Some applications of trace functions in analytic number theory

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a pôt-pourri of joint works with
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Peter Sarnak 61th birthday conference, Princeton

The horocycle flow at prime times

We start with following classical (q always denote a prime number)

Theorem (Equidisitribution of Hecke points)

As $q \to \infty$, the integral points of the closed horocycle of height $1/q$

$$\{\frac{n+i}{q},\ n = 1, \cdots, q\} \subset \{x + \frac{i}{q},\ x \in]0,1]\} \subset \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$$

become equidistributed on the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ wrt the hyperbolic measure $\mu_{\mathrm{Hyp}} = \frac{3}{\pi}\frac{dxdy}{y^2}$.

The horocycle flow at prime times

A very special case of Sarnak's general "Moebius Disjointness conjecture for zero entropy flows" is that one can impose primality conditions on the *n* variable and still be equidistributed

Conjecture (Equidisitribution of prime Hecke points)

As $q \to \infty$, the prime Hecke points

$$\{\frac{p+i}{q},\ p \text{ prime} < q\} \subset \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$$

become equidistributed on the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ wrt the hyperbolic measure μ_{Hyp}.

The horocycle flow at prime times

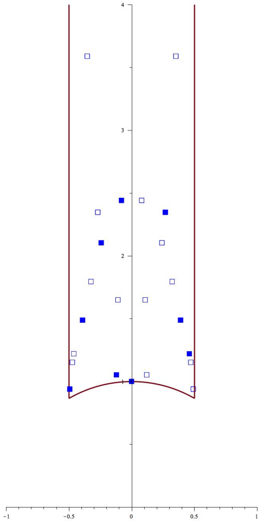

Figure: Prime Hecke points $q = 61$

The horocycle flow at prime times

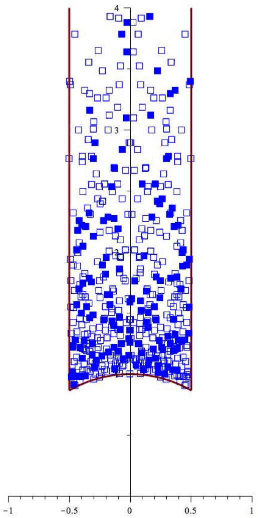

Figure: Prime Hecke points $q = 1229$

The horocycle flow at prime times

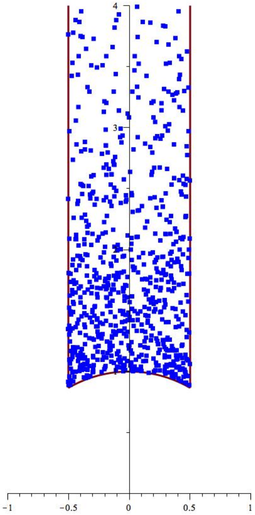

Figure: Prime Hecke points $q = 7919$

The horocycle flow at prime times

P. Sarnak and A. Ubis obtained several striking results towards this conjecture; for instance

Theorem (Sarnak-Ubis)

Assume the Ramanujan-Petterson conjecture. As $q \to \infty$, any weak$-*$ limit μ_{prime} of the uniform probability measure supported at the prime Hecke points

$$\{\frac{p+i}{q},\ p\ prime < q\} \subset \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$$

satisfy

$$\frac{1}{5}d\mu_{\mathrm{Hyp}} \leq \mu_{prime} \leq \frac{9}{5}d\mu_{\mathrm{Hyp}}.$$

Trying to improve upon these result was irresitible so we (FKM) tried...

The horocycle flow at prime times

and came to the problem of obtaining the following type of bounds

$$\sum_{n \ll q} \lambda_f(n) \mathrm{Kl}_2(n;q) \ll q^{1-\delta}, \ \delta > 0$$

with $(\lambda_f(n))_{n \geq 1}$ the Hecke eigenvalues of some weight 0 Hecke-eigenform f and $\mathrm{Kl}_2(n;q)$ is the normalized Kloosterman sum

$$\mathrm{Kl}_2(n;q) = \frac{1}{q^{1/2}} \sum_{\substack{x,y \in \mathbb{F}_q \\ xy = n}} e_q(x+y)$$

and eventually we succeeded... Unfortunately there was a (gross) miscalculation and the above bound, as nice as it looks, does not seem to have a direct impact on improving the work of Sarnak-Ubis...

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Algebraically weighted horocycles

Nevertheless, this bound has an interpretation in terms of *equidistribution*: for any modular form one has

$$\frac{1}{q-1}\sum_{n=1\cdots q-1} f(\frac{n+i}{q})e_q(-n^{-1}) \to 0$$

or in other terms

Theorem

The (signed) measure of total variation 1*, supported on the integral points* $\frac{n+i}{q}$ *of the horocycle of height* $1/q$ *and weighted by* $e_q(-n^{-1})$*, weak-*$\star$ *converge to the* 0 *measure.*

There is already one instance one such equidistribution for integral horocycle points weigthed by an algebraic function which follows from subconvexity

Algebraically weighted horocycles

One has

$$L(f\otimes(\frac{\cdot}{q}),1/2)\ll q^{1/2-\delta}\Rightarrow\sum_{n\ll q}\lambda_f(n)(\frac{n}{q})\ll q^{1-\delta}$$

$$\Rightarrow\frac{1}{q-1}\sum_{n=1\cdots q-1}f(\frac{n+i}{q})(\frac{n}{q})\to 0$$

or equivalently

Theorem (Equidisitribution of quadratic Hecke points)

As $q\to\infty$, the quadratic Hecke points

$$\{\frac{n^2+i}{q},\ 1\leq n<q\}\subset \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$$

become equidistributed wrt the hyperbolic measure μ_{Hyp}.

Quadratic Hecke points

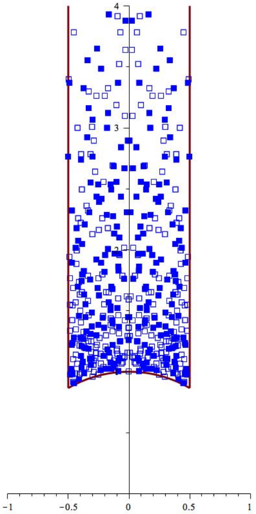

Figure: quadratic Hecke points $q = 1229$

Trace functions

This equidistribution statement is valid for a wide class of functions $F : \mathbb{F}_q \to \mathbb{C}$ named *trace functions*:

$$x \in \mathbb{F}_q \mapsto F(x) = \mathrm{tr}(\mathrm{Frob}_q, \mathcal{F}_x) = \mathrm{tr}(\mathrm{Frob}_x, \mathcal{F}_{\overline{\eta}}^{I_x})$$

where $\mathcal{F}$ is a constructible middle-extension ℓ-adic sheaf on $\mathbb{A}^1_{\mathbb{F}_q}$ (aka a finite dimensional ℓ-adic representation

$$\rho_{\mathcal{F}} : \mathrm{Gal}(\mathbb{F}_q(X)) \mapsto \mathrm{GL}(\mathcal{F}_{\overline{\eta}})\ )\text{ which is}$$

- Pure of weight 0 (at any point where $\mathcal{F}$ is lisse) which implies (Deligne) that for $x \in \mathbb{F}_q$

$$|F(x)| \leq \mathrm{r}k(\mathcal{F}).$$

- Geometrically irreducible (better isotypic) as a representation of $\mathrm{Gal}(\overline{\mathbb{F}_q}(X))$.

Trace functions

To such $\mathcal{F}$ is associated an integer, the "conductor" measuring the "complexity" of the Galois representation underlying $\mathcal{F}$:

$$C(\mathcal{F}) = \mathrm{r}k(\mathcal{F}) + \sum_{x \in \mathbb{P}^1(\overline{\mathbb{F}_q})} \mathrm{swan}_x(\mathcal{F}) + \mathrm{drop}_x(\mathcal{F}).$$

Any such function defines a q-periodic arithmetic function on $\mathbb{Z}$:

$$F : \mathbb{Z} \mapsto \mathbb{Z}/q\mathbb{Z} \mapsto \mathbb{C}$$

and the general objective is to determine how such functions correlate with other arithmetic functions when $q \to \infty$ while $C(\mathcal{F})$ remains bounded.

Trace functions

Example

- Additive chracters: $\psi : (\mathbb{F}_q, +) \mapsto \mathbb{C}^\times$, associated to the Artin-Schreier sheaf $\mathcal{L}_\psi$.
- Multiplicative characters: $\chi : (\mathbb{F}_q^\times, \times) \mapsto \mathbb{C}^\times$, associated to the Kummer sheaf of $\mathcal{L}_\chi$.
- (normalized) Kloosterman sums associated to the (Tate-twisted) Kloosterman sheaf $\mathcal{K}l_k$

$$x \mapsto \mathrm{Kl}_k(x) = \frac{1}{q^{\frac{k-1}{2}}} \sum_{x_1 \cdot \cdots \cdot x_k = x} e_q(x_1 + \cdots + x_k)$$

- We will also meet for $a \in \mathbb{F}_q$, the normalized Dirac function $q^{1/2}\delta_{a \pmod q}$

Construction of Trace functions

One can construct new trace functions from existing ones; the associated sheaves have conductors bounded in terms of the initial conductors

- Pullback: for $f : \mathbb{P}^1_{\mathbb{F}_q} \mapsto \mathbb{P}^1_{\mathbb{F}_q}$, $f^*F(x) = F(f(x))$ correspond to the Pullback sheaf $f^*\mathcal{F}$ (the restriction of $\rho_{\mathcal{F}}$ to some finite index subgroup).
- Dual: $\overline{F} : x \mapsto \overline{F}(x)$ correspond to the dual sheaf $\check{\mathcal{F}}$ (the dual representation $\check{\rho}_{\mathcal{F}}$)
- tensor product: $x \mapsto F(x)G(x)$ correspond to the tensor product $\mathcal{F} \otimes \mathcal{G}$ (a.k.a the tensor product representation).
- Push-forward: $f : Y \mapsto \mathbb{A}^1_{\mathbb{F}_q}$,

$$f_*F(x) = q^{-\frac{\dim Y_{\overline{\eta}}}{2}} \sum_{f(y)=x} F(y)$$

  should correspond to $f_*\mathcal{F}(\dim Y_{\overline{\eta}}/2)$ but, in general, one obtains a complex of sheaves.

Construction of Trace functions

Two important examples of push-forward for $Y = \mathbb{A}^2_{\mathbb{F}_q}$ (Deligne, Laumon, Katz):

- The Fourier transform:

$$\hat{F}(x) = q^{-1/2} \sum_{y \in \mathbb{F}_q} F(y) e_q(xy)$$

  corresponding to the "naive" Fourier transform sheaf $\hat{\mathcal{F}} = NFT_{e_q}(\mathcal{F})(1/2)$ if $\mathcal{F}$ is "Fourier" (not geometrically an Artin-Schreier sheaf $\mathcal{L}_\psi$).

- Multiplicative convolution:

$$F \star G(x) = q^{-1/2} \sum_{x_1 x_2 = x} F(x_1) G(x_2)$$

  correspond to some multiplicative convolution of sheaves $\mathcal{F} \star \mathcal{G}(1/2)$ if $\mathcal{F}$ and $\mathcal{G}$ are non-geometrically Kummer sheaves $\mathcal{L}_\chi$.

Correlation of trace functions

The main result making it possible to do analytic number theory with trace functions is

Theorem (Deligne, Weil II)

For $\mathcal{F}$ and $\mathcal{G}$ as above,

$$\frac{1}{q}\sum_{x\in\mathbb{F}_q} F(x)\overline{G}(x) = \alpha_{\mathcal{F},\mathcal{G}} + O(C(\mathcal{F})C(\mathcal{G})q^{-1/2})$$

with

$$|\alpha_{\mathcal{F},\mathcal{G}}| = \begin{cases} 1 & \text{if } \mathcal{F} \simeq_{geom} \mathcal{G} \\ 0 & \text{otherwise.} \end{cases}$$

In particular given $C \geq 1$ and $q \gg C^4$, a trace function F uniquely determine its associated sheaf $\mathcal{F}$ amongst all the sheaves of conductor $\leq C$.

Correlation of trace functions: the automorphism group

The group PGL_2 acts on $\mathbb{P}^1_{\mathbb{F}_q}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$$

hence on ℓ-adic sheaves and traces functions by pullback $\gamma^* F(x) = F(\gamma.x)$. Applying Deligne's Theorem to the pair $(F, G) = (F, \gamma^* F)$ one is led to introduce

Definition

The automorphism group of $\mathcal{F}$

$$\mathrm{Aut}_{\mathcal{F}}(\mathbb{F}_q) = \{\gamma \in \mathrm{PGL}_2(\mathbb{F}_q),\ \gamma^*\mathcal{F} \simeq_{geom} \mathcal{F}\} \subset \mathrm{PGL}_2(\mathbb{F}_q).$$

Correlation of trace functions: the automorphism group

An important fact for us is that this group is pretty small

Lemma

One of the following holds

- $|\mathrm{Aut}_{\mathcal{F}}(\mathbb{F}_q)| \leq 60$,
- $\mathrm{Aut}_{\mathcal{F}}(\mathbb{F}_q)$ *is contained in the normalizer of a maximal torus*
- *p divides* $|\mathrm{Aut}_{\mathcal{F}}(\mathbb{F}_q)|$.

In the later case, and if $q \gg_{C(\mathcal{F})} 1$, $\mathcal{F} \simeq_{geom} \gamma^* \mathcal{L}_\psi$ and $\mathrm{Aut}_{\mathcal{F}}(\mathbb{F}_q)$ is conjugate to $N(\mathbb{F}_q)$.

Lemma

If $q \gg_{C(\mathcal{F})} 1$ and $\mathcal{F} \not\simeq_{geom} \mathcal{L}_\psi \otimes \mathcal{L}_\chi$ then

$$|\mathrm{Aut}_{\mathcal{F}}(\mathbb{F}_q) \cap B(\mathbb{F}_q)| = O_{C(\mathcal{F})}(1).$$

Trace functions arising in analytic number theory

- The Obvious ones: ψ, χ, $q^{1/2}\delta_{a \pmod q}$,
- Less obvious: Kloosterman sums arise from the Petterson-Kuznetzov formula; more generally, local q-adic integrals in trace formulas are trace functions (an essential ingredient for Ngo's proof of the fundamental lemma).
- New trace function can be formed from elementary manipulations:
  - Poisson:

$$\sum_{n\sim X} F(n) \sim \frac{X}{q^{1/2}} \sum_{n \ll q/X} \widehat{F}(n)$$

  - Multi-Poisson (a.k.a Voronoi a.k.a Hecke-Godement):

$$\sum_{n\sim X} d_k(n) F(n) = \sum_{n_1.\cdots.n_k \sim X} F(n_1 \cdots n_k) \sim \frac{X}{q^{k/2}} \sum_{n \ll q^k/X} \widetilde{F}(n)$$

$$\widetilde{F}(n) = q^{-1/2} \sum_{x \in \mathbb{F}_q} F(x) \mathrm{Kl}_k(nx; q)$$

Trace functions arising in analytic number theory

- Applications of the Cauchy-Schwarz or Hoelder inequality may result in expressions involving multi-shifted product of trace functions

$$\sum_{x\in\mathbb{F}_q} F(\gamma_1 x)\overline{F}(\gamma'_1 x)\cdots F(\gamma_p x)\overline{F}(\gamma'_p x)$$

and determining whether the sheaf

$$\bigotimes_i \gamma_i^*\mathcal{F}\otimes{\gamma'}_i^*\check{\mathcal{F}}$$

contain the trivial representation or not may require a good knowledge of the geometry monodromy group of $\mathcal{F}$ (usually computed by Katz).

The Polya-Vinogradov method

One of the most basic question is to evaluate the sum of a trace function over an interval of length $< q$ (ie. the correlation of F with the characteristic function $1_{[1,X]},\ X < q$).

Proposition (The Polya-Vinogradov method)

If $\mathcal{F} \not\simeq_{geom} \mathcal{L}_\psi$

$$\sum_{n \leq X} F(n) \ll_{C(\mathcal{F})} q^{1/2} \log q.$$

Proof. By Plancherel

$$\sum_{n \leq X} F(n) = \sum_{u \in \mathbb{F}_q} \widehat{F}(u) \widehat{1_{[1,X]}}(u)$$

$$\ll_{C(\mathcal{F})} \sum_{u \in \mathbb{F}_q} |\widehat{1_{[1,X]}}(u)| \ll_{C(\mathcal{F})} q^{1/2} \log q$$

□

Trace function vs. Intervals

This bound is non trivial as long as $X \gg q^{1/2} \log q$ and it a fundamental question to have nontrivial bounds for smaller values of X. This is know for very specific cases $\chi(x)\psi(x)$, $\chi(x)\psi(P(x))$, $P \in \mathbb{F}_q[X]$ (work of Burgess, ..., Pierce-Heath-Brown).

In general, one can improve slightly on Polya-Vinogradov by complementing PV with interval shifting techniques

Theorem (F-K-M-Raju-Rivat-Soundararajan)

For $\mathcal{F} \not\simeq \mathcal{L}_\psi$

$$\sum_{n \leq X} F(n) \ll_{C(\mathcal{F})} q^{1/2} \log(X/q^{1/2})$$

which is non-trivial as long as $X \gg_{C(\mathcal{F})} q^{1/2}$.

Type II sums of Trace function

Other standard sums one would like to evaluate are bilinear sums $M, N \leq q$

$$\Sigma(\alpha,\beta) = \sum\sum_{m\leq M, n\leq N} \alpha_m \beta_n F(mn) \ll_{C(\mathcal{F})} (MN)^{1/2} \|\alpha\|_2 \|\beta\|_2$$

Proposition (F-K-M)

For $\mathcal{F} \not\simeq \mathcal{L}_\psi \otimes \mathcal{L}_\chi$

$$\Sigma(\alpha,\beta) \ll_{C(\mathcal{F})} (MN)^{1/2} \|\alpha\|_2 \|\beta\|_2 \left(\frac{1}{q^{1/2}} + \frac{1}{M} + \frac{q^{1/2}\log q}{N}\right)^{1/2}$$

which is non-trivial as long as $M \gg 1$, $N \gg q^{1/2} \log q$.

Type II sums of trace functions

Proof. By Cauchy-Schwarz we are led to

$$\sum_{m_1,m_2}\sum \alpha_{m_1}\overline{\alpha}_{m_2} \sum_{n\leq N} F(m_1 n)\overline{F}(m_2 n)$$

and by the P-V method we need to evaluate the Fourier transform

$$\mathcal{C}(m_1, m_2; h) = \sum_{x\in\mathbb{F}_q} F(m_1 x)\overline{F}(m_2 x)e_q(hx) = \sum_{y\in\mathbb{F}_q} \widehat{F}(y)\overline{\widehat{F}}(\gamma.y),$$

with $\gamma = \begin{pmatrix} m_1 & -m_1 h \\ 0 & m_2 \end{pmatrix}$. Since $|\mathrm{Aut}_{\widehat{\mathcal{F}}}(\mathbb{F}_q) \cap B(\mathbb{F}_q)| = O_{C(\mathcal{F})}(1)$ we conclude that

$$\mathcal{C}(m_1, m_2; h) \ll q^{1/2}$$

for all but $O_{C(\mathcal{F})}(1)$ values of m_1/m_2. □

Type II sums of trace function: a large sieve type inequality

Theorem (Ping-Sawin)

For $\mathcal{F} \not\simeq \mathcal{L}_\chi$, $M, N \leq q$

$$\sum\sum_{m \leq M, n \leq N} \alpha_m \beta_n F(mn) \ll_{C(\mathcal{F})} q^{1/2} \|\alpha\|_2 \|\beta\|_2$$

which is non-trivial as long as $MN \gg q$.

The proof is based on Katz's theory of multiplicative convolution.

Trace function vs. modular forms

Theorem (F-K-M)

For $\mathcal{F} \not\simeq \mathcal{L}_\psi$ and f a Hecke-eigenform (Eisenstein or cuspidal)

$$\sum_{n \leq X} F(n)\lambda_f(n) \ll_{C(\mathcal{F})} q^{1-1/16+o(1)}.$$

Corollary (Equidistribution of algebraically weighted Hecke points)

For $\mathcal{F} \not\simeq \mathcal{L}_\psi$ and f a Hecke-eigenform (Eisenstein or cuspidal)

$$\frac{1}{q}\sum_{n \leq q} \widehat{F}(n) f\left(\frac{n+i}{q}\right) = o_{C(\mathcal{F})}(1)$$

For instance, the Hecke points $\frac{n+i}{q}$ with $n \pmod q$ in the image of a non-constant polynomial map on $\mathbb{F}_q$ of bounded degree are equidistributed.

Trace function vs. modular forms: idea of the proof

If $F = \chi$ the bound

$$\sum_{n \leq X} F(n)\lambda_f(n) \ll_{C(\mathcal{F})} q^{1-1/16+o(1)}$$

is equivalent to a subconvex bound for $L(f \otimes \chi, 1/2)$ and it is no surprise that here too we proceed by *amplification*. Most proofs, amplify the character χ, using its quasi-invariance under multiplication. This is not available for a general F; instead we follow an alternative proof due to Bykovsky and amplify the form f (amongst forms of level q) using the Hecke-invariance of f.

Trace function vs. modular forms: idea of the proof

In the end, the amplifyer produce matrices $\gamma \in \mathrm{GL}_2(\mathbb{Q}) \cap M_2(\mathbb{Z})$ and the correlation sums

$$\mathcal{C}(\widehat{F}, \gamma) = \sum_{x \in \mathbb{F}_q} \widehat{F}(x)\overline{\widehat{F}}(\gamma.x)$$

and the aim is to show that "often" these sums are bounded by $\ll q^{1/2}$; equivalently that "often"

$$\gamma \pmod q \notin \mathrm{Aut}_{\widehat{\mathcal{F}}}(\mathbb{F}_q)$$

which we do using that $\mathrm{Aut}_{\widehat{\mathcal{F}}}(\mathbb{F}_q)$ is not big and repulsion arguments (eg. Linnik's basic lemma).

Trace functions vs. GL_3 forms

In an recent "tour de force" (to appear soon), R. Munshi obtained a subconvex bound for character twist L-functions of GL_3 automorphic form:

Theorem (Munshi)

There exists an absolute constant $\delta > 0$ such that for f a $\mathrm{SL}_3(\mathbb{Z})$-Hecke cuspform, one has

$$L(f \otimes \chi, 1/2) \ll q^{3/4-\delta}$$

or essentially equivalently

$$\sum_{n \sim q^{3/2}} \lambda_f(n,1)\chi(n) \ll q^{3/2-\delta}.$$

Trace functions vs. GL_3 forms

Munshi's proof which is based on a very difficult elaborate variant of te δ-symbol method and does not really use the fact that χ is a character: one has the following direct generalization of Munshi's theorem

Theorem (K-M-Nelson)

For f as above and F a trace function, one has

$$\sum_{n\sim q^{3/2}} \lambda_f(n,1)F(n) \ll_{C(\mathcal{F})} q^{3/2-\delta}.$$

As before this result has an interpretation in terms of algebraically twisted equidistribution on $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})$.

Trace function vs. modular forms: Eisenstein series

For application to classical analytic number theory the most interesting case of the above bound is when $f = E$ is an Eisenstein series: this translates into bounds of the shape

$$\sum_{n \sim X} F(n)d(n) \ll_{C(\mathcal{F})} q^{1-1/8+o(1)}$$

or (more useful) the smooth bilinear sum

$$\sum\sum_{m \sim M, n \sim N} F(mn) \ll_{C(\mathcal{F})} q^{1-1/8+o(1)}$$

which are non-trivial if $X, MN \gg q^{1-1/8+o(1)}$. In particular one obtain a non-trivial bound for $M = N = q^{1/2-1/16+o(1)}$ which is below the P-V range $M = N = q^{1/2}$ but general F!

Trace function vs. primes

One can use the above results to investigate how trace function correlate with the primes:

Theorem (F-K-M)

For $\mathcal{F} \not\simeq \mathcal{L}_\psi \otimes \mathcal{L}_\chi$

$$\sum_{p \leq q} F(p) \ll_{C(\mathcal{F})} q^{1-1/48+o(1)}.$$

Proof. We use a combinatorial decomposition of the primes to reduce the bounds to various type of bilinear sums:

1. Type I: $\sum\sum_{m\sim M, n\sim N} \alpha(m)F(mn)$, $M \leq q^\eta, N \geq q^{1-\eta}$ with η small.
2. Type II: $\sum\sum_{m\sim M, n\sim N} \alpha_m\beta_n F(mn)$, $q^\eta N \leq q^{1/2-\eta}$, $q^{1/2+\eta}N \leq q$ with η small.
3. Type I1/2: $\sum\sum_{m\sim M, n\sim N} F(mn)$, $q^{1/2-\eta}M, N \geq q^{1/2}$ with η small.

□

Primes in large arithmetic progressions to large moduli

In the spring of 2013, Y. Zhang shocked the world by proving the existence of bounded gaps between primes. Shortly after the paper was released several people gathered aroung the Polymath8 project coordinated by T. Tao to look to improve on Zhang's initial constant which was 70.10^6. The key ingredient of the proof was a Bombieri-Vinogradov type theorem for smooth moduli going beyond the $1/2$ level of distribution of primes in the spirit of earlier works of Fouvry-Iwaniec and Bombieri-Friedlander-Iwaniec and is a technical masterpiece using heavily results concerning trace functions. Eventually Polymath8 could imporve on Zhang's result and other technique of him to get Zhang's constant to ≤ 5000.

Primes in large arithmetic progressions to large moduli

But in the fall of the same year a second revolution occured with the work of J. Maynard who provided a radically different and more powerfull approach to the Goldston-Pintz-Yildirim sieveing argument and proved that Zhang's constant was ≤ 600 using "only" the Bombieri-Vinogradov theorem. J. Maynard joined Polymath8 and 600 was further reduced to 246. Nevertheless, Zhang theorem on primes in arithmetic progressions (although it plays no role in the 246) is still an landmark result: a version of this theorem is (Zhang had 1/584)

Theorem (Zhang, Polymath8)

For any any $a \geq 1$, any $\eta < 7/300$ and $A \geq 1$

$$\sum_{\substack{q \text{ smooth} \\ q \leq X^{1/2+\eta}}} |\pi(x; q, a) - \frac{1}{\varphi(q)}\pi(x)| \ll_{A,\eta} \frac{x}{\log^A x}.$$

Ideas of the proof

The proof proceed by a combinatorial decomposition of the primes and by using the dispersion method of Linnik. This split the problem into the one of bounding non-trivially three types of multilinear exponential sums (I, II, III) with smooth moduli. The smoothness enter critically in evaluation of these sums to shorten the size of the moduli relative to the size of the variable of the argument (passing the P-V limit) using the q-vdC method.

The q-van der Corput method

The q-vdC method is a method to bound non-trivially and beyond the Polya-Vinogradov range (ie. $X \ll q^{1/2}$) sums of the shape

$$\Sigma(F;X) = \sum_{n \leq X} F(n;q)$$

where F is periodic of period q and $q = \prod_{i=1}^{k} q_i$ a highly factorable squarefree integer. We assume that F is of the shape

$$F(.;q) = \prod_i F_i(.)$$

were F_i is a trace function modulo q_i (which may depend on $\hat{q}_i = q/q_i$):

The q-van der Corput method

The method produces expressions of the shape

$$\sum_{x \in \mathbb{F}_{q_i}} F_i(x)\overline{F_i(x+l)}\psi(hx), \; l, h \in \mathbb{F}_q.$$

which need be bounded by $O(q^{1/2})$

Theorem

For q admitting suitable factorizations, the q-vdC method works whenever

$$\mathcal{F}_i \not\simeq \mathcal{L}_{\psi(P(.))}$$

for P a polynomial of degree ≤ 2 (ie. F_i is not proportional to $x \mapsto \psi(P(x))$).

The q-van der Corput method

In the polymath8 project this method has been used for

$$F_i(x) = \psi(f(x))$$

with $f \in \mathbb{F}_{q_i}(X)$ non-polynomial,

$$F_i(x) = \mathrm{Kl}_3(x)$$

and

$$F_i(x) = \frac{1}{p^{1/2}} \sum_{\substack{y \in \mathbb{F}_{q_i} \\ (y+x)y \neq 0}} \psi\left(\frac{1}{(y+x)y} + ey\right).$$

The later function is not present in Katz's books; it should be possible to study the corresponding sheaf but there is a nice trick: the Fourier transform of F_i is a pullback of the Kloosterman sum Kl_3 which therefore satisfies the q-vdC criterion; but the criterion is invariant under Fourier transform...

Double q-van der Corput method

As pointed out by two participants of the polymath blog, is it possible to refine the van der Corput method further and to introduce a "double" vdC-method. A succesful application of that method require bounds of the shape

$$\sum_{x,y\in\mathbb{F}_q} \mathrm{Kl}_3(f(x,y))\overline{\mathrm{Kl}}_3(f(x+l,y+l'))\psi(hx+h'y) \ll q$$

where f is an explicit rational monomial fraction.

Theorem (Polymath8 + W. Sawin)

Zhang's theorem on prime in large arithmetic progressions with smooth modili holds with 7/300 *replaced by a strictly larger constant.*

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The fourth moment of Dirichlet L-functions

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Abstract

We compute the fourth moment of Dirichlet L-functions at the central point for prime moduli, with a power savings in the error term.

1. Introduction

Estimating moments of families of L-functions is a central problem in number theory due in large part to extensive applications. Yet, these moments are seen to be natural objects to study in their own right as they illuminate structure of the family and display beautiful symmetries.

The Riemann zeta function has by far garnered the most attention from researchers. Ingham [Ing26] proved the asymptotic formula

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = a_4(\log T)^4 + O((\log T)^3)$$

The fourth moment of Dirichlet L-functions

$$M_4(q) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} |L(\chi, 1/2)|^4$$

Theorem (Young/FKM-Blomer-Milicevic)

One has

$$M_4(q) = P_4(\log q) + O(q^{-\eta + o(1)})$$

$$\eta = 5/512 \ (Y), \ \text{or} \ \eta = 1/32 \ (BFKMM).$$

The fourth moment of Dirichlet L-functions: variants

Besides improving of existing error terms, our methods allow for new results: an evaluation of a "mixed" moment involving Eisenstein and cuspidal L-functions:

$$M_{f,E}(q) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} L(f \otimes \chi)\overline{L(\chi, 1/2)}^2.$$

By treating more complicated sums of products of Kloosterman sums we have

Theorem (BFKMM)

For f cuspidal

$$M_{f,E}(q) = \frac{L(f,1)}{\zeta(2)} + O_f(q^{-1/68+o(1)})$$

The fourth moment of Dirichlet L-functions: variants

We also propose a tentative approach to evaluate the second moment of $L(f \otimes \chi, 1/2)$ for f cuspidal:

Theorem (BFKMM)

For f cuspidal

$$M_{f,f}(q) = \frac{1}{\varphi^*(q)} \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} |L(f \otimes \chi, 1/2)|^2$$
$$= P_{1,f}(\log q) + O_f(q^{-1/144+o(1)}),$$

assuming good bounds (squareroot saving) on certain three-dimensional sums of product of Kloosterman sums.

The fourth moment of Dirichlet L-functions: proof

By partition of unity and averaging over $\chi \pmod q$ one is reduced to evaluate sums of the shape

$$q^{1/2} \sum\sum_{\substack{m\sim M, n\sim N\\ m\equiv n \pmod q}} d(m)d(n)$$

with

$$M \leq N, MN = q^{2+o(1)}$$

One want to evaluate this sum with a main term up to an error of size $O(q^{3/2-\eta+o(1)})$.
Applying Voronoi in the n variable one obtains (up to a main term) a sum of the shape

$$\frac{N}{q} \sum\sum_{m\sim M, n\sim q^2/N} d(m)d(n)\mathrm{Kl}_2(mn)$$

If $M \ll q^{1/2-o(1)}$ the trivial bound is OK.

The fourth moment of Dirichlet L-functions: proof

We need to cover the full range

$$q^{1/2+o(1)} \leq M,\ 1 \leq N/M \leq q^{1/2+o(1)},\ MN = q^{2+o(1)}.$$

When $M \geq q^{1/2+\delta}$ we rewrite the sum is an instance of the shifted convolution problem

$$\sum_{h \ll N/q} \sum\sum_{\substack{m \sim M, n \sim N \\ m-n=qh}} d(m)d(n)$$

which is handled by spectral methods and may depend on an approximation to the Ramanujan-Petersson conjecture. However Blomer-Milicevic have an approach giving non trivial bounds independently of R-P. This method is sufficient to handle non-trivially all the range

$$1 \leq N/M \leq q^{1/2-o(1)}.$$

The fourth moment of Dirichlet L-functions: proof

The remaining range

$$q^{1/2-o(1)} \leq N/M \leq q^{1/2+o(1)},\ M, N' = q^2/N \sim q^{1/2+o(1)}$$

is exactly the P-V range but neither variables are smooth (because weighted by the divisor function).
Opening the divisor function we write the sum

$$\sum\sum_{m\sim M, n\sim q^2/N} d(m)d(n)\mathrm{Kl}_2(mn)$$

as a quadrilinear smooth sum

$$\sum\sum_{\substack{m_1\sim M_1, \mathfrak{m}_2\sim M_2 \\ n_1\sim N_1, n_2\sim N_2}} \mathrm{Kl}_2(m_1 m_2 n_1 n_2)$$

with $M_1M_2 = q^{1/2+o(1)},\ N_1N_2 = q^{1/2+o(1)},\ M_1 \leq M_2,\ N_1 \leq N_2.$

Grouping variables, we can then apply the Type II sum bound. The bounds are good unless $M_1 = N_1 = q^{o(1)}$ but then the last sum is

$$\sum_{\substack{m_1,n_1 \sim q^{o(1)} \\ m_2,n_2 \sim q^{1/2-o(1)}}} \sum \mathrm{Kl}_2(m_1 n_1 m_2 n_2) \ll q^{1-1/8+o(1)}$$

The mixed moment

This case requires evaluating non-trivially a more general type I sum within (and below) the polynomial range:

$$\sum_{m,n\sim q^{1/2}} \alpha_m \mathrm{Kl}_k(mn) \ll \|\alpha\| q^{3/4-1/24+o(1)},\ k=2.$$

For this, we use the Vinogradov-Karatsuba "shift by ab" technique. This leads to bounding two-variable sums of product of Kloosterman sums for $k=2$ and for suitable values of the parameters b_1, b_2, b_1', b_2', h

$$\Sigma(\mathrm{Kl}_k;q) := \sum\sum_{r,s \pmod q} \mathrm{Kl}_k(s(r+b_1))\mathrm{Kl}_k(s(r+b_2)) \times \overline{\mathrm{Kl}_k(s(r+b_1'))\mathrm{Kl}_k(s(r+b_2'))} e_q(hs) \ll q.$$

The mixed moment

In subsequent work with W. Sawin, this bound was extended to more general k:

Theorem (KM-W. Sawin)

One has for $k = 2$ or $k > 2$ and odd

$$\sum_{m,n \sim q^{1/2}} \alpha_m \mathrm{Kl}_k(mn) \ll \|\alpha\| q^{3/4-1/24+o(1)}$$

and it is expected that a small modification of the argument will give the case of even k as well.
Such bounds have interesting consequences to the study of the distribution of arithmetic functions n arithmetic progressions.

Summary

Always read attentively Peter Sarnak's papers: whether you understand them or not and even if you mess them up, something interesting will always come out of them !

HAPPY BIRTHDAY PETER !