bar{\tau}}$$

$$(\text{so } |q|, |\bar{q}| < 1)$$

holomorphic/anti-holomorphic

$c, \bar{c}$ = left/right conformal charges (property of CFT)

often $\bar{c} = c$

$h, \bar{h}$ = conformal weights of state conformal invariance

$$Z\left(\frac{p\tau+q}{r\tau+s}, \frac{p\bar{\tau}+q}{r\bar{\tau}+s}\right) = Z(\tau, \bar{\tau})$$

$$\text{for } \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z})$$

In terms of usual generators of $SL_2(\mathbb{Z})$,

$$Z(\tau+1, \bar{\tau}+1) = Z(\tau, \bar{\tau})$$

$$Z(-1/\tau, -1/\bar{\tau}) = Z(\tau, \bar{\tau})$$

Each state has

measures behavior under rescaling

scaling dimension $\Delta = h + \bar{h}$

spin $\ell = h - \bar{h}$

under rotation

$$Z(\tau+1, \bar{\tau}+1) = Z(\tau, \bar{\tau})$$

implies $\ell \in \mathbb{Z}$,

but Δ can be any element of $[0, \infty)$.

(unitarity: $h, \bar{h} \geq 0$)

The CFT has a conformal algebra acting on it.

smallest possibility:

generic case

Virasoro algebra Vir^c (central extension of algebra of infinitesimal conformal transformations).

could also be affine Lie algebra, e.g., $U(1)^c$

extra symmetry

free bosons

action of $\mathrm{Vir}^{c} \times \mathrm{Vir}^{\bar{c}}$

or $U(1)^{c} \times U(1)^{\bar{c}}$

(or others)

groups together

primary fields and descendants under algebra

In partition function, descendants combine to give characters of Verma modules.

$$Z(\tau,\bar{\tau}) = \sum_{\text{primaries}} \underbrace{\chi_{\ell,\Delta}(\tau,\bar{\tau})}_{\text{character}}$$

(must take multiplicities into account)

Vacuum is unique state w/ $\Delta = 0$

What's the next smallest Δ?

<u>spectral gap</u> Δ_1 of CFT

Why do we care?

Among other things,

expect "pure quantum gravity" to have

conformal algebra Vir^c

and $\Delta_1 \gtrsim c/12$

as $\underbrace{c \to \infty}_{\#\text{ particles} \to \infty}$.

Does such a theory exist?
(Can Δ_1 be that large?)

It's unclear whether pure quantum gravity even is consistent, or how to identify such a CFT.

Can we rule it out, or pin down properties it must have?

Toy version:

conformal algebra $\quad U(1)^c \times U(1)^c$

theory of c free bosons

Narain gave a lattice construction, which is conjectured to give all $U(1)^c$ CFTs.

let's take $\bar{c} = c$.

Write elements of $\mathbb{R}^{2c}$ as $(x, y) \in (\mathbb{R}^c)^2$.

<u>Def.</u> A <u>Narain lattice</u> Λ is an even unimodular lattice of signature (c, c)

(i.e., under quadratic form $|x|^2 - |y|^2$).

Always exists, unique up to action of $O(c, c)$.

Narain lattice Λ

$\Lambda \longleftrightarrow U(1)^c$ CFT

$(x,y) \in \Lambda$ — primary field

$\frac{|x|^2+|y|^2}{2}$ — Δ

$\frac{|x|^2-|y|^2}{2}$ — ℓ

Narain lattice $\Longrightarrow \ell \in \mathbb{Z}$
(but not Δ)

action of $O(c) \times O(c)$ preserves CFT

Maximizing spectral gap for Narain lattice (conj., for all $U(1)^c$ CFTs)

$\cong$

sphere packing problem restricted to Narain lattices in $\mathbb{R}^{2c}$

Best Narain lattices known:

TABLE 1. Putatively optimal Narain CFTs and the best lattice packing known (i.e., optimizing among all lattices of determinant 1, without the condition of being an even lattice of signature (c, c)).

c	Δ_1	Name	Best lattice packing
1	$1/2$	$SU(2)_1$ WZW	$\sqrt{1/3} = 0.5773\ldots$
2	$2/3$	$SU(3)_1$ WZW	$\sqrt{1/2} = 0.7071\ldots$
3	$3/4$	$SU(4)_1$ WZW	$\sqrt[6]{1/3} = 0.8326\ldots$
4	1	$SO(8)_1$ WZW	1
5	1	$SO(10)_1$ WZW	$\sqrt[10]{4/3} = 1.0291\ldots$
6	$\sqrt{4/3} = 1.1547\ldots$	Coxeter-Todd	$\sqrt{4/3} = 1.1547\ldots$
7	$\sqrt{4/3} = 1.1547\ldots$		$\sqrt[14]{64/3} = 1.2443\ldots$
8	$\sqrt{2} = 1.4142\ldots$	Barnes-Wall	$\sqrt{2} = 1.4142\ldots$

How do they compare as $c \to \infty$?

(Existence proofs are almost as good.)

Modular bootstrap

=

use conformal invariance of $Z(\tau, \bar{\tau})$ to produce bounds

=

LP bounds!

(Hartman, Mazáč, Rastelli: 2019)

For $U(1)^c$,

spinless modular bootstrap = sphere packing LP bound in $\mathbb{R}^{2c}$

spinning modular bootstrap = new bound for Narain lattices

The Virasoro versions are slightly different.

We'll focus on $U(1)^c$ here.

$U(1)^c$ has characters $\dfrac{q^h}{\eta(\tau)^c}$

as in Danylo's course

So for $U(1)^c \times U(1)^{\bar{c}}$

we get

$$\sum_{\text{primaries}} \frac{q^h \bar{q}^{\bar{h}}}{\eta(\tau)^c \eta(-\bar{\tau})^{\bar{c}}}.$$

For the <u>spinless</u> <u>modular</u> <u>bootstrap</u>, we set $\bar{\tau} = -\tau$.
$q = \bar{q}$

This specialization preserves $\tau \mapsto -1/\tau$ invariance, but <u>not</u>

$\tau \mapsto \tau + 1$.

Set $\bar{\tau} = -\tau$. Then

$$\mathcal{Z}(\tau) = Z(\tau, -\tau)$$

$$= \sum_{\text{primaries}} \frac{q^{\Delta}}{\eta(\tau)^{c+\bar{c}}}$$

satisfies

$$\mathcal{Z}(-1/\tau) = \mathcal{Z}(\tau).$$

But $\eta(-1/\tau) = (\tau/i)^{1/2}\eta(\tau)$,

so we can remove the η factor.

Conclusion from conformal invariance:

if n_Δ is the multiplicity of states w/ scaling dim. Δ, and we set $d = c + \bar{c}$, then

$$\sum_{\Delta} n_\Delta e^{2\pi i \tau \Delta}$$

$$= \sum_{\Delta} n_\Delta \left(\frac{i}{\tau}\right)^{d/2} e^{2\pi i (-1/\tau)\Delta}$$

whenever $\operatorname{Im} \tau > 0$.

$$\sum_{\Delta} n_{\Delta} e^{2\pi i \tau \Delta}$$

$$= \sum_{\Delta} n_{\Delta} \left(\frac{i}{\tau}\right)^{d/2} e^{2\pi i (-1/\tau)\Delta}$$

For $\mathrm{Im}\,\tau > 0$, $x \mapsto e^{2\pi i \tau (|x|^2/2)}$ is a (complex) Gaussian on $\mathbb{R}^d$ with Fourier transform

$$x \mapsto (i/\tau)^{d/2} e^{2\pi i (-1/\tau)(|x|^2/2)}.$$

I.e., setting $\Delta = |x|^2/2$ makes this look like Poisson summation.

For every radial Schwartz fn. $f: \mathbb{R}^d \to \mathbb{R}$,

$$\sum_{\Delta} n_\Delta f(\sqrt{2\Delta}) = \sum_{\Delta} n_\Delta \hat{f}(\sqrt{2\Delta}).$$

This holds for all f since complex Gaussians span a dense subset of radial Schwartz fns.

Analogue of Poisson summation.

But we've seen that the LP bound needs only such an identity to work.

(In fact, this is a special case, the -1 eigenfunction uncertainty principle, conjectured to be equivalent.)

So $U(1)^c \times U(1)^{\bar{c}}$ spinless modular bootstrap

$=$ LP bound in $\mathbb{R}^{c+\bar{c}}$.

The modular bootstrap originated from a quite different derivation.

(see suggested reading)

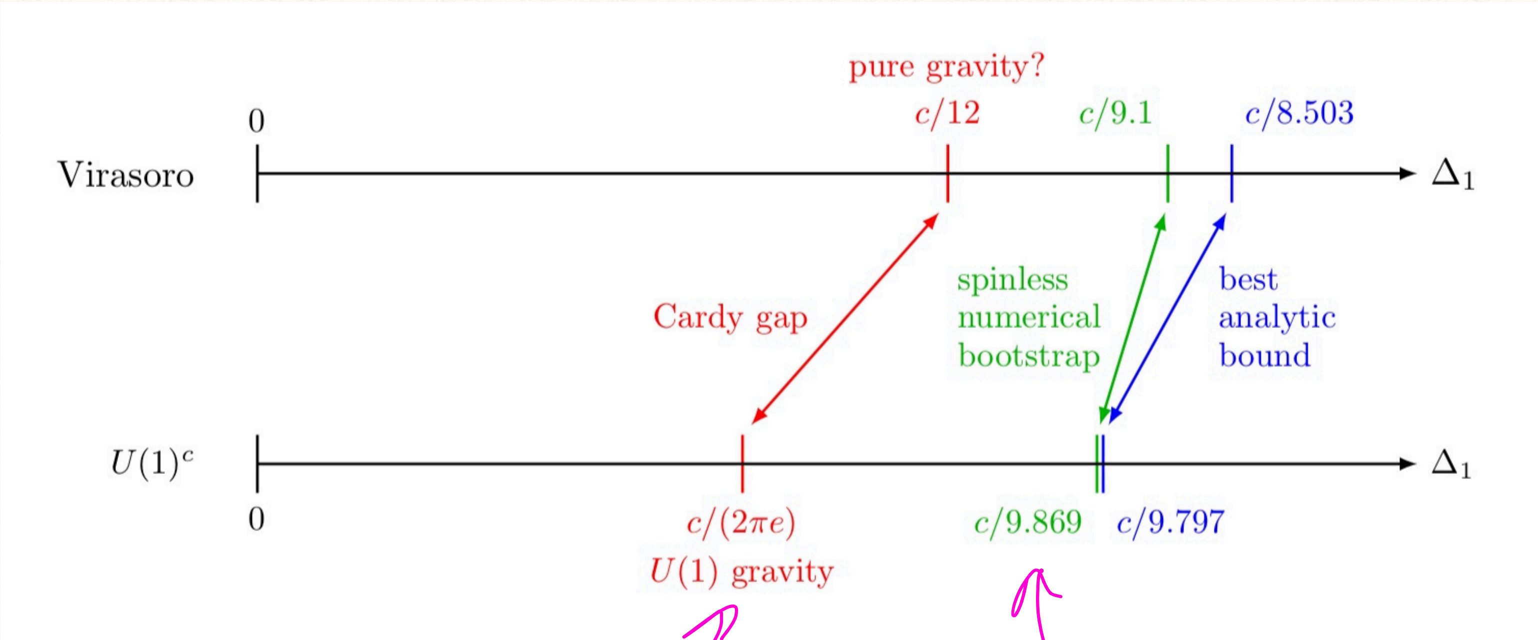
But this fact about d-dimensional Fourier transforms is the key to the equivalence.

Physicists care most
about large systems
(i.e., $\mathbb{R}^d$ with $d \to \infty$).

The Virasoro case is
a little different.
(See suggested reading.)

This is a particularly
important asymptotic
question.

Asymptotics so far



Minkowski-Hlawka for Narain lattices

Conj. $\frac{c}{\pi^2}$

Could some sort of SDP bounds help?

What about the spinning modular bootstrap?

It amounts to adapting LP bounds to Narain lattices.

To prove $\Delta_1 < \Delta_{gap}$, find

$$f : \mathbb{R}^c \times \mathbb{R}^c \longrightarrow \mathbb{R}$$ Schwartz fn.

s.t. $\hat{f} = -f$

$f(0,0) > 0$

Key difference

$f(x,y) \geq 0$ whenever $|x|^2 - |y|^2 \in 2\mathbb{Z}$ and $|x|^2 + |y|^2 \geq 2\Delta_{gap}$

To prove $\Delta_1 < \Delta_{gap}$, find

$$f : \mathbb{R}^c \times \mathbb{R}^{\bar{c}} \longrightarrow \mathbb{R}$$ Schwartz fn.

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$f(0,0) > 0$

$f(x,y) \geq 0$ whenever $|x|^2 - |y|^2 \in 2\mathbb{Z}$ and $|x|^2 + |y|^2 \geq 2\Delta_{gap}$

Proof: For Narain Λ,

$$\sum_{(x,y)\in\Lambda} f(x,y) = \sum_{(x,y)\in\Lambda} \hat{f}(x,y)$$

by Poisson. Then apply inequalities.

QED.

How good is the spinning modular bootstrap?

Set $\bar{c} = c$.

Sharp for $c = 1$

(Afkhami-Jeddi, Cohn, Hartman, Tajdini)

$c = 4$

(spinless, Viazovska)

Conjecture: sharp for $c = 2$

$$\Delta_1 = 2/3$$

SU(3)$_1$ WZW model

(tensor product $A_2 \otimes A_2$ of lattices)

Similar to $\mathbb{R}^2$ LP bound

Questions

- $c=2$ spinning bound sharp for $U(1)^c$?
  (new magic function?)
- asymptotics for spinless, spinning $U(1)^c$ bounds as $c \to \infty$?
- Virasoro asymptotics
  (most interesting for physics)
- other affine Lie algebras?
- higher-order bounds?

Suggested reading

- T. Hartman, D. Mazáč, and L. Rastelli, Sphere packing and quantum gravity.

- N. Afkhami-Jeddi, H. Cohn, T. Hartman, D. de Laat, and A. Tajdini, High-dimensional sphere packing and the modular bootstrap.

- N. Afkhami-Jeddi, H. Cohn, T. Hartman, and A. Tajdini, Free partition functions and an averaged holographic duality.