SOLUTION TO EXERCISE 6 FOR LEONARDO'S EXERCISE SESSION

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1. Solutions

By writing $\widehat{Y}(s,n) = \widehat{X}(s,n) - \widehat{X_M}(s,n)$, we see that $\widehat{Y}(s,n)$ satisfies

$$d\widehat{Y}(s,n) = d\widehat{X}(s,n) - \frac{M}{\langle n \rangle}\widehat{Y}(s,n)ds.$$

By solving this SDE, we obtain

$$\widehat{Y}(s,n) = \int_0^s e^{-\frac{M}{\langle n \rangle}(s-s')} d\widehat{X}(s,n)$$

for $|n| \leq M$ and $Y(s,n) = \widehat{X}(s,n)$ for $|n| > M$. Thus,

$$\widehat{X_M}(s,n) = \widehat{X}(s,n) - \widehat{Y}(s,n) = \int_0^s \big(1 - e^{-\frac{M}{\langle n \rangle}(s-s')}\big) d\widehat{X}(s,n).$$

i. By independence and Ito's isometry ,

$$\begin{aligned}
\mathbb{E}[X_M^2(x)] &= \sum_{|n|\leq M} \frac{1}{\langle n \rangle^2} \int_0^1 \big(1 - e^{-\frac{M}{\langle n \rangle}s}\big)^2 ds \\
&\lesssim \sum_{|n|\leq M} \frac{1}{\langle n \rangle^2} \\
&\sim \begin{cases} \log M & \text{if } d = 2 \\ M & \text{if } d = 3. \end{cases}
\end{aligned}$$

For the lower bound, we simply note that for $|n| \leq M$,

$$1 - e^{-\frac{M}{\langle n \rangle}s} \geq 1 - e^{-\frac{1}{2}s}.$$

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ii. By independence, Ito's isometry, and the Cauchy, we have

$$\begin{aligned}
&\mathbb{E}\Big[2\int_{\mathbb{T}^d} XX_M dx - \int_{\mathbb{T}^d} X_M^2 dx\Big] \\
&\quad= \mathbb{E}\Big[2\int_{\mathbb{T}^d} YX_M dx + \int_{\mathbb{T}^d} X_M^2 dx\Big] \\
&\quad= \int_{\mathbb{T}^d} \mathbb{E}[X_M^2(x)]dx + 2\int_{\mathbb{T}^d} \mathbb{E}[Y(x)X_M(x)]dx \\
&\quad= (2\pi)^d \mathbb{E}[X_M^2(x)] + 2(2\pi)^d \sum_{|n|\le M} \frac{1}{\langle n\rangle^2}\int_0^1 e^{-\frac{M}{\langle n\rangle}s}\big(1-e^{-\frac{M}{\langle n\rangle}s}\big)ds \\
&\quad\le (2\pi)^d \mathbb{E}[X_M^2(x)] + C(2\pi)^d \sum_{|n|\le M} \frac{1}{\langle n\rangle M} \\
&\quad\sim \sum_{|n|\le M} \frac{1}{\langle n\rangle^2}.
\end{aligned}$$

iii. By Plancherel's identity and independence, we have

$$\begin{aligned}
\mathbb{E}\Big[\Big|\int_{\mathbb{T}^d} X_M f dx\Big|^2\Big] &= \mathbb{E}\Big[\Big|\sum_{|n|\le M} \widehat{X_M}(n)\widehat{f}(n)\Big|^2\Big] \\
&= \sum_{|n|\le M} \mathbb{E}\big[|\widehat{X_M}(n)|^2\big]|\widehat{f}(n)|^2 \\
&\lesssim \sum_{|n|\le M} \frac{1}{\langle n\rangle^2}|\widehat{f}(n)|^2 \\
&\le \|f\|_{H^{-1}}.
\end{aligned}$$

iv. Note that one can write $:Y^2:$ as the following multiple stochastic integral:

$$:Y^2:= \sum_{n_1,n_2\in\mathbb{Z}^d}\int_0^1\int_0^1 \widehat{Y}(t_1,n_1)\widehat{Y}(t_2,n_2)dW_{n_1}(s_1)dW_{n_2}(s_2)e^{i(n_1+n_2)x}.$$

By Ito's isometry for the multiple stochastic integral, we have

$$\begin{aligned}
\mathbb{E}\Big[\Big|\int_{\mathbb{T}^d} :(X-X_M)^2: dx\Big|^2\Big] &= \mathbb{E}\Big[\Big|\int_{\mathbb{T}^d} :Y^2: dx\Big|^2\Big] \\
&= \sum_{n\in\mathbb{Z}^d} \mathbb{E}\big[|\widehat{Y}(1,n)|^2\big]\mathbb{E}\big[|\widehat{Y}(1,-n)|^2\big] \\
&= \sum_{|n|\le M} \frac{1}{\langle n\rangle^4}\Big(\int_0^1 e^{-\frac{2M}{\langle n\rangle}s}ds\Big)^2 + \sum_{|n|>M}\frac{1}{\langle n\rangle^4} \\
&\lesssim M^{-2}\sum_{|n|\le M}\frac{1}{\langle n\rangle^2} + M^{4-d} \\
&\lesssim \begin{cases} M^{-2}\log M & \text{if } d=2 \\ M^{-1} & \text{if } d=3. \end{cases}
\end{aligned}$$

v. By Plancherel's identity, the SDE solved by X_M, and Ito's isometry, we have

$$\begin{aligned}
\mathbb{E}\bigg[\int_0^1 \Big\|\frac{d}{ds}X_M(s)\Big\|_{H^1}^2 ds\bigg] &= \sum_{|n|\leq M} \langle n\rangle^2 \mathbb{E}\bigg[\int_0^1 \Big|\frac{d}{ds}\widehat{X_M}(s,n)\Big|^2 ds\bigg] \\
&= M^2 \sum_{|n|\leq M} \frac{1}{\langle n\rangle^2}\mathbb{E}\bigg[\int_0^1 \big|\widehat{Y}(s,n)\big|^2 ds\bigg] \\
&= M^2 \sum_{|n|\leq M} \frac{1}{\langle n\rangle^2}\int_0^1 e^{-\frac{2M}{\langle n\rangle}s} ds \\
&\lesssim M \sum_{|n|\leq M} \frac{1}{\langle n\rangle} \\
&\lesssim M^d.
\end{aligned}$$

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