Equidistribution, L-functions and Ergodic Theory: on some problems of Y. V. Linnik

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Linnik's Equidistribution Problems

In the late 50's, Y. V. Linnik investigated the distribution properties of the representations of a large integer by a ternary quadratic form.

$Q(A, B, C)$ an integral ternary quadratic form; for $d \in \mathbf{Z}$

$$R_Q(d) = \{(a, b, c) \in \mathbf{Z}^3,\ Q(a, b, c) = d\}$$

the set of representations of d by Q.

$$V_{Q,\pm}(\mathbf{R}) = \{(a, b, c) \in \mathbf{R}^3,\ Q(a, b, c) = \pm 1\}.$$

Basic question : *how is the radial projection*

$$|d|^{-1/2}.R_Q(d) \subset V_{Q,\pm}(\mathbf{R})$$

distributed as $|d| \to \infty$ *(*$\pm = \mathrm{sign}(d)$*)?*

Linnik considered the following special cases

1. The diagonal form $Q(A, B, C) = -A^2 - B^2 - C^2$. $V_{Q,-}(\mathbf{R})$ is the sphere.
2. The discriminant quadratic form $Q(A, B, C) = B^2 - 4AC$ for $d < 0$. $V_{Q,-}(\mathbf{R})$ is the two sheeted hyperboloid.
3. The discriminant quadratic form $Q(A, B, C) = B^2 - 4AC$ for $d > 0$. $V_{Q,+}(\mathbf{R})$ is the one sheeted hyperboloid.

On $V_{Q,\pm}(\mathbf{R})$ fix a $SO(Q)$-invariant Borel measure (unique up to scalar), $\mu_{Q,\pm}$.

In the 60s, Linnik and Skubenko partially solved the problems:

Theorem (Linnik,Skubenko)

$p > 2$ a fixed prime; $D_{Q,\pm,p}$ the set of discriminants of sign $\pm$, s.t. $R_Q(d) \neq \emptyset$, and such that p is split in the quadratic field $\mathbf{Q}(\sqrt{d})$. As $d \to \infty$ in $D_{Q,\pm,p}$, the sets $|d|^{-1/2}.R_Q(d)$ are equidistributed on $V_{Q,\pm}(\mathbf{R})$ w.r.t. $\mu_{Q,\pm}$.

- The proof is based on Linnik's "ergodic method".
- In the late 80's, the splitting condition at p was removed by Duke by an entierely different approach (harmonic analysis and the spectral theory of automorphic forms).

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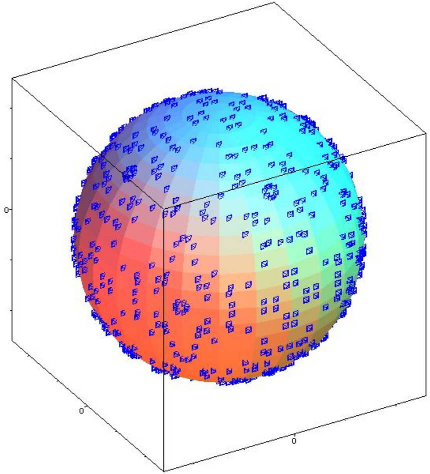

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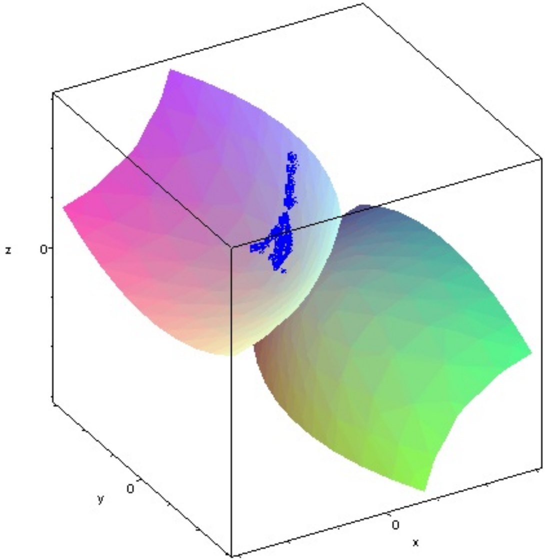

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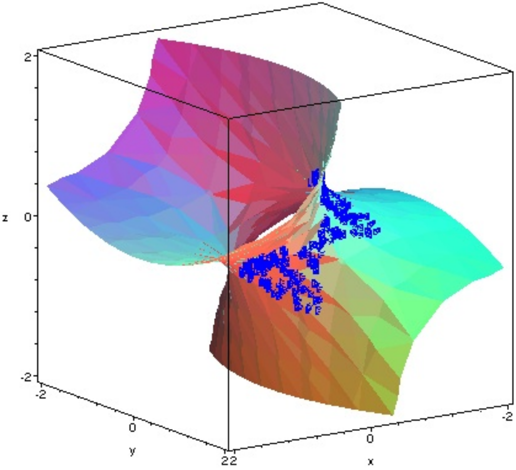

The hyperbolic model

Identify $R_Q(d)$ with the integral binary quadratic forms of discriminant d

$$(a,b,c) \in R_{B^2-4AC}(d) \longleftrightarrow q_{a,b,c}(X,Y) := aX^2 + bXY + cY^2;$$

- For $d < 0$, associate to the $SL_2(\mathbf{Z})$-orbit of $q_{a,b,c}$, the image, $z_{[a,b,c]}$ in $SL_2(\mathbf{Z})\backslash\mathbf{H} =: Y_0(1)$ of the root $z_{a,b,c} \in \mathbf{H}$ of $q_{a,b,c}(X,1)$. One gets $h(d)$ Heegner points

$$\mathcal{H}_d = \{z_{[a,b,c]},\ b^2 - 4ac = d\} \subset Y_0(1).$$

- For $d > 0$, associate to each $SL_2(\mathbf{Z})$-orbit of $q_{a,b,c}$ the image $\gamma_{[a,b,c]}$ in $Y_0(1)$ of the geodesic line (half-circle) in $\mathbf{H}$ joining the real roots of $q_{a,b,c}(X,1)$. One gets $h(d)$ closed geodesics

$$\Gamma_d = \{\gamma_{[a,b,c]},\ b^2 - 4ac = d\} \subset Y_0(1),\ |\Gamma_d| = h(d).$$

- For $Q = -A^2 - B^2 - C^2$, and $d < 0$ set $\mathcal{G}_d := |d|^{-1/2}.R_Q(d)$

By Gauss, Dirichlet's class number formula and Siegel's theorem

$$|\mathcal{H}_d| = |d|^{1/2+o(1)},\ \mathrm{length}(\Gamma_d) = d^{1/2+o(1)},$$

$$|\mathcal{G}_d| \neq 0 \Rightarrow |\mathcal{G}_d| = |d|^{1/2+o(1)}.$$

The Linnik/Skubenko theorem (without Linnik's condition) is equivalent to :

Theorem (Duke)

For $|d| \to \infty$ amongst the discriminants, $\mathcal{G}_d$, $\mathcal{H}_d$ and Γ_d is equidistributed on S^2, $Y_0(1)$, $S^1_*(Y_0(1))$, w.r.t. the Lebesgue, the hyperbolic and the Liouville measure respectively.

Corollary

For $\ell > 0$, let Γ_ℓ be the union of the closed geodesics of length ℓ. As $\ell \to \infty$ (in the length spectrum), Γ_ℓ is e.d. on $S^1_*(Y_0(1))$ w.r.t. the Liouville measure.

To be compared with the Theorem of Bowen/Margulis (valid in a much more general context):

$$\text{as } L \to +\infty \text{ the set } \Gamma_{\leqslant L} \text{ is e.d.}$$

Higher rank generalization

Recall the identification

$$S^1_*(Y_0(1)) \simeq X_2 = PGL_2(\mathbf{Z})\backslash PGL_2(\mathbf{R})$$

closed geodesic $\gamma \leftrightarrow$ compact $\mathrm{diag}_2(\mathbf{R})$-orbits in X_2

More generaly:
$X_n = PGL_n(\mathbf{Z})\backslash PGL_n(\mathbf{R}) =$ lattices in $\mathbf{R}^n$/homothety.
μ_{Haar} the $PGL_n(\mathbf{R})$-invariant probability measure on X_n.
$H =$diagonal $n \times n$ matrices/scalars.
Problem How are the compact H-orbits distributed in X_n ?

- Compact H-orbits in $X_n \to$ maximal compact flats of the variety $Y_n = X_n/PO_n(\mathbf{R})$.

For $n = 3$, one has the following analog to Duke's Thm:

Theorem (Einsiedler/Lindenstrauss/M./Venkatesh)

For $V > 0$, let Γ_V the union of compact H-orbits in X_3 of volume V; as $V \to \infty$, Γ_V is e.d. w.r.t μ_{Haar}.

- Answers an old question of Linnik and imply the analog of the Bowen/Margulis thm. on the ed. of $\Gamma_{\leqslant V}$ which was not known.

Fundamental fact:

Compact H-orbits in X_n are parametrized by arithmetic data:

- K tot. real of degree n, $\theta : K \hookrightarrow \mathbf{R}^n$, $\bar{I}$ a K-homothety class of a lattice $I \subset K$. $\mathcal{O}_I :=$ the *order* of I
- $[K, \theta, \bar{I}] \to \Gamma_{\bar{I}} := \overline{\theta(I)}.H$ is compact by Dirichlet units thm
- invariants: $\mathrm{vol}(\Gamma_{\bar{I}}) = \mathrm{reg}(\mathcal{O}_I)$, $\mathrm{disc}(\Gamma_{\bar{I}}) := \mathrm{disc}(\mathcal{O}_I)$.
- Grouping together the orbits with the same order $\mathcal{O}$ we obtain $\Gamma_{\mathcal{O}}$, a "packet" of H-orbits parametrized by some class group.

- A explanation for ed. is that $\mathcal{G}_d, \mathcal{H}_d, \Gamma_{\mathcal{O}}$ are homogeneous under the action of some (extended) class group.
- In a fancier language, the sets $\mathcal{G}_d, \mathcal{H}_d, \Gamma_{\mathcal{O}}$ are described as a (projection of an) orbit of a torus in an *adelic* space:

$$T_{\mathcal{O}}(\mathbf{Q})\backslash T_{\mathcal{O}}(\mathbf{A}).g_{\mathcal{O}} \mapsto G(\mathbf{Q})\backslash G(\mathbf{A})/\mathbf{K} := X_{\mathbf{K}}$$

with

- $G = PD^{\times}$ with D either the Hamilton quaternions, the 2×2-matrices or the 3×3-matrices,
- $\mathbf{K} = \mathbf{K}_{\infty}.\mathbf{K}_f$ a compact subgroup; $\mathbf{K}_f$ open compact ($K_f = \mathcal{O}_D \otimes \widehat{\mathbf{Z}}$),
- $T_{\mathcal{O}}$ a torus $\simeq \mathrm{res}_{K/\mathbf{Q}} K^{\times}/\mathbf{Q}^{\times}$, for $K \hookrightarrow D$ a number field and $\mathcal{O} = K \cap \mathcal{O}_D$ the associated order.

The harmonic analytic approach: Duke's proof

Weyl's ed. criterion + Spectral decomposition of $L^2(X_{\mathbf{K}})$:
For any φ in an orthonormal basis of Hecke-eigenforms evaluate the period of φ along the toric orbit,

$$W(d,\varphi) := \int_{T_d(\mathbf{Q})\backslash T_d(\mathbf{A})} \varphi(t.g_d)dt \overset{?}{\to} 0, \text{ as } d \to \infty.$$

- A formula of Waldspurger (here Maass) relates $W(d,\varphi)$ to the $|d|-$th Fourier coefficient of a metaplectic form $\tilde{\varphi}$:

$$W(d,\varphi) = \rho_{\tilde{\varphi}}(|d|)|d|^{-1/4+o_{\varphi}(1)}$$

- The bound $\rho_{\tilde{\varphi}}(|d|) \ll |d|^{1/4-1/28}$ was proven by Iwaniec in the holomorphic case and by Duke in general.

The harmonic analytic approach: subconvexity bounds.

Given χ a suitable character of $T_d(\mathbf{Q})\backslash T_d(\mathbf{A})$, consider the twisted period

$$W(\chi,\varphi) := \int_{T_d(\mathbf{Q})\backslash T_d(\mathbf{A})} \chi(t)\varphi(t.g_d)dt.$$

- (Waldspurger, Gross, Zhang, Clozel/Ullmo, Popa)

$$|W(\chi,\varphi)|^2 = L(\pi_\varphi \otimes \pi_\chi, 1/2)|d|^{-1/2+o_\varphi(1)}.$$

- In particular

$$L(\pi_\varphi \otimes \pi_\chi, 1/2) \ll |d|^{1/2-\eta},\ \eta > 0 \Rightarrow W(\chi,\varphi) \to 0$$

This is a (known) example of a subconvex bound. □

Remark : the twist by a character χ allows to prove ed. of *strict suborbits of the torus* $\rightsquigarrow$ *sparse equidistribution*.

The subconvexity problem

For $\Pi = \Pi_\infty \otimes \bigotimes_p \Pi_p$ an "automorphic object"

$$L(\Pi, s) = \prod_p L(\Pi_p, s) = \prod_p \prod_{i=1}^{d} (1 - \frac{\alpha_{\Pi,i}(p)}{p^s})^{-1};$$

$$L(\Pi_\infty, s)L(\Pi, s) = \varepsilon(\Pi) q_\Pi^{\frac{1-2s}{2}} L(\check{\Pi}_\infty, 1-s)L(\check{\Pi}, 1-s)$$

The convexity bound w.r.t. the conductor q_Π is often known:

$$\text{for } \Re e s = 1/2,\ L(\Pi, s) \ll_s q_\Pi^{1/4+o(1)}.$$

The subconvexity problem (ScP) (conductor aspect): *improve $1/4$ to $1/4 - \eta$ for some $\eta > 0$.*

Theorem

Let F be a fixed number field. The ScP w.r.t. the conductor is solved for the following L-functions:

$$L(\chi, s),\ L(\chi \otimes \pi_0, s),\ L(\pi, s),\ L(\pi \otimes \pi_0, s)$$

where π_0 is a fixed $GL_2(\mathbf{A}_F)$-automorphic representation, χ a $GL_1(\mathbf{A}_F)$-a. r. π a $GL_2(\mathbf{A}_F)$-a. r. with $q_\chi,\ q_\pi \to \infty$.

- Over $\mathbf{Q}$: Burgess, Duke/Friedlander/Iwaniec, Kowalski/M./Vanderkam, Sarnak, M., Harcos, Blomer.
- Over F: Cogdell/Piatetski-Shapiro/Sarnak, Venkatesh, M./Venkatesh.

Subconvexity via Dirichlet series

- The central value L-function is approximated by a Dirichlet polynomial of length $q_\Pi^{1/2}$:

$$L(\Pi, 1/2) \simeq \sum_{n \ll q_\Pi^{1/2}} \frac{\lambda_\Pi(n)}{n^{1/2}}$$

- Use the method of moments supplemented by the amplification method of Friedlander/Iwaniec (cf. Friedlander's ICM Zürich lecture):

Goal: for a well choosen family $\mathcal{B}$ containing Π and suitable k ($|\mathcal{B}| = q_\Pi^{k/4+o(1)}$),

$$|\sum_{n\geqslant 1} \frac{\lambda_\Pi(n)}{n^{1/2}}|^k |\sum_{l\leqslant L} x_l \lambda_\Pi(l)|^2 \leqslant \sum_{\Pi\in\mathcal{B}} |\sum_{n\geqslant 1} \frac{\lambda_\Pi(n)}{n^{1/2}}|^k |\sum_{l\leqslant L} x_l \lambda_\Pi(l)|^2$$

$$\overset{?}{\leqslant} |\mathcal{B}|^{1+o(1)} \sum_{l\leqslant L} |x_l|^2.$$

The proof of such a bound makes extensive use of spectral theory of automorphic forms (e.g. Kuznetzov formula in both directions, spectral large sieve, exponential decay of triple products).

Noticeable facts:

1. For some L-functions, subconvexity follows ultimately from the spectral gap for GL_2.
2. For other L-functions, subconvexity follows from subconvexity of "simpler" L-function (hence ultimately from the spectral gap).

Subconvexity via periods

Inspired, in part, by the work of Bernstein/Reznikov on triple products, Venkatesh developped another method to establish subconvexity which "explains" much of the earlier methods and generalize to a variety of contexts; in particular it generalizes the subconvexity to number fields:

- the central value is approximated by a period: e.g. for RS L-functions,

$$q_\pi^{-1/2} L(\pi \otimes \pi_0, 1/2) \simeq \int_{PGL_2} \varphi(g)\varphi_0(g)E(g)dg =: \langle \overline{\varphi}, \varphi_0 E\rangle$$

To show that $\langle \overline{\varphi}, \varphi_0 E\rangle \to 0$, Venkatesh uses *geometric bounds* supported by *ergodic* ideas.

- Equidistribution of orbits of the Hecke operator T_{q_π} ($\Leftarrow$ Spectral gap).
- Mixing properties of Hecke operators *supported at small primes* $\rightsquigarrow$ *Amplification.*

Suppose $q_\pi = q_\psi$ is prime, and $q_{\pi_0} = q_{\varphi_0} = 1$

$$\begin{aligned} |\langle\overline{\varphi}, \varphi_0 E\rangle|^2 \leqslant \langle\varphi_0 E, \varphi_0 E\rangle &= \int_{PGL_2} \varphi_0.E(g)\overline{\varphi_0.E}(g)dg \\ &= \int_{PGL_2} \varphi_0.\overline{\varphi_0}(g)E.\overline{E}(g)dg = \langle\varphi_0.\overline{\varphi_0}, E.\overline{E}\rangle \\ &= \langle\varphi_0.\varphi_0\rangle\langle E, \overline{E}\rangle + \sum_{\psi\neq 1}\langle\varphi_0.\overline{\varphi_0}, \psi\rangle\langle\psi, E.\overline{E}\rangle + \dots \end{aligned}$$

Because of $\varphi_0.\overline{\varphi_0}$, ψ has level 1; moreover $\langle\psi, E.\overline{E}\rangle$ is small either because

1. If $\chi_\pi = 1$, $\langle\psi, E.\overline{E}\rangle$ is a matrix coefficient for $\mathrm{Diag}(q_\pi, 1) \rightsquigarrow$ Spectral Gap, or
2. If $\chi_\pi \neq 1$, since E is Eisenstein, $\langle\psi, E.\overline{E}\rangle \simeq q_\chi^{-1/2} L(\pi_\psi \otimes \chi, 1/2 + it) \rightsquigarrow$ Subconvexity

$\Longrightarrow$ The non-constant contribution $\sum_{\psi\neq 1}$ is small.

The constant term $(\langle\varphi_0.\varphi_0\rangle\langle E, \overline{E}\rangle)$ is not small. To make it small $\rightsquigarrow$ Amplification. □

Applications:

Let F be tot. real. A formula of Baruch/Mao + Results of Schulze-Pillot + Subconvexity implies the last remaining case of Hilbert 11th problem :

Theorem (Cogdell/Piatetski-Shapiro/Sarnak)

Q a tot. positive integral ternary quadratic form $/F$. For all but finitely many tot. positive squarefree $d \in \mathcal{O}_F$, d is representable by Q iff d is everywhere locally representable. Moreover in that case

$$|R_Q(d)| = N_{F/\mathbf{Q}}(d)^{1/2+o(1)}.$$

Linnik's problems can be generalized to *quaternionic varieties*: ie. for B/F a quaternion algebra

$$G = \mathrm{res}_{F/\mathbf{Q}} PB^{\times}, X_{B,\mathbf{K}} := G(\mathbf{Q})\backslash G(\mathbf{A})/\mathbf{K};$$

to orders $\mathcal{O}$ in a quadratic extensions K/F, is associated the torus $T_{\mathcal{O}} \simeq \mathrm{res}_{F/\mathbf{Q}} K^{\times}/F^{\times} \subset G$ and a *quadratic cycle*

$$\Gamma_{\mathcal{O}} = [T_{\mathcal{O}}(\mathbf{Q})\backslash T_{\mathcal{O}}(\mathbf{A}).g_{\mathcal{O}}] \subset X_{B,\mathbf{K}}$$

Formulas of Waldspurger's type + *Subconvexity for Hecke and RS L-functions imply in many case:*

Statement (Zhang, Popa + M./Venkatesh)

For $\mathrm{disc}(\mathcal{O}) \to \infty$*, the* $\Gamma_{\mathcal{O}}$ *are e.d. on* $X_{B,\mathbf{K}}$*. Moreover "sufficiently big suborbits"* $\Gamma'_{\mathcal{O}} \subset \Gamma_{\mathcal{O}}$ *(ie.* $\mathrm{vol}(\Gamma'_{\mathcal{O}}) \geqslant \mathrm{vol}(\Gamma_{\mathcal{O}})^{1-1/10000}$*) are ed. as well.*

When $X_{\mathbf{K}}$ is a Shimura variety and K/F tot. imaginary: $\Gamma_{\mathcal{O}}$ corresponds to a set of CM abelian var. $\rightsquigarrow$ further arithmetic interpretations.

- The Galois orbit of a CM point is a strict suborbit $\Gamma'_{\mathcal{O}} \subset \Gamma_{\mathcal{O}}$. Zhang's ε-conjecture predicts that $\Gamma'_{\mathcal{O}}$ is "sufficiently big" in the generic case

Zhang's ε-conjecture $\Rightarrow$ ed. of Galois orbits of generic CM points.

- Surjectivity of the reduction of *CM* abelian varieties on the super-singular locus.
- Non-vanishing results for RS L-functions and rank of "modular varieties/F" over Hilbert class fields of quadratic extensions.

At present, the equidistribution of the packet of compact H-orbits

$$\Gamma_{\mathcal{O}} \subset X_3 \text{ for } \mathrm{disc}(\mathcal{O}) \to \infty$$

($\mathcal{O}$ a tot. real cubic order) does not seem accessible to a purely harmonic analytic approach:

- Work of Gan/Gross/Savin express the Weyl sums in terms of Fourier coefs. of G_2-automorphic forms; but these seem hard to bound.
- no simple formula relating Weyl sums for X_3 to L-functions is expected...excepted in one very special case. For it
- one can evaluate the Weyl sum and *bootstrap* that little piece of information to equidistribution. In this, the following subconvex bound is crucial

Theorem (Duke/Friedlander/Iwaniec(+B.H.M.), M.V.)

For K/F a cubic extension, for $\Re es = 1/2$

$$\zeta_K(s) \ll_F \mathrm{disc}(K)^{1/4-\eta}, \ \eta > 0$$

Linnik's ergodic method revisited

Linnik's "ergodic method" was certainly in advance on its time but constitute quite an intricate mix of ergodic and number theoretic arguments.

Goal of a joint project with M. Einsiedler, E. Lindenstrauss and A. Venkatesh:

1. Capture the "essence" of Linnik's method. Separate ergodic arguments from number theoretic ones.
2. From there simplify the proofs and strenghten the results.
3. Extend to more complicated situations.

$G = PD^{\times}$ with $D = M_n$ or a division algebra/$\mathbf{Q}$ of degree n,

$$X_{\mathbf{K}} = G(\mathbf{Q})\backslash G(\mathbf{A})/\mathbf{K},$$

We wish to investigate the distribution properties of sequence of orbits

$$T_d.g_d := [T_d(\mathbf{Q})\backslash T_d(\mathbf{A}).g_d] \subset X_{\mathbf{K}}$$

with $g_d \in G(\mathbf{A})$ and $T_d \subset G$ are maximal $\mathbf{Q}$-anisotropic tori. Set $\mu_d = \mu_{Haar, T_d.g_d}$.

Problem: what are the possible weak-* limits of the μ_d ?

Expectation (Clozel/Ullmo): any weak-* limit of the μ_d is *homogeneous, ie. is the Haar measure supported by a periodic adelic orbit of a subgroup $H \subset G$.*

Linnik's condition "*p a fixed prime splits in* $K_d = \mathbf{Q}(\sqrt{d})$"

- Arithmetical interpretation: p factors in K_d, $p.\mathcal{O}_K = \mathfrak{P}.\mathfrak{P}'$; the subgroup of the class group generated by $[\mathfrak{P}]$ has a non-trivial action on $\mathcal{G}_d, \mathcal{H}_d$ or Γ_d which has to be exploited.
- Dynamical interpretation: the torus $T_d = \mathrm{res}_{K_d/\mathbf{Q}} K_d^\times/\mathbf{Q}^\times$ is split at p. In particular, the non-compact p-adic torus $T_d(\mathbf{Q}_p) \simeq \mathrm{Diag}_2(\mathbf{Q}_p)$ has non-trivial dynamics.

More generaly, fix a place v of $\mathbf{Q}$ s.t. D_v is split.

Problem (simpler version): *same problem as above but restricted to orbits associated to sequences of tori* T_d *s.t.* $T_{d,v}$ *is split and* $g_{d,v}^{-1} T_{d,v} g_{d,v} = \mathrm{Diag}_n(\mathbf{Q}_v)$*).*

Ergodic theory: Pick any weak-$*$ limit, μ, of the μ_d; then μ is invariant under $\mathrm{Diag}_n(\mathbf{Q}_v)$.
Problem: classify such measures possibly under extra assumptions
$\rightsquigarrow$ Measure rigidity for toric actions (cf. Lindenstrauss's lecture).
Fundamental quantity: the (metric) entropy of μ, $h_\mu(t)$, w.r.t some $t \in T(\mathbf{Q}_v)$.

- Fact 1: If T_v has rank 1 and $h_\mu(t)$ is maximal then $\mu = \mu_{Haar}$.
- Fact 2: If T_v has rank > 1, μ is ergodic and $h_\mu(t)$ is positive for some $t \Rightarrow \mu$ is algebraic (Einsiedler, Katok, Lindenstrauss). In particular, if n is prime, $\mu = \mu_{Haar}$.

Number theory: the verification of the entropy condition use heavily the arithmetic and global structure nature of the ambient space.
$\rightsquigarrow$ verify at various degrees of precision the following Linnik's principle: *the distinct components of* $[T_d(\mathbf{Q})\backslash T_d(\mathbf{A}).g_d]$ *are well-spaced*.

Rank 1: Linnik's problems

[ELMV] give a "new" ergodic theoretic proof of Linnik's and Skubenko's Thms along with some improvements.

- Apply Fact 1 to a fixed place v splitting all T_d: Maximal entropy is deduced from:

Linnik's principle: *For $X =$ either S^2, $Y_0(1)$ or $S^1_*(Y_0(1))$, the μ-mass of a δ-neighborhood of the diagonal $\Delta_\delta \subset X \times X$ is nearly as small as expected, ie.*

$$\mu(\Delta_\delta) = O(\delta^{2-\varepsilon}),\ = O(\delta^{2-\varepsilon}),\ \text{or}\ = O(\delta^{3-\varepsilon}).$$

- Number theory enters through a version of Siegel mass formula:

The number of representations of an int. binary quadratic form q by a fixed ternary form Q (mod. $SO_Q(\mathbf{Z})$), is small. Moreover

- 1st and second problems: $v = p$ and the spectral gap for the p-th Hecke operator is needed.
- 3rd problem: $v = \infty$ (the T_d are $\mathbf{R}$-split); as only one splitting place is sufficient, Linnik's condition at some prime p is not necessary.

Rank > 1: ed. of compact H-orbits in X_3

$D = M_3$, $G = PGL_3$,
$X_3 = PGL_3(\mathbf{Q})\backslash PGL_3(\mathbf{R})$,
$H =$ diagonal 3×3 matrices/scalar.
$\mathcal{O} \subset K$ a tot. cubic order,

$$\Gamma_{\mathcal{O}} = \bigcup_{\bar{I}, \mathcal{O}_I = \mathcal{O}} \Gamma_{\bar{I}}$$

the collection of compact H-orbits with order $\mathcal{O}$.

Theorem (Einsiedler/Lindenstrauss/M./Venkatesh)

As $\operatorname{disc}(\mathcal{O}) \to \infty$, *the* $\Gamma_{\mathcal{O}}$ *become ed. w.r.t* μ_{Haar}.

Remark Related results for compact H-orbits in compact quotient of $PGL_3(\mathbf{R})$ but to some extend less satisfactory (quasi-ed.).

Two issues: let μ be a weak-$*$ limit of the $\mu_{\mathcal{O}}$

1. Is μ a probability measure (ie. are the measures $\mu_{\mathcal{O}}$ tight) ?
2. Fact 2: does a. e. ergodic component of μ have > 0 entropy w.r.t. some $t \in H$. ?

$\Rightarrow$ Harmonic analysis: we use the Siegel-Eisenstein series to build good non-negative test functions, $\mathbf{E}$ s.t. either

1. $\mathbf{E}$ dominates the characteristic function of $\mathcal{K}^{\complement}$ for any compact $\mathcal{K}$ (using Mahler compactness criterion).
2. $\mathbf{E}$ dominates dominate the characteristic function of any δ-ball in $X_{\mathbf{K}}$.

and we evaluate the Weyl sums

$$W(T_{\mathcal{O}}, \mathbf{E}) = \int_{T_{\mathcal{O}}(\mathbf{Q})\backslash T_{\mathcal{O}}(\mathbf{A})} \mathbf{E}(t.g_{\mathcal{O}})dt.$$

One can compute these periods in terms of L-functions (Hecke)

$$W(T_{\mathcal{O}}, \mathbf{E}) = \mu_{Haar}(\mathbf{E}) + \mathrm{disc}(\mathcal{O})^{-1/4+o(1)} \times \int_{\Re e s = 1/2} |\zeta_K(s) s^{-2006}||ds| \times \text{local int.}$$

Subconvexity for ζ-fct of cubic fields + local subconvexity $\Rightarrow$

$$W(T_{\mathcal{O}}, \mathbf{E}) \to \mu_{Haar}(\mathbf{E}) \Rightarrow$$

$$\mu(\mathcal{K}^{\complement}) \to 0 \text{ if } \mu_{Haar}(\mathcal{K}^{\complement})) \to 0, \Rightarrow \text{ the } \mu_{\mathcal{O}} \text{ are tight},$$

$$\mu(\delta - \text{ball}) = O(\delta^3)(= O(\delta^{2+\eta}),\ \eta > 0) \Rightarrow h_{\mu_x^{\mathcal{E}}}(t) > 0, a.e.x$$

□

Remark Siegel/Eisenstein series are used crucially in other contexts (Eskin/Margulis/Mozes, Veech)

Harmonic Analysis vs. Ergodic Theory

G semi simple group/$\mathbf{Q}$, $\{H_d\}$ a sequence of subgroups $\rightsquigarrow$ investigate the distribution properties of $H_d(\mathbf{Q})\backslash H_d(\mathbf{A})$-orbits in $G(\mathbf{Q})\backslash G(\mathbf{A})$.

Interesting in many situations in number theory: eg. CM-points on Shimura varieties are organized into toric orbits (with $T(\mathbf{R})$ compact), cf. Vatsal's lecture.

Two approaches:

- Harmonic analytic: try to evaluate the periods : $\varphi \in L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$

$$W_G(H_d, \varphi) = \int_{H_d(\mathbf{Q})\backslash H_d(\mathbf{A})} \varphi(t.g_d)dt.$$

- Ergodic theoretic: fix a finite set S of places and use measure rigidity results for the (limits of the) $H_d(\mathbf{Q}_S)$-invariant Haar measure on $H_d.g_d$.

The H.A. approach may succeed if the H_d are "big enough". Then

1. $W_G(H_d, \varphi)$ may be expressed in terms of matrix coefficients, L-functions and/or periods on other groups $\rightsquigarrow$ harmonic analysis techniques (bounds on matrix coefficients, amplification).
2. $\rightsquigarrow$ quantitative e.d. results.

BUT

1. many interesting cases escape H.A.
2. H.A. cannot analyze sub-orbits of "too small" a size.

The E.T. approach is more robust

1. If H_d are generated by unipotent (or degenerate to unipotents) $\rightsquigarrow$ use Ratner's theory.
2. If $H_d = T_d$ are generic tori $\rightsquigarrow$ try to apply the emerging rigidity theory of toric actions.
3. E.T. allows for control ed. of orbits of size beyond the possibilities of H.A.

BUT

1. not quantitative so far
2. for tori, one need a Linnik's type condition: a fixed set of places s.t. $T_{d,S}$ has large S-rank.
3. In particular E.T. use only a "small part" of the action of an adelic torus
$$T(\mathbf{A}) = \prod_v{}' T(\mathbf{Q}_v)$$

In this and a large class of related problems, we hope that more progress can be obtained from further interplay between the H.A. and the E.T.
For instance it could be useful to transpose to E.T. a feature of H.A. that has apparently no equivalent:
Functoriality $\rightsquigarrow$ identities between periods along different groups:
The success of Duke's proof: Waldspurger formula express the period $W(T_d, \varphi)$ along a non-split torus in terms of another period along a split group (Diag_2 or PGL_2) and this prove being essential.
The same feature apply for ed. on X_3.