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Heegner points and non-vanishing of Rankin/Selberg L-functions

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Abstract. We discuss the nonvanishing of central values $L(\frac{1}{2}, f \otimes \chi)$, where f is a fixed automorphic form on GL(2) and χ varies through class group characters of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$, as D varies; we prove results of the nature that at least $D^{1/5000}$ such twists are nonvanishing. We also discuss the related question of the rank of a fixed elliptic curve $E/\mathbb{Q}$ over the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$, as D varies. The tools used are results about the distribution of Heegner points, as well as subconvexity bounds for L-functions.

1. Introduction

The problem of studying the non-vanishing of central values of automorphic L-functions arise naturally in several contexts ranging from analytic number theory, quantum chaos and arithmetic geometry and can be approached by a great variety of methods (ie. via analytic, geometric spectral and ergodic techniques or even a blend of them).

Amongst the many interesting families that may occur, arguably one of the most attractive is the family of (the central values of) *twists by class group characters*: Let f be a modular form on PGL(2) over $\mathbb{Q}$ and K a quadratic field of discriminant D. If χ is a ring class character associated to K, we may form the L-function $L(s, f \otimes \chi)$: the Rankin-Selberg convolution of f with the θ-series $g_\chi(z) = \sum_{\{0\} \neq \mathfrak{a} \subset O_K} \chi(\mathfrak{a}) e(N(\mathfrak{a})z)$. Here g_χ is a holomorphic Hecke-eigenform of weight 1 on $\Gamma_0(D)$ with Nebentypus χ_K and a cusp form iff χ is not a quadratic character[1].

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[1]Equivalently, one can define $L(s, f \otimes \chi)$ as $L(s, \Pi_f \otimes \chi)$, where Π_f is the base-change to K of the automorphic representation underlying f, and χ is regarded as a character of $\mathbb{A}_K^\times / K^\times$.

We will always assume that the conductor of f is coprime to the discriminant of K. In that case the sign of the functional equation equals $\pm\left(\frac{-D}{N}\right)$, where one takes the $+$ sign in the case when f is Maass, and the $-$ sign if f is weight 2 holomorphic (these are the only cases that we shall consider).

Many lovely results have been proved in this context: we refer the reader to §1.3 for a review of some of these results. A common theme is the use, implicit or explicit, of the equidistribution properties of special points. The purpose of this paper is to give an informal exposition (see §1.1) as well as some new applications of this idea. Since our goal is merely to illustrate what can be obtained along these lines we have not tried to reach the most general results that can be obtained and, in particular, we limit ourselves to the non-vanishing problem for the family of unramified ring class characters of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ of large discriminant D.

We prove

Theorem 1. *Let $f(z)$ be a weight 0, even, Maass (Hecke-eigen) cuspform on the modular surface $X_0(1)$; then, for any $0 < \delta < 1/2700$, one has the lower bound*

$$\left|\{\chi \in \widehat{\mathrm{Cl}_K},\ L(f\otimes\chi, 1/2) \neq 0\}\right| \gg_{\delta,f} D^{\delta}$$

Theorem 2. *Let q be a prime and $f(z)$ be a holomorphic Hecke-eigen cuspform of weight 2 on $\Gamma_0(q)$ such that q remains inert in K; then, for any $0 < \delta < 1/2700$, one has the lower bound*

$$\left|\{\chi \in \widehat{\mathrm{Cl}_K},\ L(f\otimes\chi, 1/2) \neq 0\}\right| \gg_{\delta,f} D^{\delta}$$

for any $\delta < 1/2700$.

The restriction to either trivial or prime level in the theorems above is merely for simplification (to avoid the occurrence of oldforms in our analysis) and extending these results to more general levels is just a technical matter. Another arguably more interesting generalization consists in considering levels q and quadratic fields K such that the sign of the functional equation is -1: then one expects that the number of χ such that the first derivative $L'(f\otimes\chi, 1/2) \neq 0$ is $\gg D^{\delta}$ for some positive absolute δ. This can be proven along the above lines at least when f is holomorphic of weight 2 by using the Gross/Zagier formulas; the proof however is significantly more difficult and will be dealt with elsewhere; interestingly the proof combines the two types of equidistribution results encountered in the proof of Theorems 1 and 2 above. In the present paper, we give, for the sake of diversity, an entirely different, purely geometric, argument of such a generalization when f corresponds to an elliptic curve. For technical reasons we need to assume a certain hypothesis "$S_{\beta,\theta}$" that guarantees there are enough small split primes in K. This is a fairly common feature of such problems (cf. [**DFI95**], [**EY03**]) and we regard it as almost orthogonal to the main issues we are considering. Given $\theta > 0$ and $\alpha \in]0, 1]$ we consider

Hypothesis $S_{\beta,\theta}$. *The number of primitive*[2] *integral ideals $\mathfrak{n}$ in O_K with $\mathrm{Norm}(\mathfrak{n}) \leqslant D^{\theta}$ is $\gg D^{\beta\theta}$.*

Actually, in a sense it is remarkable that Theorems 1 and 2 above *do not* require such a hypothesis. It should be noted that $S_{\beta,\theta}$ is always true under the generalized

[2]That is, not divisible by any nontrivial ideal of the form (m), with $m \in \mathbb{Z}$.

Lindelöf hypothesis and can be established unconditionally with any $\alpha \in]0, 1/3[$ for those Ds whose largest prime factor is a sufficiently small power of D by the work of Graham/Ringrose [**GR90**](see [**DFI95**] for more details).

THEOREM 3. *Assume $S_{\beta,\theta}$. Let E be an elliptic curve over $\mathbb{Q}$ of squarefree conductor N, and suppose D is odd, coprime to N, and so that all primes dividing N split in the quadratic extension $\mathbb{Q}(\sqrt{-D})$. Then the Mordell-Weil rank of E over the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$ is $\gg_\epsilon D^{\delta-\epsilon}$, where $\delta = \min(\beta\theta, 1/2 - 4\theta)$.*

Neither the statement nor the proof of Theorem 3 make any use of automorphic forms; but (in view of the Gross/Zagier formula) the proof actually demonstrates that the number of nonvanishing central derivatives $L'(f_E \otimes \chi, 1/2)$ is $\gg D^\alpha$, where f_E is the newform associated to E. Moreover, we use the ideas of the proof to give another proof (conditional on $S_{\beta,\theta}$) of Thm. 1.

We conclude the introduction by describing the main geometric issues that intervene in the proof of these Theorems. Let us consider just Theorem 1 for clarity. In that case, one has a collection of Heegner points in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$ with discriminant $-D$, parameterized by Cl_K. The collection of values $L(\frac{1}{2}, f \otimes \chi)$ reflects – for a fixed Maass form f, varying χ through $\widehat{\mathrm{Cl}_K}$ – the distribution of Heegner points. More precisely, it reflects the way in which the distribution of these Heegner points interacts with the subgroup structure of Cl_K. For example, if there existed a subgroup $H \subset \mathrm{Cl}_K$ such that points in the same H-coset also tend to cluster together on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$, this would cause the L-values to be distributed unusually. Thus, in a sense, whatever results we are able to prove about these values are (geometrically speaking) assertions that the group structure on Cl_K does not interact at all with the "proximity structure" that arises from its embedding into $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$.

REMARK 1.1. Denote by $\mathrm{Cl}_K = \mathrm{Pic}(O_K)$ the class group of O_K and by $\widehat{\mathrm{Cl}_K}$ its dual group. We write $h_K = |\mathrm{Cl}_K| = |\widehat{\mathrm{Cl}_K}|$ for the class number of O_K. By Siegel's theorem one has

$$h_K \gg_\varepsilon D^{1/2-\varepsilon} \tag{1}$$

(where the constant implied is not effective) so the lower bounds of Theorems 1 and 2 are far from giving a constant proportion of nonvanishing values. (In the case where f is Eisenstein, Blomer has obtained much better results: see Sec. 1.3). Moreover, both proofs make use of (1) so the constants implied are ineffective.

1.1. Nonvanishing of a single twist. Let us introduce some of the main ideas of the present paper in the most direct way, by sketching two very short proofs that *at least one* twist is nonvanishing in the context of Theorem 1. We denote by $\mathbf{H}$ the upper-half plane. To the quadratic field $K = \mathbb{Q}(\sqrt{-D})$ – where we always assume that $-D$ is a fundamental discriminant – and each ideal class x of the maximal order O_K of $\mathbb{Q}(\sqrt{-D})$ there is associated a Heegner point $[x] \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$. [3]

One can describe the collection $He_K := \{[x] : x \in \mathrm{Cl}_K\}$ using the moduli description of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$: if one identifies $z \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$ with the isomorphism

[3]Namely, $[x]$ is represented by the point $\frac{-b+\sqrt{-D}}{2a}$, where $au^2+buv+cv^2$ is a quadratic form of discriminant $-D$ corresponding to the ideal class x, i.e. there exists a fractional ideal $\mathfrak{J}$ in the class x and a $\mathbb{Z}$-basis α, β for $\mathfrak{J}$ so that $\mathrm{Norm}(u\alpha + v\beta) = \mathrm{Norm}(\mathfrak{J})(au^2 + buv + cv^2)$.

class of elliptic curves over $\mathbb{C}$, via $z \in \mathbf{H} \to \mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$, then He_K is identified with the set of elliptic curves with CM by O_K.

If f is a Maass form and χ a character of Cl_K, one has associated a twisted L-function $L(s, f \times \chi)$, and it is known, from the work of Waldspurger and Zhang [**Zha01, Zha04**] that

$$L(f \otimes \chi, 1/2) = \frac{2}{\sqrt{D}} \Big| \sum_{x \in \mathrm{Cl}_K} \overline{\chi}(x) f([x]) \Big|^2. \tag{2}$$

In other words: the values $L(\frac{1}{2}, f \otimes \chi)$ are the squares of the "Fourier coefficients" of the function $x \mapsto f([x])$ on the finite abelian group Cl_K. The Fourier transform being an isomorphism, in order to show that there exists *at least one* $\chi \in \widehat{\mathrm{Cl}_K}$ such that $L(1/2, f \otimes \chi)$ is nonvanishing, it will suffice to show that $f([x]) \neq 0$ for at least one $x \in \mathrm{Cl}_K$. There are two natural ways to approach this (for D large enough):

(1) Probabilistically: show this is true for a random x. It is known, by a theorem of Duke, that the points $\{[x] : x \in \mathrm{Cl}_K\}$ become equidistributed (as $D \to \infty$) w.r.t. the Riemannian measure on Y; thus $f([x])$ is nonvanishing for a random $x \in \mathrm{Cl}_K$.

(2) Deterministically: show this is true for a special x. The class group Cl_K has a distinguished element, namely the identity $e \in \mathrm{Cl}_K$; and the corresponding point $[e]$ looks very special: it lives very high in the cusp. Therefore $f([e]) \neq 0$ for obvious reasons (look at the Fourier expansion!)

Thus we have given two (fundamentally different) proofs of the fact that there exists χ such that $L(\frac{1}{2}, f \otimes \chi) \neq 0$! Soft as they appear, these simple ideas are rather powerful. The main body of the paper is devoted to quantifying these ideas further, i.e. pushing them to give that *many* twists are nonvanishing.

Remark 1.2. The first idea is the standard one in analytic number theory: to prove that a family of quantities is nonvanishing, compute their average. It is an emerging philosophy that many averages in analytic number theory are connected to equidistribution questions and thus often to ergodic theory.

Of course we note that, in the above approach, one does not really need to know that $\{[x] : x \in \mathrm{Cl}_K\}$ become equidistributed as $D \to \infty$; it suffices to know that this set is becoming *dense*, or even just that it is not contained in the nodal set of f. This remark is more useful in the holomorphic setting, where it means that one can use *Zariski dense* as a substitute for *dense*. See [**Cor02**].

In considering the second idea, it is worth keeping in mind that $f([e])$ is *extremely* small – of size $\exp(-\sqrt{D})$! We can therefore paraphrase the proof as follows: the L-function $L(\frac{1}{2}, f \otimes \chi)$ admits a certain canonical square root, which is not positive; then the sum of all these square roots is very small but known to be nonzero!

This seems of a different flavour from any analytic proof of nonvanishing known to us. Of course the central idea here – that there is always a Heegner point (in fact many) that is very high in the cusp – has been utilized in various ways before. The first example is Deuring's result [**Deu33**] that the failure of the Riemann hypothesis (for ζ) would yield an effective solution to Gauss' class number one problem; another particularly relevant application of this idea is Y. André's lovely proof [**And98**] of the André–Oort conjecture for products of modular surfaces.

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1.2. Quantification: nonvanishing of many twists. As we have remarked, the main purpose of this paper is to give quantitative versions of the proofs given in §1.1. A natural benchmark in this question is to prove that a *positive proportion* of the L-values are nonzero. At present this seems out of reach in our instance, at least for general D. We can compute the first but not the second moment of $\{L(\frac{1}{2}, f \otimes \chi) : \chi \in \widehat{\mathrm{Cl}_K}\}$ and the problem appears resistant to the standard analytic technique of "mollification." Nevertheless we will be able to prove that $\gg D^{\alpha}$ twists are nonvanishing for some positive α.

We now indicate how both of the ideas indicated in the previous section can be quantified to give a lower bound on the number of χ for which $L(\frac{1}{2}, f \otimes \chi) \neq 0$. In order to clarify the ideas involved, let us consider the *worst case*, that is, if $L(\frac{1}{2}, f \otimes \chi)$ was only nonvanishing for a single character χ_0. Then, in view of the Fourier-analytic description given above, the function $x \mapsto f([x])$ is a linear multiple of χ_0, i.e. $f([x]) = a_0\chi_0(x)$, some $a_0 \in \mathbb{C}$. There is no shortage of ways to see that this is impossible; let us give two of them that fit naturally into the "probabilistic" and the "deterministic" framework and will be most appropriate for generalization.

(1) Probabilistic: Let us show that in fact $f([x])$ cannot behave like $a_0\chi_0(x)$ for "most" x. Suppose to the contrary. First note that the constant a_0 cannot be too small: otherwise $f(x)$ would take small values everywhere (since the $[x] : x \in \mathrm{Cl}_K$ are equidistributed). We now observe that the twisted average $\sum f([x])\overline{\chi_0(x)}$ must be "large": but, as discussed above, this will force $L(\frac{1}{2}, f \otimes \chi_0)$ to be large. As it turns out, a *subconvex* bound on this L-function is precisely what is needed to rule out such an event. [4]
(2) Deterministic: Again we will use the properties of certain distinguished points. However, the identity $e \in \mathrm{Cl}_K$ will no longer suffice by itself. Let $\mathfrak{n}$ be an integral ideal in O_K of small norm (much smaller than $D^{1/2}$). Then the point $[\mathfrak{n}]$ is still high in the cusp: indeed, if we choose a representative z for $[\mathfrak{n}]$ that belongs to the standard fundamental domain, we have $\Im(z) \asymp \frac{D^{1/2}}{\mathrm{Norm}(\mathfrak{n})}$. The Fourier expansion now shows that, under some mild assumption such as $\mathrm{Norm}(\mathfrak{n})$ being odd, the sizes of $|f([e])|$ and $|f([\mathfrak{n}])|$ must be wildly different. This contradicts the assumption that $f([x]) = a_0\chi(x)$.

As it turns out, both of the approaches above can be pushed to give that a large number of twists $L(\frac{1}{2}, f \otimes \chi)$ are nonvanishing. However, as is already clear from the discussion above, the "deterministic" approach will require some auxiliary ideals of O_K of small norm.

[4]Here is another way of looking at this. Fix some element $y \in \mathrm{Cl}_K$. If it were true that the function $x \mapsto f([x])$ behaved like $x \mapsto \chi_0(x)$, it would in particular be true that $f([xy]) = f([x])\chi_0(y)$ for all x. This could not happen, for instance, if we knew that the collection $\{[x], [xy]\}_{x \in \mathrm{Cl}_d} \subset Y^2$ was equidistributed (or even dense). Actually, this is evidently not true for all y (for example $y = e$ or more generally y with a representative of small norm) but one can prove enough in this direction to give a proof of many nonvanishing twists if one has enough small split primes. Since the deterministic method gives this anyway, we do not pursue this.

1.3. Connection to existing work. As remarked in the introduction, a considerable amount of work has been done on nonvanishing for families $L(f\otimes\chi,1/2)$ (or the corresponding family of derivatives). We note in particular:

(1) Duke/Friedlander/Iwaniec and subsequently Blomer considered the case where $f(z)=E(z,1/2)$ is the standard non-holomorphic Eisenstein series of level 1 and weight 0 and $\Xi=\widehat{\mathrm{Cl}_K}$ is the group of unramified ring class characters (ie. the characters of the ideal class group) of an imaginary quadratic field K with large discriminant (the central value then equals $L(g_\chi,1/2)^2=L(K,\chi,1/2)^2$). In particular, Blomer [**Blo04**], building on the earlier results of [**DFI95**], used the mollification method to obtain the lower bound

$$|\{\chi\in\widehat{\mathrm{Cl}_K},\ L(K,\chi,1/2)\neq 0\}|\gg\prod_{p|D}(1-\frac{1}{p})\widehat{\mathrm{Cl}_K}\text{ for }|\mathrm{disc}(K)|\to+\infty. \tag{3}$$

This result is evidently much stronger than Theorem 1.

Let us recall that the mollification method requires the asymptotic evaluation of the first and second (twisted) moments

$$\sum_{\chi\in\widehat{\mathrm{Cl}_K}}\chi(\mathfrak{a})L(g_\chi,1/2),\quad\sum_{\chi\in\widehat{\mathrm{Cl}_K}}\chi(\mathfrak{a})L(g_\chi,1/2)^2$$

(where $\mathfrak{a}$ denotes an ideal of O_K of relatively small norm) which is the main content of [**DFI95**]. The evaluation of the second moment is by far the hardest; for it, Duke/Friedlander/Iwaniec started with an integral representation of the $L(g_\chi,1/2)^2$ as a double integral involving two copies of the theta series $g_\chi(z)$ which they averaged over χ; then after several tranformations, they reduced the estimation to an equidistribution property of the Heegner points (associated with O_K) on the modular curve $X_0(N_{K/\mathbb{Q}}(\mathfrak{a}))(\mathbb{C})$ which was proven by Duke [**Duk88**].

(2) On the other hand, Vatsal and Cornut, motivated by conjectures of Mazur, considered a nearly orthogonal situation: namely, fixing f a holomorphic cuspidal newform of weight 2 of level q, and K an imaginary quadratic field with $(q,\mathrm{disc}(K))=1$ and fixing an auxiliary unramified prime p, they considered the non-vanishing problem for the central values

$$\{L(f\otimes\chi,1/2),\ \chi\in\Xi_K(p^n)\}$$

(or for the first derivative) for $\Xi_K(p^n)$, the ring class characters of K of exact conductor p^n (the primitive class group characters of the order O_{K,p^n} of discriminant $-Dp^{2n}$) and for $n\to+\infty$ [**Vat02, Vat03, Cor02**]. Amongst other things, they proved that if $p\nmid 2q\mathrm{disc}(K)$ and if n is large enough – where "large enough" depends on f,K,p – then $L(f\otimes\chi,1/2)$ or $L'(f\otimes\chi,1/2)$ (depending on the sign of the functional equation) is non-zero for *all* $\chi\in\Xi_K(p^n)$.

The methods of [**Cor02, Vat02, Vat03**] look more geometric and arithmetic by comparison with that of [**Blo04, DFI95**]. Indeed they combine the expression of the central values as (the squares of) suitable periods on Shimura curves, with some equidistribution properties of CM points which are obtained through ergodic arguments (i.e. a special case of Ratner's theory on the classification of measures invariant under unipotent orbits), reduction and/or congruence arguments to pass from the "definite case" to the "indefinite case" (i.e. from the non-vanishing of central values to the non-vanishing of the first derivative at 1/2) together with the invariance property of non-vanishing of central values under Galois conjugation.

1.4. Subfamilies of characters; real qudratic fields. There is another variant of the nonvanishing question about which we have said little: given a subfamily $\mathscr{S} \subset \widehat{\mathrm{Cl}_K}$, can one prove that there is a nonvanishing $L(\frac{1}{2}, f \otimes \chi)$ for some $\chi \in \mathscr{S}$? Natural examples of such $\mathscr{S}$ arise from cosets of subgroups of $\widehat{\mathrm{Cl}_K}$. We indicate below some instances in which this type of question arises naturally.

(1) If f is holomorphic, the values $L(\frac{1}{2}, f \otimes \chi)$ have arithmetic interpretations; in particular, if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $L(\frac{1}{2}, f^\sigma \otimes \chi^\sigma)$ is vanishing if and only if $L(\frac{1}{2}, f \otimes \chi)$ is vanishing. In particular, if one can show that one value $L(\frac{1}{2}, f \otimes \chi)$ is nonvanishing, when χ varies through the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(f))$-orbit of some fixed character χ_0, then they are *all* nonvanishing.

This type of approach was first used by Rohrlich, [**Roh84**]; this is also essentially the situation confronted by Vatsal. In Vatsal's case, the Galois orbits of χ in question are precisely cosets of subgroups, thus reducing us to the problem mentioned above.

(2) Real quadratic fields: One can ask questions similar to those considered here but replacing K by a *real* quadratic field. It will take some preparation to explain how this relates to cosets of subgroups as above.

Firstly, the question of whether there exists a class group character $\chi \in \widehat{\mathrm{Cl}_K}$ such that $L(\frac{1}{2}, f \otimes \chi) \neq 0$ is evidently not as well-behaved, because the size of the class group of K may fluctuate wildly. A suitable analogue to the imaginary case can be obtained by replacing Cl_K by the *extended class group*, $\widetilde{\mathrm{Cl}_K} := \mathbb{A}_K^\times/\mathbb{R}^* U K^\times$, where $\mathbb{R}^*$ is embedded in $(K \otimes \mathbb{R})^\times$, and U is the maximal compact subgroup of the finite ideles of K. This group fits into an exact sequence $\mathbb{R}^*/O_K^\times \to \widetilde{\mathrm{Cl}_K} \to \mathrm{Cl}_K$. Its connected component is therefore a torus, and its component group agrees with Cl_K up to a possible $\mathbb{Z}/2$-extension.

Given $\chi \in \widehat{\widetilde{\mathrm{Cl}_K}}$, there is a unique $s_\chi \in \mathbb{R}$ such that χ restricted to the $\mathbb{R}_+^*$ is of the form $x \mapsto x^{is_\chi}$. The "natural analogue" of our result for imaginary quadratic fields, then, is of the following shape: For a fixed automorphic form f and sufficiently large D, there exist χ with $|s_\chi| \leqslant C$ – a constant depending only on f – and $L(\frac{1}{2}, f \otimes \chi) \neq 0$.

One may still ask, however, the question of whether $L(\frac{1}{2}, f \otimes \chi) \neq 0$ for $\chi \in \widehat{\mathrm{Cl}_K}$ if K is a real quadratic field which happens to have large class group – for instance, $K = \mathbb{Q}(\sqrt{n^2+1})$. We now see that this *is* a question of the flavour of that discussed above: we can prove nonvanishing in the large family $L(\frac{1}{2}, f \otimes \chi)$, where $\chi \in \widehat{\widetilde{\mathrm{Cl}_K}}$, and wish to pass to nonvanishing for the subgroup $\widehat{\mathrm{Cl}_K}$.

(3) The split quadratic extension: to make the distinction between $\widetilde{\mathrm{Cl}_K}$ and Cl_K even more clear, one can degenerate the previous example to the split extension $K = \mathbb{Q} \oplus \mathbb{Q}$.

In that case the analogue of the θ-series χ is given simply by an Eisenstein series of trivial central character; the analogue of the L-functions $L(\frac{1}{2}, f \otimes \chi)$ are therefore $|L(\frac{1}{2}, f \otimes \psi)|^2$, where ψ is just a usual Dirichlet character over $\mathbb{Q}$.

Here one can see the difficulty in a concrete fashion: even the asymptotic as $N \to \infty$ for the square moment

$$\sum_{\psi} |L(\frac{1}{2}, f \otimes \psi)|^2, \tag{4}$$

where the sum is taken over Dirichlet characters ψ of conductor N, is not known in general; however, if one adds a small auxiliary t-averaging and considers instead

$$\sum_{\psi} \int_{|t| \ll 1} |L(\frac{1}{2} + it, f \otimes \psi)|^2 dt. \tag{5}$$

then the problem becomes almost trivial.[5]

The difference between (4) and (5) is precisely the difference between the family $\chi \in \mathrm{Cl}_K$ and $\chi \in \widetilde{\mathrm{Cl}_K}$.

2. Proof of Theorem 1

Let f be a primitive even Maass Hecke-eigenform (of weight 0) on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$ (normalized so that its first Fourier coefficient equals 1); the proof of theorem 1 starts with the expression (2) of the central value $L(f \otimes \chi, 1/2)$ as the square of a twisted period of f over H_K. From that expresssion it follows that

$$\sum_{\chi} L(f \otimes \chi, 1/2) = \frac{2h_K}{\sqrt{D}} \sum_{\sigma \in \mathrm{Cl}_K} |f([\sigma])|^2.$$

Now, by a theorem of Duke [**Duk88**] the set $He_K = \{[x] : x \in \mathrm{Cl}_K\}$ becomes equidistributed on $X_0(1)(\mathbb{C})$ with respect to the hyperbolic measure of mass one $d\mu(z) := (3/\pi)dxdy/y^2$, so that since the function $z \to |f(z)|^2$ is a smooth, square-integrable function, one has

$$\frac{1}{h_K} \sum_{\sigma \in \mathrm{Cl}_K} |f([\sigma])|^2 = (1 + o_f(1)) \int_{X_0(1)(\mathbb{C})} |f(z)|^2 d\mu(z) = \langle f, f \rangle (1 + o_f(1))$$

as $D \to +\infty$ (notice that the proof of the equidistribution of Heegner points uses Siegel's theorem, in particular the term $o_f(1)$ is not effective). Hence, we have

$$\sum_{\chi} L(f \otimes \chi, 1/2) = 2\frac{h_K^2}{\sqrt{D}} \langle f, f \rangle (1 + o_f(1)) \gg_{f,\varepsilon} D^{1/2-\varepsilon}$$

by (1). In particular this proves that for D large enough, there exists $\chi \in \widehat{\mathrm{Cl}_K}$ such that $L(f \otimes \chi, 1/2) \neq 0$. In order to conclude the proof of Theorem 1, it is sufficient to prove that for any $\chi \in \widehat{\mathrm{Cl}_K}$

$$L(f \otimes \chi, 1/2) \ll_f D^{1/2-\delta},$$

for some absolute $\delta > 0$. Such a bound is known as a *subconvex* bound, as the corresponding bound with $\delta = 0$ is known and called the convexity bound (see [**IS00**]). When χ is a quadratic character, such a bound is an indirect consequence

[5] We thank K. Soundararajan for an enlightening discussion of this problem.

of [**Duk88**] and is essentially proven in [**DFI93**] (see also [**Har03, Mic04**]). When χ is not quadratic, this bound is proven in [**HM06**].

REMARK 2.1. The theme of this section was to reduce a question about the average $L(\frac{1}{2}, f\otimes\chi)$ to equidistribution of Heegner points (and therefore to subconvexity of $L(\frac{1}{2}, f\otimes\chi_K)$, where χ_K is the Dirichlet character associated to K). This reduction can be made precise, and this introduces in a natural way triple product L-functions:

$$\begin{aligned}(6)\quad \frac{1}{h_K}\sum_{\chi\in\widehat{\mathrm{Cl}_K}} L(1/2, f\otimes\chi) &\sim \frac{1}{h_K}\sum_{x\in\mathrm{Cl}_K}|f([x])|^2\\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}}|f(z)|^2dz + \sum_g\langle f^2, g\rangle\sum_{x\in\mathrm{Cl}_K}g([x])\end{aligned}$$

Here $\sim$ means an equality up to a constant of size $D^{\pm\varepsilon}$, and, in the second term, the sum over g is over a basis for $L^2_0(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H})$. Here L^2_0 denotes the orthogonal complement of the constants. This g-sum should strictly include an integral over the Eisenstein spectrum; we suppress it for clarity. By Cauchy-Schwarz we have a majorization of the second term (continuing to suppress the Eisenstein spectrum):

$$(7)\qquad \left|\sum_g\langle f^2, g\rangle\sum_{x\in\mathrm{Cl}_K}g([x])\right|^2 \leqslant \sum_g\left|\langle f^2, g\rangle\right|^2\left|\sum_{x\in\mathrm{Cl}_K}g([x])\right|^2$$

where the g-sum is taken over $L^2_0(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H})$, again with suppression of the continuous spectrum. Finally, the summand corresponding to g in the right-hand side can be computed by period formulae: it is roughly of the shape (by Watson's identity, Waldspurger/Zhang formula (2), and factorization of the resulting L-functions)

$$\frac{L(1/2, \mathrm{sym}^2 f\otimes g)L(1/2, g)^2L(1/2, g\otimes\chi_K)}{\langle g, g\rangle^2\langle f, f\rangle}.$$

By use of this formula, one can, for instance, make explicit the dependence of Theorem (1) on the level q of f: one may show that there is a nonvanishing twist as soon as $q < D^A$, for some explicit A. Upon GLH, $q < D^{1/2}$ suffices. There seems to be considerable potential for exploiting (7) further; we hope to return to this in a future paper. We note that similar identities have been exploited in the work of Reznikov [**Rez05**].

One can also prove the following twisted variant of (6): let $\sigma_{\mathfrak{l}}\in\mathrm{Cl}_K$ be the class of an integral ideal $\mathfrak{l}$ of O_K coprime with D. Then one can give an asymptotic for $\sum_\chi\chi(\sigma_{\mathfrak{l}})L(f\otimes\chi, 1/2)$, when the norm of $\mathfrak{l}$ is a sufficiently small power of D. This again uses equidistribution of Heegner points of discriminant D, but at level $\mathrm{Norm}(\mathfrak{l})$.

3. Proof of Theorem 2

The proof of Theorem (2) is in spirit identical to the proof of Theorem (1) that was presented in the previous section. The only difference is that the L-function is the square of a period on a quaternion algebra instead of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$. We will try to set up our notation to emphasize this similarity.

For the proof of Theorem (2) we need to recall some more notations; we refer to [**Gro87**] for more background. Let q be a prime and B_q be the definite quaternion algebra ramified at q and ∞. Let O_q be a choice of a maximal order. Let S be the set of classes for B_q, i.e. the set of classes of left ideals for O_q. To each $s \in S$ is associated an ideal I and another maximal order, namely, the right order $R_s := \{\lambda \in B_q : I\lambda \subset I\}$. We set $w_s = \#R_s^\times/2$. We endow S with the measure ν in which each $\{s\}$ has mass $1/w_s$. This is not a probability measure.

The space of functions on S becomes a Hilbert space via the norm $\langle f, f\rangle^2 = \int |f|^2 d\nu$. Let $S_2^B(q)$ be the orthogonal complement of the constant function. It is endowed with an action of the Hecke algebra $\mathbf{T}^{(q)}$ generated by the Hecke operators T_p $p \nmid q$ and as a $\mathbf{T}^{(q)}$-module $S_2^B(q)$ is isomorphic with $S_2(q)$, the space of weight 2 holomorphic cusp newforms of level q. In particular to each Hecke newform $f \in S_2(q)$ there is a corresponding element $\tilde{f} \in S_2^B(q)$ such that

$$T_n\tilde{f} = \lambda_f(n) \cdot \tilde{f}, \quad (n, q) = 1.$$

We normalize $\tilde{f}$ so that $\langle \tilde{f}, \tilde{f}\rangle = 1$.

Let K be an imaginary quadratic field such that q is inert in K. Once one fixes a special point associated to K, one obtains for each $\sigma \in G_K$ a "special point" $x_\sigma \in S$, cf. discussion in [**Gro87**] of "x_a" after [**Gro87**, (3.6)].

One has the Gross formula [**Gro87**, Prop 11.2]: for each $\chi \in \widehat{\mathrm{Cl}_K}$,

$$L(f \otimes \chi, 1/2) = \frac{\langle f, f\rangle}{u^2\sqrt{D}} \left|\sum_{\sigma\in \mathrm{Cl}_K} \tilde{f}(x_\sigma)\chi(\sigma)\right|^2 \tag{8}$$

Here u is the number of units in the ring of integers of K. Therefore,

$$\sum_{\chi\in\widehat{\mathrm{Cl}_K}} L(f \otimes \chi, 1/2) = \frac{h_K\langle f, f\rangle}{u^2\sqrt{D}} \sum_{\sigma\in \mathrm{Cl}_K} \left|\tilde{f}(x_\sigma)\right|^2$$

Now we use the fact that the Cl_K-orbit $\{x_\sigma,\ \sigma \in \mathrm{Cl}_K\}$ becomes equidistributed, as $D \to \infty$, with respect to the (probability) measure $\frac{\nu}{\nu(S)}$: this is a consequence of the main theorem of [**Iwa87**] (see also [**Mic04**] for a further strengthening) and deduce that

$$h_K^{-1}\sum_{\sigma} \left|\tilde{f}(x_\sigma)\right|^2 = (1 + o_q(1))\frac{1}{\nu(S)}\int |\tilde{f}|^2 d\nu \tag{9}$$

In particular, it follows from (1) that, for all $\varepsilon > 0$

$$\sum_{\chi} L(f \otimes \chi, 1/2) \gg_{f,\varepsilon} D^{1/2-\varepsilon}.$$

Again the proof of theorem 2 follows from the subconvex bound

$$L(f \otimes \chi, 1/2) \ll_f D^{1/2-\delta}$$

for any $0 < \delta < 1/1100$, which is proven in [**Mic04**].

4. Quantification using the cusp; a conditional proof of Theorem 1 and Theorem 3 using the cusp.

Here we elaborate on the second method of proof discussed in Section 1.1.

4.1. Proof of Theorem 1 using the cusp. We note that $S_{\beta,\theta}$ implies that there are $\gg D^{\beta\theta-\epsilon}$ distinct primitive ideals with odd norms with norm $\leqslant D^{\theta}$. Indeed $S_{\beta,\theta}$ provides many such ideals without the restriction of odd norm; just take the "odd part" of each such ideal. The number of primitive ideals with norm $\leqslant X$ and the same odd part is easily verified to be $O(\log X)$, whence the claim.

PROPOSITION 4.1. *Assume hypothesis $S_{\beta,\theta}$, and let f be an even Hecke-Maass cusp form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$. Then $\gg D^{\delta-\epsilon}$ twists $L(\frac{1}{2}, f\otimes\chi)$ are nonvanishing, where $\delta = \min(\beta\theta, 1/2-4\theta)$.*

PROOF. Notations being as above, fix any $\alpha < \delta$, and suppose that precisely $k-1$ of the twisted sums

$$\sum_{x\in \mathrm{Cl}_K} f([x])\chi(x) \tag{10}$$

are nonvanishing, where $k < D^{\alpha}$. In particular, $k < D^{\beta\theta}$. We will show that this leads to a contradiction for large enough D.

Let $1/4+\nu^2$ be the eigenvalue of f. Then f has a Fourier expansion of the form

$$f(x+iy) = \sum_{n\geqslant 1} a_n (ny)^{1/2} K_{i\nu}(2\pi n y)\cos(2\pi n x), \tag{11}$$

where the Fourier coefficients $|a_n|$ are polynomially bounded. We normalize so that $a_1 = 1$; moreover, in view of the asymptotic $K_{i\nu}(y) \sim (\frac{\pi}{2y})^{1/2}e^{-y}(1+O_{\nu}(y^{-1}))$, we obtain an asymptotic expansion for f near the cusp. Indeed, if $z_0 = x_0+iy_0$ belongs to the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, the standard asymptotics show that – with an appropriate normalization –

$$f(z) = \text{const.}\cos(2\pi x)\exp(-2\pi y)(1+O(y^{-1})) + O(e^{-4\pi y}) \tag{12}$$

Let $\mathfrak{p}_j, \mathfrak{q}_j$ be primitive integral ideals of O_K for $1\leqslant j\leqslant k$, all with odd norm, so that $\mathfrak{p}_j$ are mutually distinct and the $\mathfrak{q}_j$ are mutually distinct; and, moreover that

$$\mathrm{Norm}(\mathfrak{p}_1) < \mathrm{Norm}(\mathfrak{p}_2) < \cdots < \mathrm{Norm}(\mathfrak{p}_k) < D^{\theta} \tag{13}$$

$$D^{\theta} > \mathrm{Norm}(\mathfrak{q}_1) > \mathrm{Norm}(\mathfrak{q}_2) > \cdots > \mathrm{Norm}(\mathfrak{q}_k). \tag{14}$$

The assumption on the size of k and the hypothesis $S_{\beta,\theta}$ guarantees that we may choose such ideals, at least for sufficiently large D.

If $\mathfrak{n}$ is any primitive ideal with norm $<\sqrt{D}$, it corresponds to a reduced binary quadratic form $ax^2+bxy+cy^2$ with $a = \mathrm{Norm}(\mathfrak{n})$ and $b^2-4ac=-D$; the corresponding Heegner point $[\mathfrak{n}]$ has as representative $\frac{-b+\sqrt{-D}}{2\mathrm{Norm}(\mathfrak{n})}$. We note that if $a = \mathrm{Norm}(\mathfrak{n})$ is odd, then

$$\left|\cos(2\pi\cdot\left(\frac{-b}{2\mathrm{Norm}(\mathfrak{n})}\right))\right| \gg \mathrm{Norm}(\mathfrak{n})^{-1}. \tag{15}$$

Then the functions $x\mapsto f([x\mathfrak{p}_j])$ – considered as belonging to the vector space of maps $\mathrm{Cl}_K\to\mathbb{C}$ – are necessarily linearly dependent for $1\leqslant j\leqslant k$, because of the assumption on the sums (10). Evaluating these functions at the $[\mathfrak{q}_j]$ shows that the matrix $f([\mathfrak{p}_i\mathfrak{q}_j])_{1\leqslant i,j\leqslant k}$ must be singular. We will evaluate the determinant of this matrix and show it is nonzero, obtaining a contradiction. The point here is that, because all the entries of this matrix differ enormously from each other in absolute value, there is one term that dominates when one expands the determinant via permutations.

Thus, if $\mathfrak{n}$ is a primitive integral ideal of odd norm $< c_0\sqrt{D}$, for some suitable, sufficiently large, absolute constant c_0, (12) and (15) show that one has the bound – for some absolute c_1, c_2 –

$$c_1 e^{-\pi\sqrt{D}/\mathrm{Norm}(\mathfrak{n})} \geqslant |f([\mathfrak{n}])| \geqslant c_2 D^{-1} e^{-\pi\sqrt{D}/\mathrm{Norm}(\mathfrak{n})}.$$

Expanding the determinant of $f([\mathfrak{p}_i\mathfrak{q}_j])_{1\leqslant i,j\leqslant k}$ we get

$$\det = \sum_{\sigma\in S_k}\prod_{i=1}^{k} f([\mathfrak{p}_i\mathfrak{q}_{\sigma(i)}])\mathrm{sign}(\sigma) \tag{16}$$

Now, in view of the asymptotic noted above, we have

$$\prod_{i=1}^{k} f([\mathfrak{p}_i\mathfrak{q}_{\sigma(i)}]) = c_3 \exp\left(-\pi\sqrt{D}\sum_i \frac{1}{\mathrm{Norm}(\mathfrak{p}_i\mathfrak{q}_{\sigma(i)})}\right)$$

where the constant c_2 satisfies $c_3 \in [(c_2/D)^k, c_1^k]$. Set $a_\sigma = \sum_i \frac{1}{\mathrm{Norm}(\mathfrak{p}_i)\mathrm{Norm}(\mathfrak{q}_{\sigma(i)})}$. Then a_σ is maximized – in view of (13) and (14) – for the identity permutation $\sigma = \mathrm{Id}$, and, moreover, it is simple to see that $a_{Id} - a_\sigma \geqslant \frac{1}{D^{4\theta}}$ for any σ other than the identity permutation. It follows that the determinant of (16) is bounded below, in absolute value, by

$$\exp(a_{Id})\left((c_2/D)^k - c_1^k k! \exp(-\pi D^{1/2-4\theta})\right)$$

Since $k < D^\alpha$ and $\alpha < 1/2 - 4\theta$, this expression is nonzero if D is sufficiently large, and we obtain a contradiction. □

4.2. Variant: the derivative of L-functions and the rank of elliptic curves over Hilbert class fields of $\mathbb{Q}(\sqrt{-D})$. We now prove Thm. 3. For a short discussion of the idea of the proof, see the paragraph after (18).

Take $\Phi_E : X_0(N) \to E$ a modular parameterization, defined over $\mathbb{Q}$, with N squarefree. If f is the weight 2 newform corresponding to E, the map

$$\Phi_E : z \mapsto \int_\tau f(w)dw, \tag{17}$$

where τ is any path that begins at ∞ and ends at z, is well-defined up to a lattice $L \subset \mathbb{C}$ and descends to a well-defined map $X_0(N) \to \mathbb{C}/L \cong E(\mathbb{C})$; this sends the cusp at ∞ to the origin of the elliptic curve E and arises from a map defined over $\mathbb{Q}$.

The space $X_0(N)$ parameterizes (a compactification) of the space of cyclic N-isogenies $E \to E'$ between two elliptic curves. We refer to [**GZ86**, II. §1] for further background on Heegner points; for now we just quote the facts we need. If $\mathfrak{m}$ is any ideal of O_K and $\mathfrak{n}$ any integral ideal with $\mathrm{Norm}(\mathfrak{n}) = N$, then $\mathbb{C}/\mathfrak{m} \to \mathbb{C}/\mathfrak{m}\mathfrak{n}^{-1}$ defines a Heegner point on $X_0(N)$ which depends on $\mathfrak{m}$ only through its ideal class, equivalently, depends only on the point $[\mathfrak{m}] \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$. Thus Heegner points are parameterized by such pairs $([\mathfrak{m}], \mathfrak{n})$ and their total number is $|\mathrm{Cl}_K| \cdot \nu(N)$, where $\nu(N)$ is the number of divisors of N.

Fix any $\mathfrak{n}_0$ with $\mathrm{Norm}(\mathfrak{n}_0) = N$ and let P be the Heegner point corresponding to $([e], \mathfrak{n}_0)$. Then P is defined over H, the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$, and we can apply any element $x \in \mathrm{Cl}_K$ (which is identified with the Galois group of H/K) to P to get P^x, which is the Heegner point corresponding to $([x], \mathfrak{n}_0)$.

Suppose $\mathfrak{m}$ is an ideal of O_K of norm m, prime to N. We will later need an explicit representative in $\mathbb{H}$ for $P^{\mathfrak{m}\mathfrak{n}_0} = ([\mathfrak{m}\mathfrak{n}_\mathfrak{o}], \mathfrak{n}_0)$. (Note that the correspondence between $z \in \Gamma_0(N)\backslash\mathbf{H}$ and elliptic curve isogenies sends z to $\mathbb{C}/\langle 1, z\rangle \mapsto \mathbb{C}/\langle 1/N, z\rangle$.) This representative (cf. [**GZ86**, eq. (1.4–1.5)]) can be taken to be

$$z = \frac{-b+\sqrt{-D}}{2a}, \tag{18}$$

where $a = \mathrm{Norm}(\mathfrak{m}\mathfrak{n}_0)$, and $\mathfrak{m}\mathfrak{n}_\mathfrak{o} = \langle a, \frac{b+\sqrt{-D}}{2}\rangle, \mathfrak{m} = \langle aN^{-1}, \frac{b+\sqrt{-D}}{2}\rangle$.

Let us explain the general idea of the proof. Suppose, first, that $E(H)$ had rank *zero*. We denote by $\#E(H)_{tors}$ the order of the torsion subgroup of $E(H)$. This would mean, in particular, that $\Phi(P)$ was a torsion point on $E(H)$; in particular $\#E(H)_{tors}.\Phi(P) = 0$. In view of (17), and the fact that P is very close to the cusp of $X_0(N)$ the point $\Phi(P) \in \mathbb{C}/L$ is represented by a nonzero element $z_P \in \mathbb{C}$ very close to 0. It is then easy to see that $\#E(H)_{tors} \cdot z_P \notin L$, a contradiction. Now one can extend this idea to the case when $E(H)$ has higher rank. Suppose it had rank *one*, for instance. Then Cl_K must act on $E(H) \otimes \mathbb{Q}$ through a character of order 2. In particular, if $\mathfrak{p}$ is any integral ideal of K, then $\Phi(P^\mathfrak{p})$ equals $\pm\Phi(P)$ in $E(H) \otimes \mathbb{Q}$. Suppose, say, that $\Phi(P^\mathfrak{p}) = \Phi(P)$ in $E(H) \otimes \mathbb{Q}$. One again verifies that, if the norm of $\mathfrak{p}$ is sufficiently small, then $\Phi(P^\mathfrak{p}) - \Phi(P) \in \mathbb{C}/L$ is represented by a nonzero $z \in \mathbb{C}$ which is sufficiently close to zero that $\#E(H)_{tors}.z \notin L$.

The $\mathbb{Q}$-vector space $V := E(H) \otimes \mathbb{Q}$ defines a $\mathbb{Q}$-representation of $\mathrm{Gal}(H/K) = \mathrm{Cl}_K$, and we will eventually want to find certain elements in the group algebra of $\mathrm{Gal}(H/K)$ which annihilate this representation, and on the other hand do not have coefficients that are too large. This will be achieved in the following two lemmas.

Lemma 4.1. *Let A be a finite abelian group and W a k-dimensional $\mathbb{Q}$-representation of A. Then there exists a basis for W with respect to which the elements of A act by integral matrices, all of whose entries are $\leqslant C^{k^2}$ in absolute value. Here C is an absolute constant.*

Proof. We may assume that W is irreducible over $\mathbb{Q}$. The group algebra $\mathbb{Q}\cdot A$ decomposes as a certain direct sum $\oplus_j K_j$ of number fields K_j; these K_j exhaust the $\mathbb{Q}$-irreducible representations of A.

Each of these number fields has the property that it is generated, as a $\mathbb{Q}$-vector space, by the roots of unity contained in it (namely, take the images of elements of A under the natural projection $\mathbb{Q}.A \to K_j$). The roots of unity in each K_j form a group, necessarily cyclic; so all the K_j are of the form $\mathbb{Q}[\zeta]$ for some root of unity ζ; and each $a \in A$ acts by multiplication by some power of ζ.

Thus let ζ be a kth root of unity, so $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(k)$ and $\mathbb{Q}(\zeta)$ is isomorphic to $\mathbb{Q}[x]/p_k(x)$, where p_k is the kth cyclotomic polynomial. Then multiplication by x on $\mathbb{Q}[x]/p_k(x)$ is represented, w.r.t. the natural basis $\{1, x, \ldots, x^{\varphi(k)-1}\}$, by a matrix all of whose coefficients are integers of size $\leqslant A$, where A is the absolute value of the largest coefficient of p_k. Since any coefficient of A is a symmetric function in $\{\zeta^i\}_{(i,k)=1}$, one easily sees that $A \leqslant 2^k$.

For any $k \times k$ matrix M, let $\|M\|$ denote the largest absolute value of any entry of M. Then one easily checks that $\|M.N\| \leqslant k\|M\|\|N\|$ and, by induction, $\|M^r\| \leqslant k^{r-1}\|M\|^r$. Thus any power of ζ acts on $\mathbb{Q}(\zeta)$, w.r.t. the basis $\{1, \zeta, \ldots, \zeta^{\varphi(k)-1}\}$, by an integral matrix all of whose entries have size $\leqslant k^k \cdot 2^{k^2} \leqslant C^{k^2}$ for some absolute C. $\square$

LEMMA 4.2. *Let assumptions and notations be as in the previous lemma; let $S \subset A$ have size $|S| = 2k$. Then there exist integers $n_s \in \mathbb{Z}$, not all zero, such that the element $\sum n_s s \in \mathbb{Z}[A]$ annihilates the A-module W. Moreover, we may choose n_s so that $|n_s| \ll C_2^{k^2}$, for some absolute constant C_2.*

PROOF. This follows from Siegel's lemma. Indeed, consider all choices of n_s when $|n_s| \leqslant N$ for all $s \in S$; there are at least N^{2k} such choices. Let $\{w_i\}_{1 \leqslant i \leqslant k}$ be the basis for W provided by the previous lemma. For each i_0, the element $(\sum n_s s)\cdot w_{i_0}$ can be expanded in terms of the basis w_i with integral coefficients of size $\leqslant (2k)C^{k^2}N$. So the number of possibilities for the collection $\{(\sum n_s s)\, w_j\}_{1 \leqslant j \leqslant k}$ is $\ll C_2^{k^3} N^k$, for some suitable absolute constant C_2. It follows that if $N \gg C_2^{k^2}$ two of these must coincide. $\square$

We are now ready to prove Theorem 3.

PROOF. (of Thm. 3). Fix $\alpha < \delta = \min(\beta\theta, 1/2 - 4\theta)$ and suppose that the rank of $E(H) \otimes \mathbb{Q}$ is k, where $k < D^\alpha$. We will show that this leads to a contradiction for D sufficiently large.

Choose $\{\mathfrak{p}_1, \dots, \mathfrak{p}_{2k}, \mathfrak{q}_1, \dots, \mathfrak{q}_{2k}\}$ satisfying the same conditions (13) and (14) as in the proof of Prop. 4.1. We additionally assume that all $\mathfrak{p}_j, \mathfrak{q}_j$ have norms coprime to N; it is easy to see that this is still possible for sufficiently large D. Recall we have fixed an integral ideal $\mathfrak{n}_0$ of norm N. Lem. 4.2 shows that there are integers n_i $(1 \leqslant i \leqslant 2k)$ such that the element

$$\Upsilon := \sum_{i=1}^{2k} n_i \cdot \mathfrak{p}_i \mathfrak{n}_0 \in \mathbb{Z}[\mathrm{Cl}_K] \tag{19}$$

annihilates $E(H) \otimes \mathbb{Q}$ and moreover $|n_i| \ll C_2^{k^2}$. In particular

$$\Upsilon \cdot \Phi_E(P^{\mathfrak{q}_j}) = 0 \quad (1 \leqslant j \leqslant k) \tag{20}$$

But $\Phi_E(P^{\mathfrak{p}_i \mathfrak{q}_j \mathfrak{n}_0})$ is the image under the map Φ_E (see (17)) of a point $z_{P,i,j} \in \mathbf{H}$ whose y-coordinate is given by (cf. (18)) $y_{P,i,j} = \frac{\sqrt{D}}{2\mathrm{Norm}(\mathfrak{p}_i \mathfrak{q}_j \mathfrak{n}_0)}$. In particular this satisfies $y_{P,i,j} \gg D^{1/2-2\theta}$.

The weight 2 form f has a q-expansion in the neighbourhood of ∞ of the form

$$f(z) = e^{2\pi i z} + \sum_{n \geqslant 2} a_n e^{2\pi i n z}$$

where the a_n are integers satisfying $|a_n| \ll n^{1/2+\epsilon}$. In particular, there exists a contour C from ∞ to $z_{P,i,j}$ so that

$$\left| \int_C f(\tau) d\tau \right| = \frac{1}{2\pi i} \exp\left(-\pi \frac{\sqrt{D}}{\mathrm{Norm}(\mathfrak{p}_i \mathfrak{q}_j \mathfrak{n}_0)}\right) \left(1 + O(\exp(-\pi D^{1/2-2\theta}))\right)$$

Thus the image of the Heegner point $P^{\mathfrak{p}_i \mathfrak{q}_j \mathfrak{n}_0}$ on $E(\mathbb{C}) = \mathbb{C}/L$ is represented by $z_{ij} \in \mathbb{C}$ satisfying $|z_{ij}| = \frac{1}{2\pi i} \exp(-\pi \frac{\sqrt{D}}{\mathrm{Norm}(\mathfrak{p}_i \mathfrak{q}_j \mathfrak{n}_0)}) \left(1 + O(\exp(-\pi D^{1/2-2\theta}))\right)$. The relation (20) shows that

$$\# E(H)_{tors} \cdot \sum n_i z_{ij} \in L.$$

Note that $\#E(H)_{tors}$ is bounded by a polynomial in D, by reducing modulo primes of H that lie above inert primes in K. Since $|n_i| \ll C_2^{k^2}$ and $k < D^\alpha$, this forces $\sum n_i z_{ij} = 0$ for sufficiently large D. This implies that the matrix $(z_{ij})_{1\leqslant i,j\leqslant 2k}$ is *singular*, and one obtains a contradiction by computing determinants, as in Sec. 4.1. $\square$

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