Indifference Pricing for Power Utilities

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Abstract

We study utility indifference pricing of claim streams with intertemporal consumption and power (CRRA) utilities. We derive explicit formulas for the derivatives of the utility indifference price with respect to claims and wealth. The simple structure of these formulas is a reflection of surprising operator identities for the derivatives of the optimal consumption stream. For example, the partial derivative of the optimal consumption stream with respect to the endowment is always a projection. Furthermore, it is an orthogonal projection with respect to a natural "economic inner product".

We find that the indifference price is a highly non-linear function of the wealth and obtain its asymptotic behavior when the claims to wealth ratio is either very small or very large.

In the large claims to wealth ratio limit, the asymptotic expansion of the indifference price is given in terms of fractional powers of the wealth. We also obtain sharp global bounds for the indifference price.

1 Introduction

Imagine that we must either

(1) as a representative of a private bank, or, hedge fund, determine the price of an over the counter derivative (option) contract,

or

(2) determine, as a representative of a corporation, the correct price for a real option (a capital investment),

or

(3) determine, as a representative of an insurance (reinsurance) company, the price of an insurance contract.

In each case, we want to determine the optimal price of a financial contract market consistently, by exploiting correlations between the payoff of our investment and the global the stock market (see, Wüthrich et al. [26] for an applied theory of market consistent pricing of insurance contracts).

For a *complete* market, with asset prices following geometric brownian motions, Black and Scholes [2] solved this pricing problem. Completeness is an essential hypothesis for their solution, because it implies that the payoff of any option can be perfectly replicated by a suitable trading strategy in stocks and bonds. The Black-Scholes price of an option is determined by arbitrage (*linearly*), and is equal to the price of the replicating strategy.

On the other hand, many aspects of the market consistent pricing problem for *incomplete* markets are still under investigation, because now, the general payoff cannot be replicated by trading in stocks and bonds.

A common dodge around market consistent pricing is to decompose the payoff (or, synonymously, contingent claim) into hedgeable and unhedgeable components. The first component is priced by arbitrage, and the second *non linearly*. In the context of insurance, the non linear component is referred to as the insurance loading, namely, the risk premium that the insured pays to the insurer. The loading depends on the risk aversion of the insurance company.

If an insurance company is "sufficiently isolated" from the financial markets, then there is a well known theoretical principle that guides the pricing of its insurance contracts. Namely, the principle of utility indifference. That is, the price of the contract is chosen so that the utility of the company is the same before and after the contract is sold. Here, we imagine that the company acts as a rational agent, maximizing a von Neumann-Morgenstern utility.

If, by contrast, the company is not isolated from the financial markets, an important modification of the basic utility indifference principle is required. Now, the company can modify its effective claims stream by choosing an appropriate trading strategy in available securities. Of course, a rational company will choose a strategy that maximizes its utility. In this context, the von Neumann-Morgenstern utility function should be replaced by the maximal utility, achievable by trading. That is, the maximal utility achievable by trading is the same before and after contracts are sold.

Pricing by maximal utility achievable by trading indifference is market consistent in the sense that perfectly hedgeable contingent claims are automatically priced by arbitrage.

The same economic reasoning applies verbatim to the pricing problems (1) and (2) from above.

Hodges and Neuberger [11] (see, also, Davis [4]) were the first to consider utility indifference pricing in incomplete markets. Since then, the interest to this topic has grown dramatically. See, e.g., [1], [3], [5], [7], [8], [9], [15], [18], [19], [20], [21], [22], [23], [24], [25]. See, also, [10] for a general survey of existing literature on this topic.

A necessary prerequisite for implementing the maximal utility achievable by trading indifference principle, is a solution to the utility maximization problem. It is impossible to analyze the implications of maximal utility indifference without good control of the maximum, that is, the optimal consumption / wealth stream. As far as we have been able to determine, very little is known about the structure of these streams in the presence of unhedgeable claims. There are general, nonconstructive existence / uniqueness results (see, e.g., Kramkov and Schachermeyer [14] and Karatzas and Zitkovic [13]).

If there are no unhedgeable claims, the Euler-Lagrange equations become linear after a change of variables. In the presence of unhedgeable claims, the Euler-Lagrange equations are genuinely nonlinear, except for the special case of exponential utilities (see, e.g., [12] and [18]). In the latter case, the equations "factor" to the point where they can be linearized after a change of variables. For this reason, almost all the analysis of indifference pricing, that we are aware of, is for exponential utilities. See, e.g., [3], [7], [8], [18], [21], [22], [24].

We illustrate "factoring" for exponential utilities in the simple case of "tradeless" indifference pricing. The "tradeless" indifference price π of a claim y is the solution to

$$u(x) \;=\; E[u(x + \pi - y)] \tag{1.1}$$

where x is the wealth of the insurance company. If $u(x) \;=\; -e^{-\gamma x}$ then

$$\pi \;=\; \gamma^{-1} \log E[e^{\gamma y}] \tag{1.2}$$

is clearly independent of x.

The conclusion of the last paragraph makes indifference pricing with exponential utilities unrealistic. Prices should depend on the wealth of the company, as already emphasized by El Karoui and Rouge [24].

In this paper, we explicitly construct the optimal consumption stream for market consistent, utility indifference pricing for all benchmark power utilities

$$u_\gamma(c) \;=\; \frac{c^{1-\gamma} - 1}{1 - \gamma} \tag{1.3}$$

In particular, the log Bernoulli utility $u_1(c) \;=\; \log c$. Power utilities are supported on $\mathbb{R}_+$, satisfy the Inada condition $u'(0) \;=\; +\infty$ as well as many other natural economic properties.

Our construction is a direct outgrowth of Malamud and Trubowitz [16], where optimal consumption streams were explicitly constructed by a recursive procedure for a new class $\mathfrak{C}$ of *incomplete* financial markets with no insurance sector. The class $\mathfrak{C}$ is characterized by several important mathematical and economic properties. For example, it is the only class for which the crucial economic properties of precautionary savings and marginal propensity to consume. From this point of view, they are the only sensible incomplete markets. The class $\mathfrak{C}$ also includes all discrete time diffusion driven incomplete markets and, consequently, almost all classical incomplete market models.

In [16], we exploited a local, recursive procedure for constructing optimal consumption streams. Here, we introduce (see, Theorem 3.7) a global construction. One important consequence of our new construction is that the *derivative* of the optimal consumption stream with respect to the endowment is a *projection at any point* in the space of endowments (see, Theorem 3.9). This is surprising. The projections are not immediately orthogonal.

However, there is an economically natural inner product, for which the projections become orthogonal. The inner product depends on the point in the space of endowments and thus introduces a natural Riemanian structure into the model.

The projection property of the derivative makes it possible to calculate the first, second and third derivatives of the indifference price in a useful, explicit form. Unexpectedly, cancellations allow us to obtain sharp, global bounds on the indifference prices (see, Theorem 7.4).

We also give a straightforward analysis of indifference prices in the small claims limit and obtain a second order expansion. This complements the results of [7], [9], [15].

The large claims limit is another matter. It is rather subtle. By homogeneity, the indifference prices of large claims are naturally expanded in small capital and the leading term is a fractional power. The fractional power (see, Theorem 6.10) is a function of both risk aversion and fixed claims. The difference between the sharp, global bounds of Theorem 7.4 and the first two terms of the small capital expansion goes to zero at an explicitly calculable rate.

Organization of the paper:

In section 2, we discuss general incomplete markets and the related utility maximization problem. We introduce the class $\mathfrak{C}$ of incomplete markets and study its basic properties.

In section 3, we construct express the solution to the utility maximization problem through an explicitly constructed non-linear map and study its properties. In particular, we show that the derivative of the optimal consumption with respect to the endowment is always a projection.

In section 4, we calculate other derivatives of the optimal consumption and establish useful algebraic identities for these derivatives.

In section 5, we introduce the utility indifference price and calculate its derivatives.

In section 6, we study the asymptotic behavior of the indifference price as the capital goes to zero.

In section 7, we establish sharp, global bounds for the indifference price.

Finally, in section 8 we calculate the second order expansion of the indifference price when the claim size is small.

2 General Incomplete markets

2.1 The structure of market incompleteness

The randomness in our model is described by a **finite**, filtered probability space $(\Omega\,,\,\mathcal{G}\,,\,\mathcal{B}\,,\,P)$ where the filtration $\mathcal{G} = (\mathcal{G}_t)_{t=0}^T$ satisfies

$$\{\emptyset\,,\,\Omega\} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_T = \mathcal{B}. \tag{2.1}$$

There are T time periods. We emphasize that everywhere in this paper the probability space Ω and time horizon T are assumed to be finite. The problem of passing to a continuous time limit is a topic of ongoing research.

Definition 2.1 *A financial asset is a pair of* $\mathcal{G}$*-adapted processes, a price process* $\mathbf{p} = (p_t\,, t = 0\,, \dots\,, T)$ *and a dividend process* $\mathbf{d} = (d_t\,, t = 0\,, \dots\,, T)$.

A financial market $(\mathcal{M}\,, \mathcal{G})$ *is a collection of financial assets.*

A τ*-period risk free bond at time* t *is the asset, whose dividend process* $d_\theta = 1$ *for* $\theta = t + \tau$ *and* $d_\theta = 0$ *otherwise.*

For the sake of brevity, we will often use (p_t) to denote a process without indicating that $t = 0\,, \dots\,, T$.

We allow for an arbitrary type of market incompleteness, except for a natural

Assumption 1 *One period risk free bonds are available for trading at each moment of time.*

Definition 2.2 *Let* $(\mathcal{M}\,, \mathcal{G}) = \{A_1\,, \dots\,, A_N\}$ *be the underlying financial market with financial assets* $A_1\,, \dots\,, A_N$. *Asset* A_i *has a price process* $(p_{i\,t})$ *and a dividend process* $(d_{i\,t})$. *The payoff subspace* $\mathcal{L}_t$ *at time* t *is defined by*

$$\mathcal{L}_t = \left\{ \sum_{i=1}^{N} x_{i\,t-1}\,(p_{i\,t} + d_{i\,t}) \mid x_{i\,t-1} \in L_2(\mathcal{G}_{t-1}) \text{ for all } i = 1\,, \dots\,, N \right\}. \tag{2.2}$$

This is the set of payoffs at time t *of all possible* $\mathcal{G}_{t-1}$*-measurable investments* $x_{i\,t-1}$ *at time* $t-1$. *We denote by* $\mathrm{P}_{\mathcal{L}_t}$ *the orthogonal projection onto the subspace* $\mathcal{L}_t$ *in the space* $L_2(\mathcal{G}_t)$. *Similarly, let* $\mathrm{P}_{\mathcal{G}^t}\,, t = 1\,, \dots\,, T$, *be the orthogonal projection (conditional expectation) from* $L_2(\Omega\,, \mathcal{B})$ *onto* $L_2(\Omega\,, \mathcal{G}_t)$. *We write* $\mathbf{P}_{\mathcal{G}}$ *for the direct sum*

$$\mathbf{P}_{\mathcal{G}} = \oplus_{t=1}^{T} \mathrm{P}_{\mathcal{G}_t}. \tag{2.3}$$

Note that, by Assumption 1, $1 \in L_2(\mathcal{G}_{t-1}) \subset \mathcal{L}_t$ and, consequently, $\mathrm{P}_{\mathcal{L}_t} 1 = 1$. Furthermore, for any $\mathcal{G}_{t-1}$-measurable Y and any $\mathcal{G}_t$-measurable X we have

$$\mathrm{P}_{\mathcal{L}_t}(X\,Y) = Y\,\mathrm{P}_{\mathcal{L}_t} X. \tag{2.4}$$

A portfolio strategy for an agent, with a $\mathcal{G}$-adapted individual endowment process, trading on the market $(\mathcal{M}\,, \mathcal{G})$ is an N dimensional, $\mathcal{G}$-adapted process $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$. Here, $\mathbf{x}_j = (x_{j\,0}\,, \dots\,, x_{j\,T-1}\,, 0)$. The random variable $x_{j\,t}$ counts the number of shares of asset A_j held at time $t+1$ before dividends are paid and assets are traded. The last component 0 formalizes the convention that no investments are made at the final time period T.

Definition 2.3 *The dividend process* $\mathbf{D}_{\mathbf{x}}$ *generated by the portfolio strategy* $\mathbf{x}$ *is*

$$D_{\mathbf{x},t} = \sum_{i=1}^{N} (d_{i\,t} + p_{i\,t})\,x_{i\,t-1} - \sum_{i=1}^{N} p_{i\,t}\,x_{i\,t} \tag{2.5}$$

for $t = 0, \dots, T$, *where* $\mathbf{d}_i$ *and* $\mathbf{p}_i$ *are the dividend and price processes of the asset* A_i. *In particular, the initial investment is* $D_{\mathbf{x},0} = -\sum_{i=1}^{N} p_{i0}\, x_{i0}$.

The process

$$X_t = X_t(\mathbf{x}) = \sum_{i=1}^{N} (d_{it} + p_{it})\, x_{j\,t-1} \tag{2.6}$$

is referred to as the wealth process of the strategy $\mathbf{x}$.

Definition 2.4 *A market* $(\mathcal{M}, \mathcal{G})$ *is arbitrage free if there is no portfolio strategy* $\mathbf{x}$ *such that* $D_{\mathbf{x},t} \geq 0$ *for all* $t = 0, \dots, T$ *and* $D_{\mathbf{x},\tau} > 0$ *for some* τ *with positive probability.*

A market $(\mathcal{M}, \mathcal{G})$ *is dynamically complete if for any* $\mathcal{G}$*-adapted process* $(Y_t, t = 1, \dots, T)$ *there exists a portfolio strategy* $\mathbf{x}$ *such that*

$$D_{\mathbf{x},t} = Y_t \tag{2.7}$$

for all $t = 1, \dots, T$.

Definition 2.5 *A* $\mathcal{G}$*-adapted process* $\mathbf{R} = (R_t)$ *is referred to as a state price density process (SPD process) for the market* $(\mathcal{M}, \mathcal{G})$ *if the identity*

$$R_t\, p_{it} = E\left[R_{t+1} \left(p_{i\,t+1} + d_{i\,t+1} \right) \Big| \mathcal{G}_t \right] \tag{2.8}$$

holds for any asset A_i, $i = 1, \dots, N$ *and any* $t = 0, \dots, T-1$.

In particular, under the standard no-bubble condition $p_{iT} = 0$, the price

$$p_{it} = R_t^{-1} E\left[\sum_{\tau=1}^{T-\tau} R_{t+\tau}\, d_{i\,t+\tau} \Big| \mathcal{G}_t \right] \tag{2.9}$$

is the discounted value of future dividends.

The following lemma summarizes some well known properties of state price densities.

Lemma 2.6 *A market* $(\mathcal{M}, \mathcal{G})$ *is arbitrage free if and only if there exists a positive SPD process.*

An arbitrage free market $(\mathcal{M}, \mathcal{G})$ *is dynamically complete if and only if there exists a unique, positive SPD process.*

A process $\mathbf{D}$ *is a dividend process of a portfolio strategy if and only if it is orthogonal to any SPD process, i.e.,*

$$E\left[\sum_{t=0}^{T} D_t\, R_t \right] = 0 \tag{2.10}$$

for any SPD process $\mathbf{R}$.

See, e.g., [6].

When markets are incomplete, there are infinitely many state price density processes. This is one of the main difficulties in the analysis of utility maximization in incomplete markets. Malamud and Trubowitz [16] introduced a unique, natural, "aggregate" state price density process and showed that all budget constraints and first order conditions can be formulated in terms of this special SPD process.

LEMMA 2.7 *Under the assumption of no arbitrage, there exists a unique, aggregate state price density process* $\mathbf{M} = (M_t)$ *such that* $M_t \in \mathcal{L}_t$ *for all* $t = 1, \ldots, T$. *Furthermore, a process* $\mathbf{R} = (R_t)$ *is a state price density process if and only if*

$$\mathrm{P}_{\mathcal{L}_t} \frac{R_t}{R_{t-1}} = \frac{M_t}{M_{t-1}} \tag{2.11}$$

for all t.

See, [16], Lemma 2.5.

The aggregate SPD process $\mathbf{M}$ is natural because it lives in the market subspace, just like the prices themselves. Note that, in general, $\mathbf{M}$ is **not positive.**[1] The main source of problems is that the projection $\mathrm{P}_{\mathcal{L}_t}$ is **not** necessarily **positivity preserving**. This fact motivated the introduction of a new class $\mathfrak{C}$ of incomplete markets in Malamud and Trubowitz [16] .

DEFINITION 2.8 *An incomplete market* $(\mathcal{M}, \mathcal{G})$ *belongs to the class* $\mathfrak{C}$ *if there exists a subfiltration* $\mathcal{H} = (\mathcal{H}_t, t = 0, \ldots, T)$ *of* $\mathcal{G}$ *such that*

- $\mathcal{H}_{t+1} \supset \mathcal{G}_t \supset \mathcal{H}_t$ *for all* t.
- *The payoff process* $(p_{i\,t} + d_{i\,t})$ *of any asset* A_i *is adapted to* $\mathcal{H}$.
- *Any* $\mathcal{H}_t$ *measurable claim* Y *can be replicated by a* $\mathcal{G}_{t-1}$ *measurable portfolio* $x_1, \ldots, x_N$ *of assets, purchased at time* $t-1$. *That is,*

$$Y = \sum_{i=1}^{N} x_i \left(p_{i\,t} + d_{i\,t}\right). \tag{2.12}$$

Equivalently, $\mathcal{L}_t = L_2(\mathcal{H}_t)$ *and* $\mathrm{P}_{\mathcal{L}_t} = \mathrm{P}_{\mathcal{H}_t} = E[\,\cdot\,|\,\mathcal{H}_t\,]$.

We refer to $\mathcal{H}$ *as the hedgeable filtration.*

It is possible to show (see, [16]) that $\mathrm{P}_{\mathcal{L}_t}$ is positivity preserving if and only if there exists a subalgebra $\mathcal{H}_t \subset \mathcal{G}_t$ such that $\mathrm{P}_{\mathcal{L}_t}$ is the conditional expectation relative to $\mathcal{H}_t$. In particular, it is possible to show (see, [16], Proposition 3.4) that the aggregate state price density process $\mathbf{M}$ is the unique *positive* state price density process adapted to $\mathcal{H}$.

The class $\mathfrak{C}$ has many interesting properties. As an illustration, we present a natural subclass of incomplete markets from the class $\mathfrak{C}$.

[1] It is possible that $M_t = 0$ with positive probability. But, in [16], $\mathbf{M}$ is constructed as a product one period stochastic discount factors M_t/M_{t-1} which are well defined.

EXAMPLE 2.9 *As an example, consider a market, consisting of assets without dividends, for which the price processes satisfy the discrete time SDE of the form*

$$p_{i\,t} \;=\; p_{i\,t-1} \;+\; \mu_{i\,t-1} \;+\; \sum_{j=1}^{N} \sigma_{i\,j\,t-1}\, X_{i\,t}, \tag{2.13}$$

where $(\sigma_{i\,j\,t})$, $(\mu_{i\,t})$ *are arbitrary,* $\mathcal{G}$*-adapted processes and the process*

$$B_{i\,\tau} \;=\; \sum_{t=0}^{\tau} X_{i\,t}$$

is a martingale with respect to $\mathcal{G}$ *for any* $i = 1, \ldots, N$. *SDE* (2.13) *is a discrete analog of the continuous time SDE*

$$dp_{i\,t} \;=\; \mu_{i\,t}\, dt \;+\; \sum_{j=1}^{N} \sigma_{i\,j\,t}\, dB_{j\,t} \tag{2.14}$$

with predictable processes $(\mu_{i\,t})$ *and* $(\sigma_{i\,j\,t})$. *Let* $\mathcal{F}$ *be the natural filtration, generated by the martingales* B_i, $i = 1, \ldots, N$ *(i.e., the minimal filtration for which the martingales are adapted). Suppose now that the martingales* B_i, $i = 1, \ldots, N$ *have the spanning property: any* $\mathcal{F}$*-martingale* Z_t *can be represented as a stochastic integral w.r.t.* B_i*s:*

$$dZ_t \;=\; Z_t \;-\; Z_{t-1} \;=\; \sum_{i=1}^{N} \xi_{i\,t-1}\, X_{i\,t}$$

Then, if the matrix $(\sigma_{i\,j\,t})_{i,j=1,\ldots,n}$ *is invertible, the market is in the class* $\mathfrak{C}$ *and the hedgeable* σ*-algebra* $\mathcal{H}_t$ *is given by* $\mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{G}_{t-1})$, *the minimal algebra, generated by* $\mathcal{F}_t$ *and* $\mathcal{G}_{t-1}$.

Recall that in a standard, diffusion driven incomplete market, price processes follow (2.14), but with B_i being Brownian motions. It is possible to show that any diffusion driven incomplete market can be approximated by a discrete time incomplete market of the above form. Here, it is important that Brownian motions naturally have the spanning property. See, [16], Section 4.2.

In the sequel, we make the following

ASSUMPTION 2 *The market* $(\mathcal{M}, \mathcal{G})$ *belongs to the class* $\mathfrak{C}$. *The corresponding hedgeable filtration is denoted by* $\mathcal{H}$. *The unique, positive, aggregate state price density process adapted to* $\mathcal{H}$ *and normalized by* $M_0 = 1$ *is denoted by* $\mathbf{M} = (M_t)$.

Assumption 2 implies that the following is true.

PROPOSITION 2.10 *For any* $\mathcal{H}$*-adapted process* (X_t) *there exists a* $\mathcal{G}$*-adapted portfolio strategy* $\mathbf{x}$ *such that* (X_t) *coincides with the wealth process of this strategy,* $X_t = X_t(\mathbf{x})$ *for all* t. *In this case,*

$$D_{\mathbf{x},\,t} \;=\; X_t \;-\; E\left[\frac{M_{t+1}}{M_t}\, X_{t+1} \,|\, \mathcal{G}_t\right] \tag{2.15}$$

for all t. Here, we use the standard convention $X_0 = X_{T+1} = 0$.

See, Lemma 3.4 in [17].

2.2 The budget set

Initial value of a process (stream) will play a special role in our considerations. For this reason, starting from this section, we will always treat the value of a random process at time zero separately and write a process $(w_t, t = 0, \dots, T)$ as $(w_0, \mathbf{w})$ where $\mathbf{w} = (w_t, t = 0, \dots, T)$.

DEFINITION 2.11 *Consider an agent endowed with an (income) stream $(w_0, \mathbf{w})$ of consumption good, trading in the financial market to achieve a desirable consumption stream $(c_0, \mathbf{c})$. A consumption stream $(c_0, \mathbf{c})$ is achievable by trading if there exists a $\mathcal{G}$-adapted portfolio stratetegy $\mathbf{x}$ such that*

$$c_t = w_t + D_{\mathbf{x},t} \tag{2.16}$$

for all $t = 0, \dots, T$.

DEFINITION 2.12 *The budget set $\mathrm{B}(w_0, \mathbf{w})$ of an agent with a $\mathcal{G}$-adapted endowment process $(w_0, \mathbf{w})$ with $\mathbf{w} = (w_t, t \geq 1)$ is the set of all positive consumption streams, that can be achieved by trading.*

By Proposition 2.10, the following is true

LEMMA 2.13 *A stream $(c_0, \mathbf{c}) \in B(w_0, \mathbf{w})$ if and only if there exists an $\mathcal{H}$-adapted wealth process (X_t) such that*

$$c_t = w_t + X_t - E\left[\frac{M_{t+1}}{M_t} X_{t+1} \,\Big|\, \mathcal{G}_t\right] > 0 \tag{2.17}$$

for all $t = 0, \dots, T$.

In applications to pricing insurance claims, we will be in the situation when the endowment stream (w_t) takes negative values. Thus, we must make sure that the budget set is non-empty.

DEFINITION 2.14 *Let $\mathbf{Y} = (Y_t, t = 1, \dots, T)$ be a $\mathcal{G}$-adapted process. The upper hedging price for $\mathbf{Y}$ at time zero is the minimal number $\mathbf{Y}_0^u \in \mathbb{R}$ such that there exists a portfolio strategy $\mathbf{x}$ satisfying*

$$D_{\mathbf{x},t} \geq Y_t \tag{2.18}$$

for all $t = 1, \dots, T$ with $-D_{\mathbf{x},0} \leq \mathbf{Y}_0^u$. A portfolio strategy $\mathbf{x}$ is called upper hedging for $\mathbf{Y}$ if (2.18) is satisfied.

In general, the calculation of the upper hedging price is a non-trivial problem. But, for incomplete markets in the class $\mathfrak{C}$, the upper hedging price can be explicitly calculated by a simple, recursive procedure.

Definition 2.15 *Let $\mathfrak{A} \subset \mathcal{B}$ be a sub-σ-algebra and $X \in L^\infty(\mathcal{B})$. The conditional supremum* $\operatorname{esssup}[X \mid \mathfrak{A}]$ *of X relative to $\mathfrak{A}$ is the unique, $\mathfrak{A}$-measurable variable Z, satisfying $Z \geq X$ such that $Z_1 \geq Z$ for any $\mathfrak{A}$-measurable variable Z_1 satisfying $Z_1 \geq X$. When the probability space is finite, we have*

$$\operatorname{esssup}[X \mid \mathfrak{A}] = \max[X \mid \mathfrak{A}]. \tag{2.19}$$

Proposition 2.16 *Let $\mathbf{Y}^u_{T+1} = 0$ and define inductively for $t \leq T$*

$$\mathbf{Y}^u_t = \max\left[Y_t + M_t^{-1} E\left[\mathbf{Y}^u_{t+1} M_{t+1} \mid \mathcal{G}_t\right] \mid \mathcal{H}_t\right] \tag{2.20}$$

for $t \geq 1$ and

$$\mathbf{Y}^u_0 = E[M_1 \mathbf{Y}^1].$$

Then, $\mathbf{Y}^u_0$ is the upper hedging price for the stream $\mathbf{Y}$. Furthermore, for any upper hedging strategy $\mathbf{x}$ for the stream $\mathbf{Y}$, the wealth process $X_t(\mathbf{x})$ satisfies

$$X_t(\mathbf{x}) \geq \mathbf{Y}^u_t \tag{2.21}$$

for all $t \geq 1$. Thus, $(\mathbf{Y}^u_t)$ is the minimal upper hedging wealth process. In particular, if $\mathbf{x}$ is an upper hedging strategy and $D_{\mathbf{x}0} = -\mathbf{Y}^u_0$ then $X_t(\mathbf{x}) = \mathbf{Y}^u_t$ for all $t = 1, \ldots, T$.

Proof. We do the proof by backward induction. The claim is obvious for $t = T+1$ (we use the convention $Y_{t+1} = 0$). Suppose that the claim is proved for all $t \geq \tau + 1$ and let us prove it for $t = \tau$. We have

$$D_{\mathbf{x}t} = X_t - M_t^{-1} E[M_{t+1} X_{t+1} \mid \mathcal{G}_t] \geq Y_t \tag{2.22}$$

if and only if

$$\begin{aligned} X_t &\geq \max\left[Y_t + M_t^{-1} E\left[X_{t+1} M_{t+1} \mid \mathcal{G}_t\right] \mid \mathcal{H}_t\right] \\ &\geq \max\left[Y_t + M_t^{-1} E\left[\mathbf{Y}^u_{t+1} M_{t+1} \mid \mathcal{G}_t\right] \mid \mathcal{H}_t\right] = \mathbf{Y}^u_t \end{aligned} \tag{2.23}$$

and $X_t = \mathbf{Y}^u_t$ implies $X_{t+1} = \mathbf{Y}^u_{t+1}$. Therefore, $X_0 = \mathbf{Y}^u_0$ if and only if $X_t = \mathbf{Y}^u_t$ for all t. The proof is complete. □

Lemma 2.17 *The budget set $B(w_0, \mathbf{w})$ is non-empty if and only if*

$$w_0 > (-\mathbf{w})^u_0 \tag{2.24}$$

In the sequel, we always assume that (2.24) holds.

2.3 Utility maximization problem

Consider an agent with a $\mathcal{G}$-adapted endowment process $(w_0\,,\,\mathbf{w})$. It is standard in the modern literature to assume that the rational behavior of the agent can be characterized by an expected, discounted, intertemporal utility

$$E\left[\sum_{t=0}^{T} e^{-\rho\, t}\, u(c_t)\right] \qquad (2.25)$$

over all consumption streams $(c_0\,,\,\mathbf{c})$ Facing his endowment stream, an agents uses financial markets to achieve the *optimal consumption stream* $(c_0\,,\,\mathbf{c})$ maximizing the above utility of all achievable consumption streams in the budget set $B(w_0\,,\,\mathbf{w})$.

Utility function $u(c)$ is assumed to be monotone increasing, concave utility and satisfying the Inada conditions

$$\lim_{c\to 0} u'(c) \;=\; +\infty \qquad , \qquad \lim_{c\to\,+\infty} u'(c) \;=\; 0. \qquad (2.26)$$

Since, by assumption, Ω is finite, Inada conditions and strict concavity guarantee existence and uniqueness of the optimal consumption stream for the objective function (2.25). The stream $(c_0\,,\,\mathbf{c})$ satisfies the standard Euler equation (see, (B.20) in [17])

$$u'(c_t)\, p_{i\,t} \;=\; E\,\left[e^{-\rho}\, u'(c_{t+1})\,\left(p_{i\,t+1} \,+\, d_{i\,t+1}\right)\,\middle|\,\mathcal{G}_t\right] \qquad (2.27)$$

for any asset $A_i\,,\; i \;=\; 1\,,\,\ldots\,,\,N$.

By definition, (2.24) is necessary for the utility maximization problem to be well defined. Standard results imply that it is also sufficient for the existence of the solution. In fact, the optimal consumption stream exists and satisfies the first order conditions under fairly general assumptions. See, Kramkov and Schachermeyer [14], Karatzas and Zitkovic [13]. Existence proof for general probability spaces is rather complicated, but in the finite dimensional setting of our model, it is a consequence of standard convex optimization.

Using Lemma 2.13, it is possible to show that, for the class $\mathfrak{C}$, the Euler equations take a special form, indicated below (see, Malamud and Trubowitz [16], Proposition 5.2). Furthermore, a direct calculation shows that

$$X_t(\mathbf{x}) \;=\; \mathrm{P}_{\mathcal{G}_t}\left[\sum_{\tau=t}^{T} D_{\mathbf{x}\,\tau}\,\frac{M_\tau}{M_t}\right] \qquad (2.28)$$

for any portfolio strategy $\mathbf{x}$. This identity allows us to rewrite the budget constraints in a form, involving only the consumption and endowment. See, [16], Theorem 2.15 and Propositions 5.1-5.2.

Proposition 2.18 *The utility maximization problem has a solution if and only if* (2.24) *is satisfied. The optimal consumption stream* $(c_0\,,\,\mathbf{c})$ *is uniquely determined by the first order conditions*

$$e^{-\rho}\, E\,[u'(c_{t+1})\,|\,\mathcal{H}_{t+1}\,] \;=\; \frac{M_{t+1}}{M_t}\, u'(c_t) \qquad (2.29)$$

and the budget constraints

$$(I - \mathrm{P}_{\mathcal{H}_t})\,\mathrm{P}_{\mathcal{G}_t}\left[\sum_{\tau=t}^{T}(c_\tau - w_\tau)\,M_\tau\right] = 0 \tag{2.30}$$

for all $t = 1, \ldots, T$ *and*

$$E\left[\sum_{\tau=0}^{T}(c_\tau - w_\tau)\,M_\tau\right] = 0. \tag{2.31}$$

Furthermore, $c_0 = c_0(w_0, \mathbf{w})$ *is monotone increasing in* w_0.

(2.29), (2.30) and (2.31) form a highly non linear and complicated system of equations. Malamud and Trubowitz [16] introduced a recursive procedure for explicitly solving the system (2.29), (2.30) and (2.31).

Note that the first order conditions can be formulated in the following equivalent form.

Corollary 2.19 *Let* $(c_0, \mathbf{c})$ *be the optimal consumption stream. Then,*

$$R_t = e^{-\rho t}u'(c_t) \tag{2.32}$$

is a state price density process for the market $(\mathcal{M}, \mathcal{G})$.

Proof. See (B.20) in [17]. □

The special state price density process (2.32) is known in economics as the subjective state price density process of the agent. Note that, since the market is incomplete, there is no unique way of valuing streams of unhedgeable claims. The intuition behind (2.32) is that the agent uses this, subjective, discounting to measure unhedgeable claims.

3 Separating consumption at time zero

From now on we will work exclusively with the power utility function

$$u(c) = \frac{c^{1-\gamma}}{1 - \gamma} \tag{3.1}$$

for $\gamma > 0$, $\gamma \neq 1$ and $u(c) = \log c$ for $\gamma = 1$. This class of utilities is often referred to as the constant relative risk aversion utility functions (abbreviated as CRRA). A large part of the analysis can be directly extended to the case of general utility functions.

Consumption at time zero plays a very special role in the structure of the optimal consumption stream. We illustrate this on the simple example of a complete market.

If the market is complete, $\mathcal{H}_t = \mathcal{G}_t$, and the first order conditions (2.29) take the simple form

$$e^{-\rho t} c_t^{-\gamma} = M_t c_0^{-\gamma} \Leftrightarrow c_t = e^{-\rho t/\gamma} M_t^{-1/\gamma} c_0. \tag{3.2}$$

Furthermore, the single, intertemporal budget constraint (2.31) implies that

$$c_0 = \frac{w_0 + E\left[\sum_{t=1}^T w_t M_t\right]}{1 + E\left[\sum_{t=1}^T e^{-\rho t/\gamma} M_t^{1-1/\gamma}\right]}. \tag{3.3}$$

Thus, endowment process $(w_0, \mathbf{w})$ only enters the optimal consumption stream through c_0, and c_0 is a linear function of the endowment stream. Thus, it is natural to write the optimal consumption stream $\mathbf{c} = (c_t, t = 1, \ldots, T)$ in the form

$$\mathbf{c} = \mathbf{c}(c_0) \qquad \text{and} \qquad c_0 = c_0(w_0, \mathbf{w}). \tag{3.4}$$

It turns out that a similar representation is possible in general incomplete markets. This representation plays a crucial role in our analysis.

3.1 Notations and definitions

Let

$$H = \oplus_{t=1}^T L_2(\Omega, \mathcal{G}_t), \tag{3.5}$$

be the Hilbert space of all adapted processes, starting at $t = 1$, equipped with the standard inner product

$$\langle \mathbf{Z}, \mathbf{Y} \rangle = \sum_{t=1}^T E[Z_t Y_t] = \sum_{t=1}^T \langle Z_t, Y_t \rangle. \tag{3.6}$$

for any $\mathbf{Z} = (Z_t)$, $\mathbf{Y} = (Y_t) \in H$. Any $\mathcal{G}$-adapted process

$$\mathbf{a} = (a_1, \ldots, a_T) \tag{3.7}$$

defines a natural multiplication operator on H via

$$\mathbf{a}\,\mathbf{Z} = \operatorname{diag}(a_t)_{t=1}^T \mathbf{Z} = (a_t Z_t) \in H. \tag{3.8}$$

We will also use the operator

$$\mathfrak{d} = \operatorname{diag}(e^{-\rho t})_{t=1}^T. \tag{3.9}$$

Depending on the context, we use boldface letters to denote both vectors and the corresponding multiplication operators.

The following special, scaled inner product plays a crucial role in our analysis.

Definition 3.1 *Fix an endowment stream* $(w_0\,,\,\mathbf{w})$. *Let*

$$(c_0\,,\,\mathbf{c}) \;=\; (c_0(w_0\,,\,\mathbf{w})\,,\,\mathbf{c}(c_0\,,\,\mathbf{w})) \tag{3.10}$$

be the corresponding optimal consumption stream, defined in Proposition 2.18. We define the scaled inner product

$$\langle \mathbf{Z}, \mathbf{Y}\rangle_c \;=\; \sum_{t=1}^{T} e^{-\rho t}\, E\left[c_t^{-\gamma-1}\, Z_t\, Y_t\right] \;=\; \left\langle \mathfrak{d}\mathbf{c}^{-\gamma-1}\,\mathbf{Z}\,,\,\mathbf{Y}\right\rangle. \tag{3.11}$$

Remark 3.2 *We emphasize that the scaled inner product depends on the endowment stream. Thus, it should be viewed as a Riemanian structure on the space of all endowment streams: in each point of the space, there is a metric, defined by the inner product* (3.11).

The following lemma in an immediate consequence of the definition.

Lemma 3.3 *Let* $\gamma \neq 1$. *Then, The norm squared of the optimal consumption stream* $\mathbf{c}$ *is given by*

$$\langle \mathbf{c}, \mathbf{c}\rangle_c \;=\; \sum_{t=1}^{T} e^{-\rho t} E\left[c_t^{1-\gamma}\right] = (1-\gamma) U(c_0\,,\,\mathbf{c}) \;-\; c_0^{1-\gamma}\,, \tag{3.12}$$

where

$$U(c_0\,,\,\mathbf{c}) \;=\; (1-\gamma)^{-1} \sum_{t=1}^{T} e^{-\rho t}\, E\left[c_t^{1-\gamma}\right] \tag{3.13}$$

is the maximal utility, achievable by trading, of an agent with endowment $(w_0\,,\,\mathbf{w})$.

In the sense of Lemma 3.3, the scaled inner product $\langle\cdot,\cdot\rangle_c$ is an economically natural inner product: the size(norm) of the consumption stream is equal to its utility.

Let $J\,:\,H\,\to\,H$ be the linear operator defined by

$$(J\mathbf{Z})_t = \sum_{\tau=1}^{t} Z_\tau. \tag{3.14}$$

for $t\;=\;1\,,\,\ldots\,,\,T$. It is easy to see that the adjoint operator J^* of J with respect to the standard inner product is given by

$$(J^*\mathbf{Z})_t \;=\; \mathrm{P}_{\mathcal{G}_t} \sum_{\tau=t}^{T} Z_\tau \quad \text{with} \quad \langle J\,\mathbf{Z}\,,\,\mathbf{Y}\,\rangle \;=\; \langle\,\mathbf{Z}\,,\,J^*\mathbf{Y}\,\rangle. \tag{3.15}$$

Let for $t = 1, \ldots, T$,

$$Q_t \;=\; \mathrm{P}_{\mathcal{G}_t} \;-\; \mathrm{P}_{\mathcal{H}_t} \tag{3.16}$$

and let $\mathbf{Q}:\,H\,\to\,H$ be the orthogonal sum

$$\mathbf{Q} = \oplus_{t=1}^{T}\, Q_t. \tag{3.17}$$

The image

$$H_0 \;=\; \mathbf{Q}H \;=\; \oplus_{t=1}^T \, Q_t \; L_2(\Omega, \mathcal{G}_t), \tag{3.18}$$

of the orthogonal projection $\mathbf{Q}$ will play an important role in our analysis. Intuitively, this is the "unhedgeable" subspace.

Let H_1 , H_2 be two Hilbert spaces. Consider a smooth map $G \,:\, H_1 \;\rightarrow\; H_2$. By definition, the derivative

$$D(G) = \frac{\partial G(\mathbf{w})}{\partial \mathbf{w}} \tag{3.19}$$

is a linear map $H_1 \;\rightarrow\; H_2$ such that for $\varepsilon \rightarrow 0$

$$G(\mathbf{w} \,+\, \varepsilon\, \mathbf{y}) \;=\; G(\mathbf{w}) \;+ \varepsilon\, \frac{\partial G(\mathbf{w})}{\partial \mathbf{w}}\, \mathbf{y} \;+\; O(\varepsilon^2). \tag{3.20}$$

The second derivative

$$D^2(G) = \frac{\partial^2 G(\mathbf{w})}{\partial \mathbf{w}^2} \tag{3.21}$$

is a bilinear map $H_1 \,\times\, H_1 \;\rightarrow\; H_2$ such that for $\varepsilon \rightarrow 0$

$$G(\mathbf{w} \,+\, \varepsilon\, \mathbf{y}) \;=\; G(\mathbf{w}) \;+ \varepsilon\; \frac{\partial G(\mathbf{w})}{\partial \mathbf{w}}\, \mathbf{y} \;+ \varepsilon^2\; \frac{\partial^2 G(\mathbf{w})}{\partial \mathbf{w}^2} (\mathbf{y} \,,\, \mathbf{y}) \;+\; O(\varepsilon^3). \tag{3.22}$$

3.2 Construction of optimal consumption streams

The goal of this section is to understand the structure of the nonlinear map

$$(w_0 \,,\, \mathbf{w}) \;\rightarrow\; (c_0 \,,\, \mathbf{c}) \tag{3.23}$$

mapping the endowment stream into the optimal consumption stream, defined in Proposition 2.18. This is analogous to the complete market case (see, (3.2)). The recursive construction of [16] explains its local structure, i.e., the dependence between c_t and c_{t+1} . In this section we introduce a new formalism that allows to treat this map in a global way and derive interesting properties of its derivatives, that can not be seen in the "local", recursive formalism of [16].

As we explain above, one of the key ideas is to decouple the initial consumption and construct the map $\mathbf{c} \;=\; \mathbf{c}(c_0 \,,\, \mathbf{w})$.

We start with a

LEMMA 3.4 *There exists a function* $A \,:\, H \;\rightarrow\; \mathbb{R}$ *such that for every* $x \in (A(\mathbf{w}) \,,\, +\infty)$ *there exists a unique number (see,* (2.24)*)*

$$w_0 \;>\; (-\,\mathbf{w})_0^u \tag{3.24}$$

for which the optimal consumption stream $(c_0 \,,\, \mathbf{c})$, *corresponding to the endowment process* $(w_0 \,,\, \mathbf{w})$, *has initial consumption* $c_0 \;=\; x$.

Proof. Fix $\mathbf{w} \in H$. By Proposition 2.18, c_0 is smooth, monotone increasing function of $w_0 \in ((-\mathbf{w})_0^u\,,\,+\infty)$. Consequently, it maps $((-\mathbf{w})_0^u\,,\,+\infty)$ onto some interval $(A(\mathbf{w})\,,\,+\infty)$. That $+\infty$ is mapped to $+\infty$ is clear because the first order conditions and finiteness of the probability space immediately imply that the consumption at time zero must go to infinity as the intertemporal wealth goes to infinity. □

Fix a consumption c_0 at time zero and let

$$\mathbf{cm} \;=\mathbf{cm}(c_0) = c_0\;\mathfrak{d}^{1/\gamma}\mathbf{M}^{-1/\gamma}. \tag{3.25}$$

be the optimal consumption stream in a *fictitious* complete market with the unique SPD process $\mathbf{M}$ (see, (3.2)).

Definition 3.5 *Let* $H_0^+ \;=\; \{\mathbf{Z}\in H_0;\; \mathbf{1}+J\mathbf{Z}>0\}$. *Define the map* $F\;:\;H_0^+\;\to\;H_0$ *via*

$$F(\mathbf{Z})\;\;=\;\;F_{c_0}(\mathbf{Z})\;\;=\;\;\mathbf{Q}J^*\;\mathbf{M}\;\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma}\,. \tag{3.26}$$

Lemma 3.6 *The map* $F\;:\;H_0^+\;\to\;F(H_0^+)\subset H_0$ *is bijective and monotone decreasing, in the sense that for all* $\mathbf{Z},\mathbf{Y}\in H_0^+$

$$\langle F(\mathbf{Z})-F(\mathbf{Y})\,,\,\mathbf{Z}\,-\,\mathbf{Y}\rangle\le 0. \tag{3.27}$$

The inequality is strict as soon as $\mathbf{Z}\neq\mathbf{Y}$.

Proof of Lemma 3.6. Denote by $D(F)\;:\;H_0\;\to\;H_0$ the derivative $\partial F(\mathbf{Z})/\partial\mathbf{Z}$. Then, by direct calculation,

$$D(F)=D(F)|_{\mathbf{Z}}=-\gamma^{-1}\mathbf{Q}\,J^*\,\mathbf{M}\;\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma-1}\;J. \tag{3.28}$$

Thus, for any $\mathbf{y}\;\in\;H_0\;\;=\;\;\mathbf{Q}\,H$ we have

$$\langle\,D(F)\,\mathbf{y}\,,\,\mathbf{y}\rangle\;\;=\;\;-\gamma^{-1}\langle\,\mathbf{M}\;\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma-1}\;J\,\mathbf{y}\,,\,J\,\mathbf{y}\rangle\;\;\le\;\;0. \tag{3.29}$$

The inequality is strict if $\mathbf{y}\;\neq\;0$. We have

$$F(\mathbf{Z})\;\;=\;\;F(\mathbf{Y})\;\;+\;\;\int_0^1\;D(F)|_{\mathbf{Y}\,+\,t(\mathbf{Z}\,-\,\mathbf{Y})}\,(\mathbf{Z}\;-\;\mathbf{Y})\,dt, \tag{3.30}$$

and therefore

$$\langle F(\mathbf{Z})-F(\mathbf{Y}),\mathbf{Z}-\mathbf{Y}\rangle=\int_0^1\langle D(F)|_{\mathbf{Y}+t(\mathbf{Z}-\mathbf{Y})}(\mathbf{Z}-\mathbf{Y}),(\mathbf{Z}-\mathbf{Y})\rangle dt\le\;0, \tag{3.31}$$

and the equality holds if and only if $\mathbf{Z}\;=\;\mathbf{Y}$.

Suppose that F is not injective, that is, there exist $\mathbf{Y}\;\neq\;\mathbf{Z}$ such that $F(\mathbf{Y})\;\;=\;\;F(\mathbf{Z})$. Then, the monotonicity condition is obviously violated. □

THEOREM 3.7 *Let* $\mathbf{w} = (w_t\,,\, t = 1\,,\,\ldots\,,\,T)$ *be an endowment process. Choose* $c_0 > A(\mathbf{w})$ *and let* w_0 *be the corresponding initial endowment. Then,*

$$\mathbf{Q}J^*\mathbf{M}\mathbf{w} \;\in\; F_{c_0}(H_0^+) \tag{3.32}$$

and the optimal consumption stream $\mathbf{c}$ *is given by*

$$\mathbf{c} \;= \mathbf{c}(c_0\,,\,\mathbf{w}) = \;\mathbf{cm}(c_0)\,(1 \,+\, J\,\mathbf{Z}(c_0\,,\,\mathbf{w}))^{-1/\gamma} \tag{3.33}$$

with

$$\mathbf{Z}(\mathbf{w}) = \mathbf{Z}(c_0\,,\,\mathbf{w}) \;=\; F_{c_0}^{-1}\,(\mathbf{Q}J^*\mathbf{M}\mathbf{w}) = F^{-1}\,(\mathbf{Q}J^*\mathbf{M}\mathbf{w}) \;\in H_0^+. \tag{3.34}$$

The derivatives of the maps $\mathbf{c}$ *and* F *are given by*

$$D(\mathbf{c}) = \frac{\partial \mathbf{c}(c_0\,,\,\mathbf{w})}{\partial \mathbf{w}} = -\gamma^{-1}\mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma}\,J\,(D(F))^{-1}\,\mathbf{Q}\,J^*\,\mathbf{M}, \tag{3.35}$$

and

$$D(F) \;=\; \frac{\partial F}{\partial \mathbf{Z}} \;=\; -\gamma^{-1}\mathbf{Q}\,J^*\,\mathbf{M}\,\mathbf{cm}^{-\gamma}\,\mathbf{c}^{1+\gamma}\,J. \tag{3.36}$$

respectively.

Proof. Existence of $\mathbf{c}$ follows from Lemma 3.4. Define the process $\mathbf{Z} = (Z_t\,,\, t = 1\,,\,\ldots\,,\,T) \in H$ via

$$c_0^{-\gamma}\,Z_t \;=\; e^{-\rho\,t}\,c_t^{-\gamma}\,M_t^{-1} \;-\; e^{-\rho\,(t-1)}\,c_{t-1}^{-\gamma}\,M_{t-1}^{-1}. \tag{3.37}$$

By (2.29), $\mathrm{P}_{\mathcal{H}_t} Z_t \;=\; 0$, that is $\mathbf{Z} \in H_0$. Summing up (3.37), we get

$$e^{-\rho\,t}\,c_t^{-\gamma}\,M_t^{-1} \;=\; c_0^{-\gamma} \;+\; c_0^{-\gamma}\sum_{\tau=1}^{t} Z_\tau \tag{3.38}$$

for all $t = 1\,,\,\ldots\,,\,T$. That is,

$$\mathbf{c} \;=\; \mathbf{cm}\,(1 \,+\, J\,\mathbf{Z})^{-1/\gamma}. \tag{3.39}$$

Substituting (3.39) into the budget constraints (2.30), we get

$$\mathbf{Q}J^*\mathbf{M}\mathbf{w} \;=\; \mathbf{Q}J^*\mathbf{M}\,\mathbf{c} \;=\; \mathbf{Q}J^*\mathbf{M}\;\mathbf{cm}\,(\mathbf{1} + J\mathbf{Z})^{-1/\gamma} \;=\; F(\mathbf{Z}), \tag{3.40}$$

which is what had to be proved.

Differentiating

$$\mathbf{c}(c_0\,,\,\mathbf{w}) = \;\mathbf{cm}\,(1 \,+\, J\,F^{-1}(\mathbf{Q}J^*\mathbf{M}\mathbf{w}))^{-1/\gamma}. \tag{3.41}$$

with respect to $\mathbf{w}$, we get

$$D(\mathbf{c}) = -\gamma^{-1}\mathbf{cm}\,(1 \,+\, J\,F^{-1}(\mathbf{Q}J^*\mathbf{M}\mathbf{w}))^{-1/\gamma-1}J(D(F))^{-1}\mathbf{Q}J^*\mathbf{M}, \tag{3.42}$$

and the identity

$$\mathbf{cm}\,(\mathbf{1} + J\mathbf{Z})^{-1/\gamma-1} = \mathbf{cm}^{-\gamma}\left(\mathbf{cm}\,(\mathbf{1} + J\mathbf{Z})^{-1/\gamma}\right)^{1+\gamma} = \mathbf{cm}^{-\gamma}\,\mathbf{c}^{1+\gamma} \tag{3.43}$$

implies (3.35). The proof of (3.36) is analogous. □

Remark 3.8 *Theorem 3.7 implies that the optimal consumption stream can be written in the form*

$$(c_0, \mathbf{c}) = (c_0(w_0, \mathbf{w})\,,\, \mathbf{c}(c_0(w_0, \mathbf{w})\,,\, \mathbf{w})). \tag{3.44}$$

Given $\mathbf{c} = \mathbf{c}(c_0\,, \mathbf{w})$, *the value* $c_0 = c_0(w_0\,, \mathbf{w})$ *is uniquely determined by the last budget constraint* (2.31). *This is similar to the complete market situation (see,* (3.2) *and* (3.3)*).*

We are now ready to state the main result of this section.

Theorem 3.9 *Let* $c_0 > A(\mathbf{w})$ *and* $(c_0, \mathbf{w}) \mapsto \mathbf{c}(c_0\,, \mathbf{w})$ *be the map defined in Theorem 3.7. Then, the derivative* $D(\mathbf{c}) = \partial \mathbf{c}/\partial \mathbf{w}$, *given in* (3.35), *is the orthogonal projection* $\mathbf{P_c}$ *onto the subspace*

$$H_c = \mathbf{M}\Big(\mathfrak{d}\mathbf{c}^{-\gamma-1}\Big)^{-1} J H_0 \tag{3.45}$$

in the Hilbert space $(H, \langle\cdot,\cdot\rangle_c)$, *equipped with the scaled inner product* (3.11).

We need an auxiliary

Lemma 3.10 *The adjoint of* $\mathbf{P_c} = D(\mathbf{c})|_{\mathbf{w}}$ *with respect to the standard inner product* $\langle\,\cdot\,,\,\cdot\rangle$ *is given by*

$$\mathbf{P}_{\mathbf{c}}^{*} = \; -\gamma^{-1}\,\mathbf{M}\,J\,D(F)^{-1}\,\mathbf{Q}\,J^{*}\,\mathbf{c}^{\gamma+1}\,\mathbf{cm}^{-\gamma}. \tag{3.46}$$

Moreover,

$$\mathbf{M}\,\mathbf{cm}^{\gamma}\,\mathbf{c}^{-\gamma-1}\,\mathbf{P_c} \;=\; \mathbf{P}_{\mathbf{c}}^{*}\,\mathbf{M}\,\mathbf{cm}^{\gamma}\,\mathbf{c}^{-\gamma-1}. \tag{3.47}$$

Proof. Let $\mathbf{y}, \mathbf{z} \in H_0 = \mathbf{Q}H$. Then,

$$\begin{aligned} \langle D(F)\,\mathbf{y}, \mathbf{z}\rangle \;&=\; \langle -\gamma^{-1}\mathbf{Q}\,J^{*}\,\mathbf{M}\;\mathbf{cm}^{-\gamma}\,\mathbf{c}^{1+\gamma}\,J\;\mathbf{y}, \mathbf{z}\rangle \qquad (3.48)\\ &=\; -\gamma^{-1}\langle J^{*}\,\mathbf{M}\;\mathbf{cm}^{-\gamma}\,\mathbf{c}^{1+\gamma}\,J\;\mathbf{y}, \mathbf{z}\rangle \\ &=\; -\gamma^{-1}\langle \mathbf{y}, J^{*}\,\mathbf{M}\;\mathbf{cm}^{-\gamma}\,\mathbf{c}^{1+\gamma}\,J\;\mathbf{z}\rangle \;=\; \langle \mathbf{y}, D(F)\;\mathbf{z}\rangle. \end{aligned}$$

Thus, $D(F) : H_0 \to H_0$ is selfadjoint with respect to the standard inner product. Consequently, $D(F)^{-1}$ is also selfadjoint and

$$\langle\, D(F)^{-1}\,\mathbf{y}\,,\,\mathbf{z}\rangle \;=\; \langle\,\mathbf{Q}\,D(F)^{-1}\,\mathbf{y}\,,\,\mathbf{z}\rangle \;=\; \langle\,\mathbf{y}\,,\,D(F)^{-1}\,\mathbf{Q}\,\mathbf{z}\rangle. \tag{3.49}$$

for any $\mathbf{y} \in H_0$ and any $\mathbf{z} \in H$. Therefore,

$$\begin{aligned} -\gamma\,\langle\,\mathbf{P_c}\,\mathbf{y}\,,\,\mathbf{z}\rangle \;&=\; \langle\,\mathbf{c}^{\gamma+1}\,\mathbf{cm}^{-\gamma}\,J\,D(F)^{-1}\,\mathbf{Q}\,J^{*}\,\mathbf{M}\,\mathbf{y}\,,\,\mathbf{z}\rangle \\ &=\; \langle\,\mathbf{y}\,,\,\mathbf{M}\,J\,D(F)^{-1}\,\mathbf{Q}\,J^{*}\,\mathbf{c}^{\gamma+1}\,\mathbf{cm}^{-\gamma}\,\mathbf{z}\rangle \qquad (3.50) \end{aligned}$$

and (3.46) follows. Identity (3.47) is verified by direct calculation. □

Proof of Theorem 3.9. The operator $\mathbf{Q}\,J^*\,\mathbf{M}$ maps H onto H_0 and therefore, $J\,(D(F))^{-1}\,\mathbf{Q}\,J^*\,\mathbf{M}$ maps H onto $J\,H_0$. Substituting the identity $\mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma} = c_0^{-\gamma}\,\mathfrak{d}^{-1}\mathbf{M}\mathbf{c}^{1+\gamma}$ into (3.35), we immediately get that the $D(\mathbf{c})$ maps H onto H_c.

It remains to prove that $D(\mathbf{c})$ is an orthogonal projection. Identity (3.36) implies that

$$\mathbf{Q} \;=\; D(F)\,(D(F))^{-1}\,\mathbf{Q} \;=\; -\gamma^{-1}\mathbf{Q}J^*\mathbf{M}\;\mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}. \quad (3.51)$$

Multiplying (3.51) from the left and right with $-\gamma^{-1}\mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}$ and $J^*\,\mathbf{M}$ respectively, we obtain

$$\begin{aligned}&-\gamma^{-1}\mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}J^*\mathbf{M} \\ &\quad= \gamma^{-2}\mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}J^*\mathbf{M}\mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma}J\;D(F)^{-1}\mathbf{Q}J^*\mathbf{M}.\end{aligned} \quad (3.52)$$

That is, $D(\mathbf{c}) \;=\; D(\mathbf{c})^2$.

It remains to prove that $D(\mathbf{c})$ is self-adjoint w.r.t. $\langle\cdot,\cdot\rangle_c$. By Lemma 3.10,

$$\mathbf{M}\;\mathbf{cm}^{\gamma}\;\mathbf{c}^{-\gamma-1}D(\mathbf{c}) = D^*(\mathbf{c})\mathbf{M}\;\mathbf{cm}^{\gamma}\;\mathbf{c}^{-\gamma-1}, \quad (3.53)$$

where $D^*(\mathbf{c})$ is the adjoint with respect to the standard inner product. Therefore, using (3.25), we get

$$\begin{aligned}\langle D(\mathbf{c})\mathbf{X},\mathbf{Y}\rangle_c \;&=\; \left\langle \mathfrak{d}\mathbf{c}^{-\gamma-1}D(\mathbf{c})\mathbf{X},\mathbf{Y}\right\rangle \;=\; \left\langle \mathbf{M}\;\mathbf{cm}^{\gamma}\;\mathbf{c}^{-\gamma-1}D(\mathbf{c})\mathbf{X},\mathbf{Y}\right\rangle \\ &=\; \left\langle \mathbf{M}\;\mathbf{cm}^{\gamma}\;\mathbf{c}^{-\gamma-1}\mathbf{X},D(\mathbf{c})\mathbf{Y}\right\rangle \;=\; \left\langle \mathfrak{d}\mathbf{c}^{-\gamma-1}\mathbf{X},D(\mathbf{c})\mathbf{Y}\right\rangle = \langle \mathbf{X},D(\mathbf{c})\mathbf{Y}\rangle_c\,.\end{aligned} \quad (3.54)$$

which is what had to be proved. □

Lemma 3.11 *Under the assumptions of Theorem 3.7, the second derivative of* $\mathbf{c}(c_0\,,\,\mathbf{w})$ *with respect to* $\mathbf{w}$ *is given by*

$$\frac{\partial^2\mathbf{c}(c_0\,,\,\mathbf{w})}{\partial\mathbf{w}^2}(\mathbf{Y},\mathbf{Y}) = (1+\gamma)\,(I\;-\;\mathbf{P_c})\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{Y})^2\,. \quad (3.55)$$

Proof of Lemma 3.11. Using (3.26) and (3.36), we get

$$\begin{aligned}F(\mathbf{Z}+\varepsilon\mathbf{Y}) &= \mathbf{Q}J^*\;\mathbf{M}\;\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z}+\varepsilon J\mathbf{Y})^{-1/\gamma} \\ &= F(\mathbf{Z}) - \varepsilon\;\gamma^{-1}\mathbf{Q}J^*\;\mathbf{M}\;\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma-1}\,J\mathbf{Y} \\ &\quad+\varepsilon^2\gamma^{-1}\left(\gamma^{-1}+1\right)\mathbf{Q}J^*\;\mathbf{M}\;\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma-2}\,(J\mathbf{Y})^2+O(\varepsilon^3) \\ &= F(\mathbf{Z})+\varepsilon\;D(F)|_{\mathbf{Z}}(\mathbf{Y})+\varepsilon^2\;D^2(F)|_{\mathbf{Z}}(\mathbf{Y},\mathbf{Y})+O(\varepsilon^3).\end{aligned} \quad (3.56)$$

Therefore,

$$D^2(F)|_{\mathbf{Z}}(\mathbf{Y},\mathbf{Y}) = \gamma^{-1}\left(\gamma^{-1}+1\right)\mathbf{Q}J^*\;\mathbf{M}\;\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma-2}\,(J\mathbf{Y})^2. \quad (3.57)$$

Let $\mathbf{X} = \mathbf{Q}J^*\mathbf{M}\mathbf{w}$. Then, by (3.36),

$$D(F^{-1})|_{\mathbf{X}} = ((D(F))|_{F^{-1}\mathbf{X}})^{-1} = -\gamma \left(\mathbf{Q}\, J^*\, \mathbf{M}\ \mathbf{cm}^{-\gamma}\mathbf{c}^{1+\gamma}\, J\right)^{-1}. \qquad (3.58)$$

Differentiating this identity and using (3.57), we arrive at

$$\begin{aligned} &D^2(F^{-1})|_{\mathbf{X}}(\mathbf{Y},\mathbf{Y}) \qquad (3.59)\\ &= -D(F^{-1})|_{\mathbf{X}} \left[D^2(F)|_{F^{-1}\mathbf{X}}\Big(D(F^{-1})|_{\mathbf{X}}(\mathbf{Y})\ ,\ D(F^{-1})|_{\mathbf{X}}(\mathbf{Y})\Big)\right]\\ &= -(D(F))^{-1}\frac{1}{\gamma}\left(\frac{1}{\gamma}+1\right)\mathbf{Q}J^*\ \mathbf{M}\ \mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma-2}\,(J(D(F))^{-1}(\mathbf{Y}))^2, \end{aligned}$$

with $\mathbf{Z} = F^{-1}(\mathbf{X})$.

Differentiating (3.35) twice with respect to $\mathbf{w}$, we obtain

$$\begin{aligned} D^2(\mathbf{c})(\mathbf{Y},\mathbf{Y}) \ &= \ \gamma^{-1}(\gamma^{-1}+1)\,\mathbf{cm}\,(1\,+\,J\,\mathbf{Z})^{-1/\gamma-2}(J(D(F))^{-1}\mathbf{Q}J^*\mathbf{M}\mathbf{Y})^2\\ &+\gamma^{-1}\ \mathbf{cm}\,(1\,+\,J\,\mathbf{Z})^{-1/\gamma-1}JD^2(F^{-1})(\mathbf{Q}J^*\mathbf{M}\mathbf{Y})^2. \qquad (3.60) \end{aligned}$$

Making use of (3.59), (3.35) and the identity

$$\mathbf{cm}\,(\mathbf{1}+J\mathbf{Z})^{-1/\gamma-2} = \mathbf{cm}^{-2\gamma}\,\mathbf{c}^{1+2\gamma}, \qquad (3.61)$$

we obtain

$$\begin{aligned} &D^2(\mathbf{c})(\mathbf{Y},\mathbf{Y})\\ &\quad = \ (\gamma+1)\ \ \mathbf{c}^{-1}\ \ (\mathbf{P_c}\mathbf{Y})^2\ -\ (\gamma+1)\ \ \mathbf{P_c}\ \mathbf{c}^{-1}\ \ (\mathbf{P_c}\mathbf{Y})^2. \qquad (3.62) \end{aligned}$$

This completes the proof. □

LEMMA 3.12 *Let* $\mathbf{w}$ *be an endowment stream and* $c_0 > A(\mathbf{w})$ *. The map* $\mathbf{c}(c_0\,,\,\mathbf{w})$ *is homogeneous of degree one, that is,*

$$\mathbf{c}(c_0\,,\,\mathbf{w}) \ = \ c_0\,\mathbf{c}(1\,,\,c_0^{-1}\,\mathbf{w}). \qquad (3.63)$$

Consequently, the Euler identity

$$\frac{\partial\mathbf{c}(c_0\,,\,\mathbf{w})}{\partial c_0} \ = \ c_0^{-1}\,\mathbf{c}(c_0\,,\,\mathbf{w}) \ - \ c_0^{-1}\,\frac{\partial\mathbf{c}(c_0\,,\,\mathbf{w})}{\partial\mathbf{w}}(\mathbf{w}) \qquad (3.64)$$

holds, as well as

$$\frac{\partial^2\mathbf{c}(c_0\,,\,\mathbf{w})}{\partial c_0\,\partial\mathbf{w}}(\mathbf{y}) = \ - \ c_0^{-1}\,\frac{\partial^2\mathbf{c}(c_0\,,\,\mathbf{w})}{\partial\mathbf{w}^2}(\mathbf{w}\,,\,\mathbf{y}), \qquad (3.65)$$

and

$$\frac{\partial^2\mathbf{c}(c_0\,,\,\mathbf{w})}{\partial c_0^2} \ = \ c_0^{-2}\,\frac{\partial^2\mathbf{c}(c_0\,,\,\mathbf{w})}{\partial\mathbf{w}^2}(\mathbf{w}\,,\,\mathbf{w}). \qquad (3.66)$$

Proof of Lemma 3.12. Proposition 2.18 implies that both $\mathbf{c}(\lambda c_0, \lambda \mathbf{w})$ and $\lambda \mathbf{c}(c_0, \mathbf{w})$ satisfy the first order conditions and budget constraints. Since the optimal stream is unique, we have

$$\mathbf{c}(\lambda c_0, \lambda \mathbf{w}) = \lambda \mathbf{c}(c_0, \mathbf{w}). \tag{3.67}$$

Differentiating this identity with respect to λ we get (3.64). Now, differentiating (3.64) with respect to $\mathbf{w}$ we get

$$\frac{\partial^2 \mathbf{c}(c_0, \mathbf{w})}{\partial c_0 \partial \mathbf{w}}(\mathbf{y}) \tag{3.68}$$
$$= c_0^{-1} \frac{\partial \mathbf{c}(c_0, \mathbf{w})}{\partial \mathbf{w}}(\mathbf{y}) - c_0^{-1} \frac{\partial^2 \mathbf{c}(c_0, \mathbf{w})}{\partial \mathbf{w}^2}(\mathbf{w}, \mathbf{y}) - c_0^{-1} \frac{\partial \mathbf{c}(c_0, \mathbf{w})}{\partial \mathbf{w}}(\mathbf{y}),$$

and (3.65) follows. Differentiating (3.64) with respect to c_0 and using formulas (3.64) and (3.65) we get

$$\frac{\partial^2 \mathbf{c}(c_0, \mathbf{w})}{\partial c_0^2} = -c_0^{-1} \frac{\partial^2 \mathbf{c}(c_0, \mathbf{w})}{\partial c_0 \partial \mathbf{w}}(\mathbf{w}) = c_0^{-2} \frac{\partial^2 \mathbf{c}(c_0, \mathbf{w})}{\partial \mathbf{w}^2}(\mathbf{w}, \mathbf{w}). \tag{3.69}$$

This completes the proof. □

4 The derivatives of the consumption at time zero

Until now, we have ignored the dependence of c_0 on the endowment and treated it as a free variable in the map $\mathbf{c}(c_0, \mathbf{w})$. To determine the value of c_0, we must impose the last budget constraint

$$c_0 + \langle \mathbf{c}(c_0, \mathbf{w}), \mathbf{M} \rangle = w_0 + \langle \mathbf{w}, \mathbf{M} \rangle. \tag{4.1}$$

and solve it for $c_0 = c_0(w_0, \mathbf{w})$. Note, that the non local structure of the map $\mathbf{c}(c_0, \mathbf{w})$ makes it difficult to study state-wise dependence of the consumption stream on c_0. But, the local in time, recursive structure of the optimal consumption stream, derived in [16], allows to address such questions. In particular, Malamud and Trubowitz [16], Theorem 5.14, show that future consumption $c_t(s) = c_t(s)(c_0, \mathbf{w})$ is monotone increasing in c_0 for each state s. Consequently, there exists a unique solution c_0 to (4.1).

Lemma 4.1 (First order derivatives, version 1) *We have*

$$\frac{\partial c_0(w_0, \mathbf{w})}{\partial \mathbf{w}}(\mathbf{y}) = \frac{\langle (I - (\mathbf{P_c})^*)\, \mathbf{M}, \mathbf{y} \rangle}{1 + \left\langle \frac{\partial \mathbf{C}}{\partial c_0}, \mathbf{M} \right\rangle}, \tag{4.2}$$

and

$$\frac{\partial c_0(w_0, \mathbf{w})}{\partial w_0} = \frac{1}{1 + \left\langle \frac{\partial \mathbf{C}}{\partial c_0}, \mathbf{M} \right\rangle}. \tag{4.3}$$

Proof. Both identities follow by differentiating (4.1). For the first identity we obtain with $\mathbf{P_c} = D(\mathbf{c})|_{\mathbf{w}} = \partial \mathbf{c}/\partial \mathbf{w}$

$$\frac{\partial c_0(w_0, \mathbf{w})}{\partial \mathbf{w}}(\mathbf{y}) + \langle \mathbf{y}, (\mathbf{P_c})^* \mathbf{M} \rangle + \langle \frac{\partial \mathbf{c}}{\partial c_0}, \mathbf{M} \rangle \frac{\partial c_0(w_0, \mathbf{w})}{\partial \mathbf{w}}(\mathbf{y}) = \langle \mathbf{y}, \mathbf{M} \rangle. \quad (4.4)$$

The second identity follows completely analogously. □

Formulae of Lemma 4.1 are very difficult to use if we want to address monotonicity and/or convexity/concavity properties of the initial consumption c_0. It is not even possible to see directly from (4.3) that the derivative $\partial c_0 / \partial w_0$ is positive. The reason is that all quantities are expressed in the wrong, "unphysical", inner product. Taking into account Theorem 3.9, we would like to have expressions for the derivatives, involving the economic inner product (3.11).

The key observation is that, due to (2.10), the standard budget constraint (4.1) still holds if we replace $\mathbf{M}$ by any other SPD process. By Corollary 2.19, $\mathbf{R} = (c_0^{-\gamma}, \mathfrak{d}\, \mathbf{c}^{-\gamma})$ is an SPD process and we arrive at

Lemma 4.2 *The optimal consumption stream satisfies*

$$(w_0 - c_0)\, c_0^{-\gamma} + \left\langle \mathbf{w} - \mathbf{c}(c_0, \mathbf{w}), \mathfrak{d}\, \mathbf{c}(c_0, \mathbf{w})^{-\gamma} \right\rangle = 0. \quad (4.5)$$

We also need the following identity.

Lemma 4.3 *Let* $\mathbf{c} = \mathbf{c}(c_0, \mathbf{w})$. *Then,*

$$\mathbf{P_c}\, \mathbf{w} = \mathbf{P_c}\, \mathbf{c} \quad (4.6)$$

Proof. The budget constraints (2.30) can be rewritten as (see, also (3.40))

$$\mathbf{Q}\, J^* \mathbf{M}\, \mathbf{c} = \mathbf{Q}\, J^* \mathbf{M}. \quad (4.7)$$

Using (3.35), we arrive at

$$\begin{aligned} \mathbf{P_c} \mathbf{w} &= -\gamma^{-1} \mathbf{c} \mathbf{m}^{-\gamma} \mathbf{c}^{1+\gamma}\, J\, (D(F))^{-1}\, \mathbf{Q}\, J^* \mathbf{M} \mathbf{w} \quad (4.8) \\ &= -\gamma^{-1} \mathbf{c} \mathbf{m}^{-\gamma} \mathbf{c}^{1+\gamma}\, J\, (D(F))^{-1}\, \mathbf{Q}\, J^* \mathbf{M} \mathbf{c} = \mathbf{P_c} \mathbf{c}. \end{aligned}$$

which is what had to be proved. □

These observations are crucial for our analysis. It allows us to write all the derivatives in a remarkably elegant way.

Proposition 4.4 (First order derivatives, version 2) *We have*

$$\frac{\partial c_0(w_0, \mathbf{w})}{\partial \mathbf{w}}(\mathbf{y}) = \frac{c_0 \langle \mathbf{y}, (I - \mathbf{P_c})(\mathbf{c}) \rangle_c}{c_0^{1-\gamma} + \|(I - \mathbf{P_c})\, \mathbf{c}\|_c^2}, \quad (4.9)$$

and

$$\frac{\partial c_0(w_0, \mathbf{w})}{\partial w_0} = \frac{c_0^{1-\gamma}}{c_0^{1-\gamma} + \|(I - \mathbf{P_c})\, \mathbf{c}\|_c^2}. \quad (4.10)$$

Proof. We first prove (4.10). Differentiating (4.5) with respect to w_0, we get

$$0 = \left(1 - \frac{\partial c_0}{\partial w_0}\right) c_0^{-\gamma} - \gamma\,(w_0 - c_0)\, c_0^{-\gamma-1} \frac{\partial c_0}{\partial w_0} \tag{4.11}$$
$$- \left\langle \frac{\partial \mathbf{c}}{\partial c_0} \frac{\partial c_0}{\partial w_0}\,,\, \mathfrak{d}\, \mathbf{c}^{-\gamma} \right\rangle - \gamma \left\langle (\mathbf{w} - \mathbf{c})\, \mathbf{c}^{-\gamma-1}\,,\, \mathfrak{d}\, \frac{\partial \mathbf{c}}{\partial c_0} \frac{\partial c_0}{\partial w_0} \right\rangle .$$

Making use of Lemma 4.3, we can rewrite (3.64) in the form

$$\frac{\partial \mathbf{c}(c_0\,,\, \mathbf{w})}{\partial c_0} = c_0^{-1}\, \mathbf{c}(c_0\,,\, \mathbf{w}) - c_0^{-1}\, \mathbf{P_c}(\mathbf{w}) = c_0^{-1}\, \mathbf{c}(c_0\,,\, \mathbf{w}) - c_0^{-1}\, \mathbf{P_c}(\mathbf{c})$$
$$= c_0^{-1}\, (I - \mathbf{P_c})\, (\mathbf{c}). \tag{4.12}$$

Therefore,

$$\left\langle \frac{\partial \mathbf{c}}{\partial c_0}\,,\, \mathfrak{d}\, \mathbf{c}^{-\gamma} \right\rangle = c_0^{-1}\, \langle\, (I - \mathbf{P_c})\, (\mathbf{c})\,,\, \mathbf{c} \rangle_c = c_0^{-1}\, \|(I - \mathbf{P_c})\, \mathbf{c}\|_c^2, \tag{4.13}$$

since $(I - \mathbf{P_c})$ is an orthogonal projection w.r.t. $\langle\cdot\,,\,\cdot\rangle$. Since $\mathbf{P_c}$ is selfadjoint with respect to $\langle\cdot\,,\,\cdot\rangle_c$, Lemma 4.3 implies that

$$\langle\, \mathbf{w} - \mathbf{c}\,,\, \mathbf{P_c}(\mathbf{c}) \rangle_c = \langle\, \mathbf{P_c}\mathbf{w} - \mathbf{P_c}\mathbf{c}\,,\, \mathbf{c} \rangle_c = 0. \tag{4.14}$$

Combining (4.14) with (4.5), we get

$$\left\langle (\mathbf{w} - \mathbf{c})\, \mathbf{c}^{-\gamma-1}\,,\, \mathfrak{d} \frac{\partial \mathbf{c}}{\partial c_0} \right\rangle = c_0^{-1}\, \langle\, \mathbf{w} - \mathbf{c}\,,\, (I - \mathbf{P_c})\, (\mathbf{c}) \rangle_c \tag{4.15}$$
$$= c_0^{-1}\, \langle\, \mathbf{w} - \mathbf{c}\,,\, \mathbf{c} \rangle_c = -\, (w_0 - c_0)\, c_0^{-\gamma-1}.$$

The required identity (4.10) follows now from (4.11), (4.13) and (4.15).

Differentiating (4.5) with respect to $\mathbf{w}$ we get

$$0 = -\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\, c_0^{-\gamma} - \gamma\,(w_0 - c_0)\, c_0^{-\gamma-1} \frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y}) \tag{4.16}$$
$$+ \left\langle \mathbf{y} - \mathbf{P_c}(\mathbf{y}) - \frac{\partial \mathbf{c}}{\partial c_0} \frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\,,\, \mathfrak{d}\, \mathbf{c}^{-\gamma} \right\rangle$$
$$- \gamma \left\langle \mathbf{w} - \mathbf{c}\,,\, \mathfrak{d}\, \mathbf{c}^{-\gamma-1} \left(\mathbf{P_c}(\mathbf{y}) + \frac{\partial \mathbf{c}}{\partial c_0} \frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y}) \right) \right\rangle .$$

Lemma 4.3 and the selfadjointness of $\mathbf{P_c}$ imply that

$$\langle\, \mathbf{w} - \mathbf{c}\,, \mathbf{P_c}(\mathbf{y}) \rangle_c = 0 \tag{4.17}$$

for any $\mathbf{y} \in H$. Using (4.12), (4.17) and (4.15), we arrive at

$$\Big\langle \mathbf{w} - \mathbf{c}\,,\, \mathfrak{d}\,\mathbf{c}^{-\gamma-1}\Big(\mathbf{P_c}(\mathbf{y}) + \frac{\partial \mathbf{c}}{\partial c_0}\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\Big)\Big\rangle \tag{4.18}$$
$$= \Big\langle \mathbf{w} - \mathbf{c}\,,\, \Big(\mathbf{P_c}(\mathbf{y}) + \frac{\partial \mathbf{c}}{\partial c_0}\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\Big)\Big\rangle_c$$
$$= \Big\langle \mathbf{w} - \mathbf{c}\,,\, \frac{\partial \mathbf{c}}{\partial c_0}\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\Big\rangle_c$$
$$= c_0^{-1}\Big\langle \mathbf{w} - \mathbf{c}\,,\, (\mathbf{c} - \mathbf{P_c}(\mathbf{c}))\,\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\Big\rangle_c$$
$$= c_0^{-1}\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\,\langle \mathbf{w} - \mathbf{c}\,,\mathbf{c}\rangle_c = -\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\,(w_0 - c_0)\,c_0^{-\gamma-1}.$$

Therefore, (4.16) takes the form

$$0 = -\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\,c_0^{-\gamma} + \Big\langle \mathbf{y} - \mathbf{P_c}(\mathbf{y}) - \frac{\partial \mathbf{c}}{\partial c_0}\frac{\partial c_0}{\partial \mathbf{w}}(\mathbf{y})\,,\mathbf{c}\Big\rangle_c\,. \tag{4.19}$$

An application of (4.12) and the fact that $\mathbf{P_c}$ is selfadjoint with respect to $\langle\cdot,\cdot\rangle_c$ complete the proof. □

Combining Lemma 4.1 and Proposition 4.4, yields the following

Corollary 4.5 *We have*

$$c_0^{\gamma}\,\mathfrak{d}\,\mathbf{c}^{-\gamma-1}\,(I - \mathbf{P_c})\,\mathbf{c} = (I - (\mathbf{P_c})^*)\,\mathbf{M}, \tag{4.20}$$

and

$$c_0^{-1+\gamma}\,\|\,(I - \mathbf{P_c})\,\mathbf{c}\,\|_c^2 = \Big\langle \frac{\partial \mathbf{c}}{\partial c_0}\,,\,\mathbf{M}\Big\rangle. \tag{4.21}$$

Surprisingly, we do not know any direct way to check these identities.

Definition 4.6 *Let*

$$\mathbf{b} = (I - \mathbf{P_c})\,(\mathbf{c})\,. \tag{4.22}$$

The random process $\mathbf{b}$ plays a very important role in our analysis and appears in almost every formula. We will need a

Lemma 4.7 *The process* $\mathbf{b}$ *is nonnegative.*

Proof. By (3.64),

$$\frac{\partial \mathbf{c}(c_0\,,\,\mathbf{w})}{\partial c_0} = c_0^{-1}\,\mathbf{b}. \tag{4.23}$$

By [16], Theorem 5.14, $\mathbf{c}$ is a coordinate-wise monotone increasing function of c_0 and the claim immediately follows. □

5 Utility indifference pricing

We start with the standard definition of the indifference price (see, e.g., [10]).

Let

$$U^{\max}(w_0\,,\,\mathbf{w}) \;=\; E\left[\sum_{t=0}^{T} e^{-\rho\, t}\,\frac{c_t^{1-\gamma}}{1-\gamma}\right] \tag{5.1}$$

where $(c_0\,,\,\mathbf{c}) \;=\; (c_0(w_0\,,\,\mathbf{w})\,,\,\mathbf{c}(c_0\,,\,\mathbf{w}))$ is the optimal consumption stream. $U^{\max}$ is strictly monotone increasing w_t for each t (see, e.g., Lemma 2.6 in [17]).

Consider now an company (private bank) with an initial capital (endowment) $w_0 \;=\; W$ and no random endowment $\mathbf{w} \;=\; 0$, that invests this capital into financial assets and trades in the market to achieve the optimal consumption (dividend) stream $(c_0(W,0)\,,\,\mathbf{c}(c_0\,,\,0))$. Suppose now that this company decides to sell insurance against a $\mathcal{G}-$adapted stream $\mathbf{Y} \;=\; (Y_t\,,\,t \;=\; 1\,,\,\ldots\,,\,T)$ of claims for an initial, nonrandom price $\pi_0 \;=\; \pi_0(\mathbf{Y})$ at time zero. Then, the endowment stream of the company becomes

$$(w_0\,,\,\mathbf{w}) \;=\; (W \,+\, \pi_0, -\,\mathbf{Y})$$

and the company will trade in the market to achieve the maximal utility $U^{\max}(W \,+\, \pi_0, -\,\mathbf{Y})$.

Definition 5.1 *The utility indifference price at time zero of the stream* $\mathbf{Y}$ *at the capital level* W *is the unique, deterministic solution* $\pi_0 \;=\; \pi_0(W\,,\,\mathbf{Y})$ *to the equation*

$$U^{\max}(W\,,\,0) \;=\; U^{\max}(W \,+\, \pi_0(W\,,\,\mathbf{Y})\,,\,-\,\mathbf{Y}), \tag{5.2}$$

provided it exists.

Interestingly enough, the indifference price does not always exist when $\gamma \,<\, 1$. In [17], we prove the following

Proposition 5.2 *There exists a continuous function* $l \;=\; l(\mathbf{Y}\,,\,\gamma)$ *such that* (5.2) *has a solution* π_0 *if and only if* $W \;>\; l$. *The lower threshold* $l(\mathbf{Y}\,,\,\gamma) \;=\; 0$ *is equal to zero if and only if either* $\gamma \;\geq\; 1$ *or* $\mathbf{Y}$ *can be replicated by trading.*

5.1 The optimal consumption stream without a random endowment

To calculate the solution to (5.2), we need to know the exact value of the left hand side. It is well known that, for a diffusion driven incomplete market (see, the discussion after Example 2.9) without a random endowment, the optimal consumption stream for a logarithmic utility ($\gamma \;=\; 1$) can be calculated explicitly. For $\gamma \,\neq 1$, no explicit expression is known.

The special structure of incomplete markets in the class $\mathfrak{C}$ allows us to explicitly solve the utility maximization problem for a CRRA utility without a random endowment.

PROPOSITION 5.3 *Let* $(c_0\,,\,\mathbf{c})$ *be the optimal consumption stream for the endowment* $(W\,,\,0)$*, maximizing* (5.1) *and* X_t *be the corresponding wealth process, i.e.,*

$$c_t \;=\; X_t \;-\; E\left[\frac{M_{t+1}}{M_t}\,X_{t+1}\,\middle|\,\mathcal{G}_t\right]. \tag{5.3}$$

Let $Z_T \;=\; 1$*. Define the process* $\mathbf{Z} \;=\; (Z_t\,,\,t\;=\;T-1\,,\,\dots\,,\,0)$ *inductively by*

$$Z_t \;=\; 1 \;+\; e^{-\rho/\gamma^{-1}}\,E\left[\left(\frac{M_{t+1}}{M_t}\right)^{1-1/\gamma}\,\left(E\,[Z_{t+1}^{\gamma}\,|\,\mathcal{H}_{t+1}\,]\right)^{1/\gamma}\,\middle|\,\mathcal{G}_t\right]. \tag{5.4}$$

Then,

$$X_t \;=\; X_{t-1}\,e^{-\rho/\gamma}\,\left(\frac{M_t}{M_{t-1}}\right)^{-1/\gamma}\,\left(E\,[Z_t^{\gamma}\,|\,\mathcal{H}_t\,]\right)^{1/\gamma}\,Z_{t-1}^{-1} \tag{5.5}$$

(with $X_0 \;=\; W$ *) and*

$$c_t \;=\; c_{t-1}\,e^{-\rho/\gamma}\,\left(\frac{M_t}{M_{t-1}}\right)^{-1/\gamma}\,\left(\frac{E\,[Z_t^{\gamma}\,|\,\mathcal{H}_t\,]}{Z_t^{\gamma}}\right)^{1/\gamma} \tag{5.6}$$

for all $t\;\ge\;1$ *and*

$$c_0 \;=\; W\,Z_0^{-1}. \tag{5.7}$$

In particular, for $\gamma\;=\;1$*,*

$$c_t \;=\; e^{-\rho\,t}\,M_t^{-1}\,c_0 \tag{5.8}$$

and

$$c_0 \;=\; \frac{W}{\sum_{t=0}^{T}\,e^{-\rho\,t}}. \tag{5.9}$$

Proof. Since the aggregate state price density process $\mathbf{M}$ is $\mathcal{H}$-adapted, (5.5) implies that (X_t) is also $\mathcal{H}$-adapted. Identities (5.5) and (5.6) imply

$$\frac{c_t}{X_t} \;=\; \frac{c_{t-1}}{X_{t-1}}\,\frac{Z_{t-1}}{Z_t}$$

and, consequently,

$$c_t \;=\; X_t\,Z_t^{-1}$$

Consequently, using (5.4), we get

$$X_{t-1} \; - \; E\left[\frac{M_t}{M_{t-1}} X_t \,\Big|\, \mathcal{G}_{t-1} \right]$$

$$= \; X_{t-1} \left(1 \; - \; E\left[\left(\frac{M_t}{M_{t-1}} \right)^{1-1/\gamma} e^{-\rho/\gamma} \left(E\left[Z_t^{\gamma} \,|\, \mathcal{H}_t \right] \right)^{1/\gamma} Z_{t-1}^{-1} \,\Big|\, \mathcal{G}_{t-1} \right] \right)$$
$$= \frac{X_{t-1}}{Z_{t-1}} \left(Z_{t-1} \; - \; E\left[\left(\frac{M_t}{M_{t-1}} \right)^{1-1/\gamma} e^{-\rho/\gamma} \left(E\left[Z_t^{\gamma} \,|\, \mathcal{H}_t \right] \right)^{1/\gamma} \,\Big|\, \mathcal{G}_{t-1} \right] \right)$$
$$= \; \frac{X_{t-1}}{Z_{t-1}} \; = \; c_{t-1},$$

and thus, (c_t) is indeed the consumption stream, corresponding to the wealth process (X_t). It follows directly from (5.6) that (c_t) satisfies the first order conditions and the claim follows. □

‘

Corollary 5.4 *If* $\gamma \neq 1$ *then*

$$U^{\max}(W\,,\,0) \; = \; (1 \, - \, \gamma)^{-1} \, W^{1-\gamma} \, Z_0^{\gamma}. \tag{5.10}$$

If $\gamma \; = \; 1$, *then*

$$U^{\max}(W\,,\,0) \; = \; \sum_{t=0}^{T} e^{-\rho\, t} \log W \; + \; \sum_{t=0}^{T} e^{-\rho\, t} \log \left(\frac{e^{-\rho\, t} M_t}{\sum_{s=0}^{T} e^{-\rho\, s}} \right). \tag{5.11}$$

Proof. By (4.5),

$$(c_0 \; - \; W)\, c_0^{-\gamma} \; + \; \langle \, \mathbf{c}\,,\, \mathfrak{d}\, \mathbf{c}^{-\gamma} \rangle \; = \; 0. \tag{5.12}$$

Substituting (5.7), we get

$$(1 \, - \, \gamma)\, U^{\max}(W\,,\,0) \; = \; W\, c_0^{-\gamma} \; = \; W^{1-\gamma}\, Z_0^{\gamma}. \tag{5.13}$$

The case $\gamma \; = \; 1$ is proved by direct calculation. □

5.2 The derivatives of the indifference price with respect to claims

The representation

$$(c_0, \mathbf{c}) \; = \; (c_0(w_0, \mathbf{w})\,,\, \mathbf{c}(c_0\,,\, \mathbf{w})) \tag{5.14}$$

of the optimal consumption stream, provided by Theorem 3.7 allows us to incorporate all the dependence of consumption on the indifference price π_0 into c_0. In particular, the defining equation (5.2) can be rewritten as

$$c_0(\pi_0)^{1-\gamma} \; + \; \langle \, \mathfrak{d}\,,\, \mathbf{c}^{1-\gamma}(c_0(\pi_0)\,,\, -\mathbf{Y}) \, \rangle \; = \; U^{\max}(W\,,\,0) \tag{5.15}$$

for $\gamma \neq 1$. Here, $c_0(\pi_0) \; = \; c_0(W \, + \, \pi_0\,,\, -\mathbf{Y})$.

Theorem 5.5 *Let $W > l(\mathbf{Y}, \gamma)$ and $\pi_0 = \pi_0(W, \mathbf{Y})$ the indifference price, solving (5.2) and $(c_0, \mathbf{c})$ be the optimal consumption stream for the endowment $(W + \pi_0(W, \mathbf{Y}), -\mathbf{Y})$. Then,*

$$\frac{\partial \pi_0}{\partial \mathbf{Y}} = \mathfrak{d}\,(\mathbf{c}/c_0)^{-\gamma}, \tag{5.16}$$

and

$$\frac{\partial^2 \pi_0}{\partial \mathbf{Y}^2}(\mathbf{y},\mathbf{y}) = \gamma\, c_0^{\gamma}\, \langle \mathbf{P_c}(\mathbf{y}), \mathbf{y}\rangle_c + \gamma\, c_0^{\gamma} \frac{\langle \mathbf{P_c}(\mathbf{c}), \mathbf{y}\rangle_c^2}{c_0^{1-\gamma} + \|(I - \mathbf{P_c})\,\mathbf{c}\|_c^2}. \tag{5.17}$$

Proof. Let $k(\mathbf{Y}) = c_0(W + \pi_0(W, \mathbf{Y}), \mathbf{Y})$. By abuse of notation, will use both c_0 and $k(\mathbf{Y})$ to denote $c_0(W + \pi_0(W, \mathbf{Y}), \mathbf{Y})$. Rewriting (5.15) in the form

$$k(\mathbf{Y})^{1-\gamma} + E\left[\sum_{t=1}^{T} e^{-\rho t}\, c_t(k(\mathbf{Y}), -\mathbf{Y})^{1-\gamma}\right] \stackrel{!}{=} U^{\max}(W, 0) \tag{5.18}$$

and differentiating (5.18) with respect to $\mathbf{Y}$, we get

$$\frac{\partial k}{\partial \mathbf{Y}}(\cdot)\left(k^{-\gamma} + \left\langle \mathfrak{d}\,\frac{\partial \mathbf{c}}{\partial c_0}, \mathbf{c}^{-\gamma}\right\rangle\right) = \left\langle \mathfrak{d}\,\mathbf{P_c}(\cdot), \mathbf{c}^{-\gamma}\right\rangle = \langle \mathbf{P_c}(\cdot), \mathbf{c}\rangle_c. \tag{5.19}$$

Note, that (5.19) also holds for $\gamma = 1$, even though (5.18) does not. Thus, all the subsequent formulae also hold for $\gamma = 1$.

By Theorem 3.9 and Lemma 4.3,

$$\left\langle \mathfrak{d}\,\mathbf{P_c}(\mathbf{y}), \mathbf{c}^{-\gamma}\right\rangle = \langle \mathbf{y}, \mathbf{P_c}(\mathbf{c})\rangle_c = \langle \mathbf{y}, \mathbf{P_c}(\mathbf{w})\rangle_c \tag{5.20}$$

for any $\mathbf{y} \in H$. By (4.23) and Theorem 3.9,

$$c_0 \left\langle \mathfrak{d}\,\frac{\partial \mathbf{c}}{\partial c_0}, \mathbf{c}^{-\gamma}\right\rangle = \langle (I - \mathbf{P_c})(\mathbf{c}), \mathbf{c}\rangle_c = \|(I - \mathbf{P_c})\,\mathbf{c}\|_c^2. \tag{5.21}$$

Consequently,

$$\frac{\partial k}{\partial \mathbf{Y}}(\mathbf{y})\left(k^{-\gamma} + k^{-1}\,\|(I - \mathbf{P_c})\,\mathbf{c}\|_c^2\right) = \langle \mathbf{P_c}(\mathbf{c}), \mathbf{y}\rangle_c. \tag{5.22}$$

Substituting

$$\frac{\partial k}{\partial \mathbf{Y}} = \frac{\partial c_0}{\partial \mathbf{Y}} + \frac{\partial c_0}{\partial w_0}\frac{\partial \pi_0}{\partial \mathbf{Y}} \tag{5.23}$$

into (5.22) and making use of Proposition 4.4 we get

$$\begin{aligned}
\frac{\partial \pi_0}{\partial \mathbf{Y}}(\mathbf{y}) &= \left(\frac{\partial c_0}{\partial w_0}\right)^{-1}\left(\frac{\partial k}{\partial \mathbf{Y}}(\mathbf{y}) - \frac{\partial c_0}{\partial \mathbf{Y}}(\mathbf{y})\right) \\
&= \frac{c_0\,\langle \mathbf{P_c}(\mathbf{c}), \mathbf{y}\rangle_c + c_0\langle \mathbf{y}, (I - \mathbf{P_c})(\mathbf{c})\rangle_c}{c_0^{1-\gamma}} &(5.24) \\
&= \frac{\langle \mathbf{y}, \mathbf{c}\rangle_c}{c_0^{-\gamma}} = \left\langle \mathfrak{d}\,(\mathbf{c}/c_0)^{-\gamma}, \mathbf{y}\right\rangle. &(5.25)
\end{aligned}$$

This proves (5.16).

Differentiating the identity

$$\frac{\partial \pi_0}{\partial \mathbf{Y}}(\mathbf{y}) \;=\; c_0^{\gamma}\, \langle \mathfrak{d}\mathbf{c}^{-\gamma}\,,\, \mathbf{y}\rangle \tag{5.26}$$

with respect to $\mathbf{Y}$ we get, using (4.23), that

$$\begin{aligned}\frac{\partial^2 \pi_0}{\partial \mathbf{Y}^2}(\mathbf{y}\,,\,\mathbf{y}) \;&=\; \gamma\, c_0^{\gamma-1}\left(\frac{\partial c_0}{\partial \mathbf{Y}}(\mathbf{y}) \;+\; \frac{\partial c_0}{\partial \pi_0}\,\frac{\partial \pi_0}{\partial \mathbf{Y}}(\mathbf{y})\right)\langle \mathbf{y}\,,\,\mathbf{c}\rangle_c \\ &-\;\gamma\, c_0^{\gamma}\left\langle \mathfrak{d}\mathbf{c}^{-\gamma-1}\left(-\,\mathbf{P}_{\mathbf{c}}(\mathbf{y}) \;+\; \frac{\partial \mathbf{c}}{\partial c_0}\left(\frac{\partial c_0}{\partial \mathbf{Y}}(\mathbf{y}) \;+\; \frac{\partial c_0}{\partial \pi_0}\,\frac{\partial \pi_0}{\partial \mathbf{Y}}(\mathbf{y})\right)\right)\,,\,\mathbf{y}\right\rangle \\ &=\;\gamma\, c_0^{\gamma}\,\langle\, \mathbf{P}_{\mathbf{c}}(\mathbf{y})\,,\,\mathbf{y}\,\rangle_c \;+\; \gamma\, c_0^{\gamma-1}\left(\frac{\partial c_0}{\partial \pi_0}\,\frac{\partial \pi_0}{\partial \mathbf{Y}}(\mathbf{y}) \;+\; \frac{\partial c_0}{\partial \mathbf{Y}}(\mathbf{y})\right)\langle \mathbf{y}\,,\,\mathbf{c}\,-\,\mathbf{b}\rangle_c\,.\end{aligned} \tag{5.27}$$

Making use of Proposition 4.4 and identity (5.25) we arrive at

$$\begin{aligned}\frac{\partial^2 \pi_0}{\partial \mathbf{Y}^2}(\mathbf{y}\,,\,\mathbf{y}) \;&=\; \gamma\, c_0^{\gamma}\,\langle\, \mathbf{P}_{\mathbf{c}}(\mathbf{y})\,,\,\mathbf{y}\,\rangle_c \\ &\quad+\;\gamma\, c_0^{\gamma-1}\left(\frac{c_0\,\langle \mathbf{y}\,,\,\mathbf{c}\,-\,\mathbf{b}\rangle_c}{c_0^{1-\gamma}\;+\;\|(I\,-\,\mathbf{P}_{\mathbf{c}})\mathbf{c}\|^2}\right)\langle \mathbf{y}\,,\,\mathbf{c}\,-\,\mathbf{b}\rangle_c \\ &=\;\gamma\, c_0^{\gamma}\,\langle\, \mathbf{P}_{\mathbf{c}}(\mathbf{y})\,,\,\mathbf{y}\,\rangle_c \;+\; \gamma\, c_0^{\gamma}\,\frac{\langle \mathbf{y}\,,\,\mathbf{b}\,-\,\mathbf{c}\rangle_c^2}{c_0^{1-\gamma}\;+\;\|(I\,-\,\mathbf{P}_{\mathbf{c}})\mathbf{c}\|_c^2}.\end{aligned} \tag{5.28}$$

□

5.3 Homogeneity of the indifference price and the derivatives with respect to capital

Here we calculate the derivatives of the indifference price with respect to the capital W. The required formulae follow from the homogeneity of the premium.

PROPOSITION 5.6 *The premium $\pi_0(W\,,\,\mathbf{Y})$ is homogeneous of degree one. That is,*

$$\pi_0(\lambda\, W\,,\,\lambda\,\mathbf{Y}) \;=\; \lambda\,\pi_0(W\,,\,\mathbf{Y}). \tag{5.29}$$

Consequently,

$$\frac{\partial \pi_0}{\partial W} \;=\; W^{-1}\,\pi_0 \;-\; W^{-1}\,\frac{\partial \pi_0}{\partial \mathbf{Y}}(\mathbf{Y}) \;\le\; 0 \tag{5.30}$$

and

$$\frac{\partial^2 \pi_0}{\partial W^2} \;=\; W^{-2}\,\frac{\partial^2\,\pi_0}{\partial \mathbf{Y}^2}(\mathbf{Y}\,,\,\mathbf{Y}) \tag{5.31}$$

and

$$\frac{\partial^2 \pi_0}{\partial W\,\partial \mathbf{Y}}(\mathbf{y}) \;=\; -\,W^{-1}\,\frac{\partial^2 \pi_0}{\partial \mathbf{Y}^2}(\mathbf{y}\,,\,\mathbf{Y}). \tag{5.32}$$

Proof. By Lemma 3.12, $\mathbf{c}(c_0\,,\,-\mathbf{Y})$ is homogeneous of degree one and homogeneity of π_0 follows directly from the definition. Identities (5.30)-(5.32) follow from the homogeneity of the indifference price in complete analogy with Lemma 3.12. In particular, (5.31) implies that π_0 is convex in W. By homogeneity,

$$\lim_{W\to\infty} \frac{\pi_0(W\,,\,\mathbf{Y})}{W} \;=\; \lim_{W\to\infty} \pi(1\,,\,\mathbf{Y}/W) \;=\; 0.$$

If $\frac{\partial \pi_0}{\partial W}(W_0\,,\,\mathbf{Y}) \,=\, a \,>\, 0$ for some $W_0 \,>\, 0$ then, by convexity

$$\pi(W\,,\,\mathbf{Y}) \;\geq\; a\,(W - W_0) \;+\; \pi(W_0\,,\,\mathbf{Y})$$

and, consequently,

$$\lim_{W\to\infty} \frac{\pi_0(W\,,\,\mathbf{Y})}{W} \;\geq\; a \;>\; 0$$

which is a contradiction. The Hessian of π_0 is

$$D^2\,\pi_0 \;=\; \begin{pmatrix} \frac{\partial^2 \pi_0}{\partial W^2} & \frac{\partial^2 \pi_0}{\partial W \partial \mathbf{Y}} \\ \left(\frac{\partial^2 \pi_0}{\partial W \partial \mathbf{Y}}\right)^T & \frac{\partial^2 \pi_0}{\partial \mathbf{Y}^2} \end{pmatrix}$$

Therefore,

$$\left\langle D^2\,\pi_0 \begin{pmatrix} v \\ y \end{pmatrix}, \begin{pmatrix} v \\ y \end{pmatrix} \right\rangle \;=\; \langle\, H\,(y \;-\; v\,\mathbf{Y}),\, (y \;-\; v\,\mathbf{Y})\,\rangle \;\geq\; 0$$

with $H \;=\; \frac{\partial^2 \pi_0}{\partial \mathbf{Y}^2}$. The proof is complete. □

Corollary 5.7 *The premium* π_0 *is jointly convex in capital and claims and is monotone decreasing in capital.*

6 The behavior of the premium for large claims to capital ratio

In this section we study the behavior of the premium when the claims size is large relative to the capital of the insurance company.

Note that, by Proposition 5.2 , this analysis only makes sense when $\gamma \geq 1$. When $\gamma \,<\, 1$ and the wealth W is below the threshold $l(\mathbf{Y})$, utility indifference premium simply does not exist. Therefore, everywhere in this section we assume that $\gamma \geq 1$.

If we multiply the claims $\mathbf{Y}$ by a parameter $\lambda \,>\, 0$ measuring the size of the claims and see what happens when $\lambda \,\to\, \infty$. Homogeneity of the premium implies that we can study its behavior as the capital changes instead of analyzing the behavior when the claims size changes (see, Proposition 5.6):

$$\pi_0(W\,,\,\lambda\,\mathbf{Y}) \;=\; \lambda\,\pi_0(\lambda^{-1}\,W\,,\,\mathbf{Y})\,. \tag{6.1}$$

Thus, we want to study the behavior of π_0 as W goes to zero.

For the sake of brevity, we denote $c_t(W + \pi_0(W, \mathbf{Y}), -\mathbf{Y})$ by $c_t(W)$. Similarly, we denote by $X_t(W)$ the corresponding wealth process. By Proposition 2.10,

$$c_t(W) = X_t(W) - Y_t - E\left[\frac{M_{t+1}}{M_t} X_{t+1}(W) \,\Big|\, \mathcal{G}_t\right] \tag{6.2}$$

for $t \geq 1$ and

$$c_0(W) = W + \pi_0(W, \mathbf{Y}) - E[M_1 X_1(W)].$$

Recall Proposition 2.16 and let $(\mathbf{Y}_t^u)$ be the minimal, upper hedging process for the claims payment stream $\mathbf{Y}$. We start with a lemma that gives the indifference premium for initial capital $W \to 0$.

Lemma 6.1 *We have*

$$\lim_{W\to 0} \pi_0(W, \mathbf{Y}) = \pi_0(0, \mathbf{Y}) = \mathbf{Y}_0^u \tag{6.3}$$

and

$$\lim_{W\to 0} X_t(W) = X_t(0) = \mathbf{Y}_t^u \tag{6.4}$$

for all $t = 1, \ldots, T$. *Consequently,*

$$\lim_{W\to 0} c_t(W) = c_t(0) = \mathbf{Y}_t^u - Y_t - E[M_{t+1}\, \mathbf{Y}_{t+1}^u\, M_t^{-1} \,|\, \mathcal{G}_t] \tag{6.5}$$

for all $t = 0, \ldots, T$. *In particular,*

$$\lim_{W\to 0} c_0(W) = c_0(0) = \mathbf{Y}_0^u - E[M_1\, \mathbf{Y}_1^u] = 0. \tag{6.6}$$

Proof. Recall that the endowment stream of the company after selling the insurance against $\mathbf{Y}$ is given by

$$w_0 = W + \pi_0, \quad \mathbf{w} = -\mathbf{Y}.$$

By Lemma 2.17, the budget set is non-empty if and only if

$$\pi_0 + W \geq \mathbf{Y}_0^u. \tag{6.7}$$

Furthermore, $\pi_0 \leq \mathbf{Y}_0^u$, because, otherwise (5.2) does not hold. Thus, (6.3) follows. By Proposition 2.16,

$$X_t(0) \geq \mathbf{Y}_t^u \tag{6.8}$$

and, since $\pi_0(0, \mathbf{Y}) = \mathbf{Y}_0^u$ is sufficient to finance (X_t), Proposition 2.16 implies that $X_t(0) = \mathbf{Y}_t^u$ for all $t = 1, \ldots, T$. □

We will need the following auxiliary

LEMMA 6.2 *Let* A, B, $M > 0$ *and* Z *be random variables. The unique random variable* X, *solving*

$$A\,X \;+\; B\,E\,[M\,X\,|\,\mathcal{G}_t] \;=\; Z \tag{6.9}$$

is given by

$$X \;=\; A^{-1}\left(Z \;-\; B\,\frac{E\,[M\,A^{-1}\,Z\,|\,\mathcal{G}_t]}{1 \;+\; E\,[M\,A^{-1}\,B\,|\,\mathcal{G}_t]}\right). \tag{6.10}$$

In particular,

$$X \;=\; \frac{A^{-1}\,Z}{1 \;+\; E\,[M\,A^{-1}\,B\,|\,\mathcal{G}_t]} \tag{6.11}$$

if both Z *and* B *are* $\mathcal{G}_t$ *-measurable.*

Proof. Multiplying both sides of (6.9) by $M\,A^{-1}$ and taking the conditional expectation $P_{\mathcal{G}_t}$, we obtain the expression for $E[M\,X\,|\,\mathcal{G}_t]$. Plugging this into (6.9) gives the required solution. □

DEFINITION 6.3 *Let* $s_t \in \mathcal{G}_t$ *be the event* $c_t(0) = 0$. *Let also* χ_{s_t} *be the indicator of the event* s_t.

Note that, by (6.6), $s_0 = \Omega$.

First order conditions (2.29) imply that $e^{-\rho}\,E[c_{t+1}^{-\gamma}\,|\,\mathcal{H}_{t+1}]$ and $c_t^{-\gamma}$ are finite or infinite simultaneously. Consequently, the following is true:

LEMMA 6.4 $s_{t+1} \subset s_t$ *for all* $t = 0, \ldots, T-1$.

Note again that we work on a finite probability space.

In Lemma 6.1, we have calculated the limit of the indifference prices as the capital W goes to zero. The next step is to calculate the expansion of the indifference price around $W = 0$.

Let $S_T = 0$. We define the following random variables S_t inductively:

- if $c_t(0) \neq 0$, let

$$K_t \;=\; E\left[c_{t+1}(0)^{-\gamma-1}\,\left(1 \;-\; E\left[\frac{M_{t+2}}{M_{t+1}}\,S_{t+1}\,\Big|\,\mathcal{G}_{t+1}\right]\right)\,\Big|\,\mathcal{H}_{t+1}\right] \tag{6.12}$$

and

$$S_t \;=\; \frac{e^{\rho}\,K_t^{-1}\,\frac{M_{t+1}}{M_t}\,c_t(0)^{-\gamma-1}}{1 \;+\; E\left[K_t^{-1}\,e^{\rho}\,\left(\frac{M_{t+1}}{M_t}\right)^2\,c_t(0)^{-\gamma-1}\,\Big|\,\mathcal{G}_t\right]} \tag{6.13}$$

where

- if $c_t(0) = 0$, let

$$K_t = E\left[\left(1 - E_{t+1}\left[\frac{M_{t+2}}{M_{t+1}} S_{t+1} \,\middle|\, \mathcal{G}_{t+1}\right]\right)^{-\gamma} \chi_{s_{t+1}} \,\middle|\, \mathcal{H}_{t+1}\right] \tag{6.14}$$

and, for $t \geq 1$, let

$$S_t = \frac{e^{-\rho\gamma^{-1}} K_t^{\gamma^{-1}} \left(\frac{M_{t+1}}{M_t}\right)^{-\gamma^{-1}}}{1 + E\left[e^{-\rho\gamma^{-1}} K_t^{\gamma^{-1}} \left(\frac{M_{t+1}}{M_t}\right)^{1-\gamma^{-1}} \,\middle|\, \mathcal{G}_t\right]} \tag{6.15}$$

- for $t = 0$,

$$S_0 = e^{-\rho\gamma^{-1}} K_0^{\gamma^{-1}} M_1^{-\gamma^{-1}} \tag{6.16}$$

Let further

$$S_t^{(\Pi)} = \prod_{\tau=0}^{t-1} S_\tau \tag{6.17}$$

and

$$C_t^{(\Pi)} = S_t^{(\Pi)} - E\left[\frac{M_{t+1}}{M_t} S_{t+1}^{(\Pi)} \,\middle|\, \mathcal{G}_t\right]. \tag{6.18}$$

Define for $\gamma > 1$

$$c_0^{(1)} = \left(\frac{Z_0^\gamma}{\sum_{t=0}^T e^{-\rho t} E\left[\left(C_t^{(\Pi)}\right)^{1-\gamma} \chi_{s_t}\right]}\right)^{1/(1-\gamma)}. \tag{6.19}$$

Let now, for each $t = 1, \ldots, T$,

$$X_t^{(1)} = S_t^{\Pi} c_0^{(1)}.$$

Note that, by definition, $X_t^{(1)} = S_{t-1} X_{t-1}^{(1)}$.

Lemma 6.5 *Let $\gamma > 1$. Then,*

$$X_t(W) = \mathbf{Y}_t^u + X_t^{(1)} W + o(W) \tag{6.20}$$

and, consequently,

$$c_t(W) = c_t(0) + c_t^{(1)} W + o(W) \tag{6.21}$$

with

$$c_t^{(1)} = X_t^{(1)} - E[M_{t+1} X_{t+1}^{(1)} M_t^{-1} \,|\, \mathcal{G}_t] \tag{6.22}$$

for all $t = 1, \ldots, T$, and $c_0^{(1)}$ is given by (6.19).

Proof. The proof is done by direct calculation. We substitute the Ansatz (6.20) $X_t(W) = X_t(0) + X_t^{(1)} W + o(W)$ into the first order conditions (2.29) and calculate the coefficients $X_t^{(1)}$ by induction backwards in time. By Lemma 6.1, $X_t(0) = \mathbf{Y}_t^u$. Therefore, it suffices to prove that $X_{t+1}^{(1)} = S_t X_t^{(1)}$ for all $t \geq 1$ and $X_1^{(1)} = S_0 c_0^{(1)}$.

The required assertion is obviously true for $t = T+1$. Suppose that the claim is valid for all time periods larger than or equal to $t+1$, and let us show the relation $X_{t+1}^{(1)} = S_t X_t^{(1)}$ it for $t \geq 1$. Let first $c_t(0) \neq 0$. The first order condition (2.29) reads

$$e^{-\rho} E\left[\left(X_{t+1} - Y_{t+1} - E\left[X_{t+2} \frac{M_{t+2}}{M_{t+1}} \,\Big|\, \mathcal{G}_{t+1}\right]\right)^{-\gamma} \Big|\, \mathcal{H}_{t+1}\right] = \frac{M_{t+1}}{M_t} \left(X_t - Y_t - E\left[X_{t+1} \frac{M_{t+1}}{M_t} \,\Big|\, \mathcal{G}_t\right]\right)^{-\gamma}. \quad (6.23)$$

Substituting the expansions $X_{t+2} = X_{t+2}(0) + S_{t+1} X_{t+1}^{(1)} W + o(W)$ (using the induction hypothesis) and $X_{t+1} = X_{t+1}(0) + X_{t+1}^{(1)} W + o(W)$ and using the Taylor approximation

$$(x + yW + o(W))^{-\gamma} = x^{-\gamma} - \gamma x^{-\gamma-1} y W + o(W), \quad (6.24)$$

we get

$$\begin{aligned} & e^{-\rho} E[c_{t+1}(0)^{-\gamma} \,|\, \mathcal{H}_{t+1}] \\ & - \gamma e^{-\rho} E\left[c_{t+1}(0)^{-\gamma-1} \left(1 - E\left[\frac{M_{t+2}}{M_{t+1}} S_{t+1} \,\Big|\, \mathcal{G}_{t+1}\right]\right) \Big|\, \mathcal{H}_{t+1}\right] X_{t+1}^{(1)} W \\ & + o(W) \\ = {} & \frac{M_{t+1}}{M_t} \left(c_t(0)^{-\gamma} - \gamma c_t(0)^{-\gamma-1} \left(X_t^{(1)} - E\left[\frac{M_{t+1}}{M_t} X_{t+1}^{(1)} \,\Big|\, \mathcal{G}_t\right]\right) W\right). \end{aligned} \quad (6.25)$$

Note that, as everywhere in this paper, the probability space Ω is assumed to be finite and therefore no special care should be taken of the $o(W)$ term. Comparing the terms of order W, we get the equation

$$e^{-\rho} \frac{M_t}{M_{t+1}} E\left[c_{t+1}(0)^{-\gamma-1} \left(1 - E\left[\frac{M_{t+2}}{M_{t+1}} S_{t+1} \,\Big|\, \mathcal{G}_{t+1}\right]\right) \Big|\, \mathcal{H}_{t+1}\right] X_{t+1}^{(1)} + c_t(0)^{-\gamma-1} E\left[\frac{M_{t+1}}{M_t} X_{t+1}^{(1)} \,\Big|\, \mathcal{G}_t\right] = c_t(0)^{-\gamma-1} X_t^{(1)}. \quad (6.26)$$

Applying Lemma 6.2, we get that $X_{t+1}^{(1)} = S_t X_t^{(1)}$ for all $t \geq 1$ with S_t given by (6.13).

Let now $c_t(0) = 0$. Then, following the same steps as above and substituting the expansions into (6.23), using

$$(yW + o(W))^{-\gamma} = y^{-\gamma} W^{-\gamma} (1 + o(1)), \quad (6.27)$$

and multiplying the identity by W^γ, we get

$$e^{-\rho}\, K_t \left(X_{t+1}^{(1)}\right)^{-\gamma} \;=\; \frac{M_{t+1}}{M_t}\; \left(X_t^{(1)} \;-\; E\left[X_{t+1}^{(1)}\, \frac{M_{t+1}}{M_t}\,\Big|\, \mathcal{G}_t\right]\right)^{-\gamma} \;+\; O(W^\gamma) \tag{6.28}$$

with K_t given by (6.14) (Here, the indicator $\chi_{s_{t+1}}$ appears because on the complement of s_{t+1}, consumption stays bounded away from zero and, after multiplication by W^γ contributes to $O(W^\gamma)$). Consequently,

$$(e^{-\rho}\, K_t)^{-\gamma^{-1}}\, X_{t+1}^{(1)} \;+\; \left(\frac{M_{t+1}}{M_t}\right)^{-\gamma^{-1}} E\left[X_{t+1}^{(1)}\, \frac{M_{t+1}}{M_t}\,\Big|\, \mathcal{G}_t\right] \;=\; \left(\frac{M_{t+1}}{M_t}\right)^{-\gamma^{-1}} X_t^{(1)} \tag{6.29}$$

and Lemma 6.2 implies that $X_{t+1}^{(1)} \;=\; S_t\, X_t^{(1)}$ with S_t given by (6.15). It remains to consider the case $t \;=\; 0$. Substituting $X_t^{(1)} \;=\; S_t^{(\Pi)}\, c_0^{(1)}$ into the Ansatz $X_t \;=\; X_t(0) \;+\; X_t^{(1)}\, W \;+\; o(W)$ and then, substituting this Ansatz into the utility indifference equation (5.2), we get for $\gamma \;>\; 1$

$$\begin{aligned}
Z_0^\gamma \;&=\; W^{\gamma\,-1}\left(c_0(W)^{1-\gamma} \;+\; E\left[\sum_{t=1}^T e^{-\rho\, t}\, c_t(W)^{1-\gamma}\right]\right)\\
&=\; (c_0^{(1)} \;+\; o(1))^{1-\gamma} \;+\; \sum_{t=1}^T e^{-\rho\, t}\, W^{\gamma-1}\, E\left[\,(c_t(0) \;+\; o(1)))^{1-\gamma}\,(1 \;-\; \chi_{s_t})\,\right]\\
&\quad+\; \sum_{t=1}^T e^{-\rho\, t}\, E\Big[\left(c_t^{(1)}(0) \;+\; o(1)\right)^{1-\gamma} \chi_{s_t}\Big]\\
&=\; \left(c_0^{(1)}\right)^{1-\gamma}\left(1 \;+\; \sum_{t=1}^T e^{-\rho\, t}\, E\Big[\left(C_t^{(\Pi)}\right)^{1-\gamma} \chi_{s_t}\Big]\right) \;+\; o(1)
\end{aligned} \tag{6.30}$$

and the identity (6.19) follows. The proof is complete. □

Note, that the only place we used that $\gamma \neq 1$ is the identity (6.30). The case $\gamma \;=\; 1$ is treated separately because (6.30) does not hold for logarithmic utility.

Let

$$\begin{aligned}
\log(c_0^{(\alpha)}) \;&=\; \left(\sum_{t=0}^T e^{-\rho\, t}\; \mathrm{Prob}[s_t]\right)^{-1}\left(\sum_{t=1}^T e^{-\rho\, t}\, \log\left(\frac{e^{-\rho\, t}\, M_t}{\sum_{t=0}^T\, e^{-\rho\, t}}\right)\right.\\
&\quad\left.-\; \sum_{t=1}^T e^{-\rho\, t}\left(E\,[\log\, c_t(0)\,(1-\chi_{s_t})\,] \;-\; E\left[\log\, C_t^{(\Pi)}\,\chi_{s_t}\,\right]\right)\right)
\end{aligned} \tag{6.31}$$

and define for each $t \;=\; 1,\, \dots\, ,\, T$,

$$X_t^{(\alpha)} \;=\; S_t^{(\Pi)}\, c_0^{(\alpha)}.$$

Lemma 6.6 *Let* $\gamma = 1$ *and*

$$\alpha = \alpha(\mathbf{Y}) = \frac{\sum_{t=0}^{T} e^{-\rho t}}{\sum_{t=0}^{T} e^{-\rho t}\, \mathrm{Prob}[s_t]} > 1\,. \tag{6.32}$$

Then,

$$X_t = \mathbf{Y}_t^u + W^\alpha X_t^{(\alpha)} + o(W^\alpha). \tag{6.33}$$

Consequently,

$$c_t(W) = c_t(0) + c_t^{(\alpha)} W^\alpha + o(W^\alpha) \tag{6.34}$$

with

$$c_t^{(\alpha)} = X_t^{(\alpha)} - E\left[\frac{M_{t+1}}{M_t} X_{t+1}^{(\alpha)} \,\Big|\, \mathcal{G}_t\right] \tag{6.35}$$

for $t \geq 1$, *and* $c_0^{(\alpha)}$ *is given by* (6.31).

Proof. The recursive relations for $X_t^{(\alpha)}$ follow by literally the same arguments as in the proof of Lemma 6.5. Substituting the Ansatz

$$c_t = c_t(0) + c_t^{(\alpha)} W^\alpha + o(W^\alpha)$$

into the utility indifference equation

$$\log(c_0) + E\left[\sum_{t=1}^{T} e^{-\rho t} \log(c_t)\right] = \sum_{t=0}^{T} e^{-\rho t} \log W + \sum_{t=0}^{T} e^{-\rho t} \log\left(\frac{e^{-\rho t} M_t}{\sum_{t=0}^{T} e^{-\rho t}}\right) \tag{6.36}$$

and using that

$$\begin{aligned}
&\log\left(c_0^{(\alpha)} W^\alpha + o(W^\alpha)\right) \\
&+ \sum_{t=1}^{T} e^{-\rho t} E\left[\log\left(c_t(0) + c_0^{(\alpha)} C_t^{(\Pi)} W^\alpha + o(W^\alpha)\right)\right] \\
&= \left(\log(c_0^{(\alpha)}) + \alpha \log W\right) \sum_{t=0}^{T} e^{-\rho t}\, \mathrm{Prob}[s_t] \\
&+ \sum_{t=1}^{T} e^{-\rho t} \left(E\left[\log c_t(0)\, (1 - \chi_{s_t})\right] + E\left[\log C_t^{(\Pi)} \chi_{s_t}\right]\right) + o(1),
\end{aligned}$$

we get the required formula for $c_0^{(\alpha)}$. □

We are now ready to calculate the "second" order of the asymptotic expansion.

Let $S_T^{(\gamma)} = 0$. We define the random variables $S_t^{(\gamma)}$ inductively.

- if $c_t(0) \neq 0$, let

$$K_{\gamma,t} = E\left[c_{t+1}(0)^{-\gamma-1}\left(1 - E\left[\frac{M_{t+2}}{M_{t+1}} S_{t+1}^{(\gamma)} \,\Big|\, \mathcal{G}_{t+1}\right]\right) \,|\, \mathcal{H}_{t+1}\right] \tag{6.37}$$

, and

$$S_t^{(\gamma)} \;=\; \frac{e^{\rho}\, K_{\gamma,t}^{-1}\, \frac{M_{t+1}}{M_t}\, c_t(0)^{-\gamma-1}}{1 \;+\; E\left[K_{\gamma,t}^{-1}\, e^{\rho} \left(\frac{M_{t+1}}{M_t}\right)^2 c_t(0)^{-\gamma-1} \,\middle|\, \mathcal{G}_t \right]}. \tag{6.38}$$

- If $c_t(0) \;=\; 0$ and $t \geq 1$, let

$$K_{\gamma,t} \;=\; E\left[\left(c_{t+1}^{(1)}\right)^{-\gamma-1} \left(1 \;-\; E_{t+1}\left[\frac{M_{t+2}}{M_{t+1}}\, S_{t+1}^{(\gamma)} \,\middle|\, \mathcal{G}_{t+1} \right]\right) \chi_{s_{t+1}} \,\middle|\, \mathcal{H}_{t+1} \right] \tag{6.39}$$

and

$$S_t^{(\gamma)} \;=\; \frac{e^{\rho}\, K_{\gamma,t}^{-1}\, \frac{M_{t+1}}{M_t} \left(c_t^{(1)}\right)^{-\gamma}}{1 \;+\; E\left[K_{\gamma,t}^{-1}\, e^{\rho} \left(\frac{M_{t+1}}{M_t}\right)^2 \left(c_t^{(1)}\right)^{-\gamma} \,\middle|\, \mathcal{G}_t \right]}. \tag{6.40}$$

- For $t \;=\; 0$

$$S_0^{(\gamma)} \;=\; e^{\rho}\, K_{\gamma,0}^{-1}\, \frac{M_{t+1}}{M_t} \left(c_0^{(1)}\right)^{-\gamma-1}. \tag{6.41}$$

Let now for each $t \;=\; 1, \ldots, T$

$$S_{t,\gamma}^{(\Pi)} \;=\; \prod_{\tau=0}^{t-1} S_\tau^{(\gamma)} \qquad \text{and} \qquad C_{t,\gamma}^{(\Pi)} \;=\; S_{t,\gamma}^{(\Pi)} \;-\; E\left[\frac{M_{t+1}}{M_t}\, S_{t+1,\gamma}^{(\Pi)} \,\middle|\, \mathcal{G}_t \right]. \tag{6.42}$$

Let also for $\gamma > 1$

$$c_0^{(\gamma)} \;=\; \frac{(\gamma-1)^{-1} \sum_{t=1}^{T} e^{-\rho t}\, E\left[(c_t(0))^{1-\gamma} \,(1 \;-\; \chi_{s_t}) \right]}{(c_0^{(1)})^{-\gamma} \;+\; \sum_{t=1}^{T} e^{-\rho t}\, E\left[(c_t^{(1)})^{-\gamma}\, C_{t,\gamma}^{(\Pi)}\, \chi_{s_t} \right]} \tag{6.43}$$

and

$$X_t^{\gamma} \;=\; S_{t,\gamma}^{(\Pi)}\, c_0^{(\gamma)} \tag{6.44}$$

for all $t \;=\; 1, \ldots, T$.

Lemma 6.7 *Let* $(X_t^{(1)})$ *and* $(c_t^{(1)})$ *be the processes of Lemma 6.5. If* $1 < \gamma < 2$, *then*

$$X_t(W) \;=\; \mathbf{Y}_t^u \;+\; X_t^{(1)}\, W \;+\; X_t^{(\gamma)}\, W^{\gamma} \;+\; O(W^2). \tag{6.45}$$

Consequently,

$$c_t \;=\; c_t(0) \;+\; c_t^{(1)}\, W \;+\; c_t^{(\gamma)}\, W^{\gamma} \;+\; O(W^2) \tag{6.46}$$

with

$$c_t^{(\gamma)} \;=\; X_t^{(\gamma)} \;-\; E\left[\frac{M_{t+1}}{M_t}\, X_{t+1}^{(\gamma)} \,\middle|\, \mathcal{G}_t \right]. \tag{6.47}$$

Proof. The proof is completely analogous to that of Lemma 6.5. Recall that

$$W^{\gamma-1}\left(c_0(W)^{1-\gamma} + E\left[\sum_{t=1}^{T} c_t(W)^{1-\gamma}\right]\right) = Z_0^{\gamma}. \tag{6.48}$$

In complete analogy with Lemma 6.5, we can substitute the Ansatz (6.45) into the first order conditions (6.23) and obtain the recursive expressions for the coefficients, using Lemma 6.2 on each step. This generates the relation (6.44), but with an unknown coefficient $c_0^{(\gamma)}$ that has to be determined from the equation (6.48). Substituting the Ansatz (6.45), (6.44) into (6.48), we get

$$\begin{aligned}(c_0^{(1)} + c_0^{(\gamma)} W^{\gamma-1} + O(W))^{1-\gamma} \\ + \sum_{t=1}^{T} e^{-\rho t} W^{\gamma-1} E\Big[(c_t(0) + O(W))^{1-\gamma}(1-\chi_{s_t})\Big] \\ + \sum_{t=1}^{T} e^{-\rho t} E\Big[\Big(c_t^{(1)} + W^{\gamma-1} C_{t,\gamma}^{(\Pi)} c_0^{(\gamma)} + O(W)\Big)^{1-\gamma}\chi_{s_t}\Big] = Z_0^{\gamma}.\end{aligned} \tag{6.49}$$

Using the expansion

$$(a + \varepsilon b)^{1-\gamma} = a^{1-\gamma} + (1-\gamma)\, a^{-\gamma}\, \varepsilon\, b + O(\varepsilon^2) \tag{6.50}$$

and gathering the terms of order $W^{\gamma-1}$ in (6.49), we get

$$\begin{aligned}0 = (c_0^{(1)})^{-\gamma} c_0^{(\gamma)} + (1-\gamma)^{-1}\sum_{t=1}^{T} e^{-\rho t} E\Big[(c_t(0))^{1-\gamma}(1-\chi_{s_t})\Big] \\ + c_0^{(\gamma)}\sum_{t=1}^{T} e^{-\rho t} E\Big[(c_t^{(1)})^{-\gamma} C_{t,\gamma}^{(\Pi)}\chi_{s_t}\Big]\end{aligned} \tag{6.51}$$

and the required identity follows. □

It remains to consider the case $\gamma \geq 2$. We only treat here the case $\gamma > 2$. The case $\gamma = 2$ must be treated separately, because the terms of order W^{γ} will enter the asymptotic expansion. Otherwise, the calculations are almost identical.

Let $S_T^{(2)} = 0$. We define the following random variables $S_t^{(2)}$ inductively:

- if $c_t(0) \neq 0$, let

$$K_{2,t} = E\left[c_{t+1}(0)^{-\gamma-1}\left(1 - E\left[\frac{M_{t+2}}{M_{t+1}} S_{t+1}^{(2)} \,\Big|\, \mathcal{G}_{t+1}\right]\right) \,|\, \mathcal{H}_{t+1}\right] \tag{6.52}$$

  and

$$S_t^{(2)} = \frac{e^{\rho} K_{2,t}^{-1} \frac{M_{t+1}}{M_t} c_t(0)^{-\gamma}}{1 + E\Big[K_{2,t}^{-1} e^{\rho}\left(\frac{M_{t+1}}{M_t}\right)^2 c_t(0)^{-\gamma} \,\Big|\, \mathcal{G}_t\Big]} \tag{6.53}$$

Let also

$$X^{(2)}_{t+1} \;=\; S^{(2)}_t\, X^{(2)}_t \;+\; R^{(2)}_t \tag{6.54}$$

with

$$R^{(2)}_t \;=\; S^{(2)}_t\, \frac{1}{2}\,(\gamma\,+\,1)$$
$$\Big(e^{-\rho}\, c_t(0)^{\gamma+1}\, M_{t-1}\, M_t^{-1}\, E\left[\, c_{t+1}^{-\gamma-2}(0)\, \left(c^{(1)}_{t+1}\right)^2 \,|\, \mathcal{H}_{t+1}\right]$$
$$-\; c_t^{-1}(0)\, \left(c^{(1)}_t\right)^2 \Big)\Big). \tag{6.55}$$

- iI $c_t(0) \;=\; 0$, let

$$K_{2\,,\,t}$$
$$=\; E\left[\left(c^{(1)}_{t+1}\right)^{-\gamma-1}\; \left(1\,-\,E_{t+1}\left[\frac{M_{t+2}}{M_{t+1}}\, S^{(2)}_{t+1} \,\Big|\, \mathcal{G}_{t+1}\right]\right)\; \chi_{s_{t+1}} \;\Big|\; \mathcal{H}_{t+1}\right] \tag{6.56}$$

and

$$S^{(2)}_t \;=\; \frac{e^{\rho}\, K^{-1}_{2\,,\,t}\, \frac{M_{t+1}}{M_t}\, \left(c^{(1)}_t(0)\right)^{-\gamma}}{1\;+\;E\left[\,K^{-1}_{2\,,\,t}\, e^{\rho}\, \left(\frac{M_{t+1}}{M_t}\right)^2\, \left(c^{(1)}_t(0)\right)^{-\gamma} \,\Big|\, \mathcal{G}_t\right]}. \tag{6.57}$$

Then, for $t \,\geq\, 1$,

$$X^{(2)}_{t+1} \;=\; S^{(2)}_t\, X^{(2)}_t \tag{6.58}$$

- for $t \;=\; 0$

$$S^{(2)}_0 \;=\; e^{\rho}\, K^{-1}_{2\,,\,0}\, \frac{M_{t+1}}{M_t}\, (c^{(1)}_0)^{-\gamma-1} \tag{6.59}$$

and

$$X^{(2)}_1 \;=\; S^{(2)}_0\, c^{(\gamma)}_0\,. \tag{6.60}$$

Let now for each $t \;=\; 1\,,\,\ldots\,,\,T\,$,

$$S^{(\Pi)}_{t\,,\,2} \;=\; \prod_{\tau=0}^{t} S^{(2)}_{\tau}$$

and

$$S^{(Q)}_{t\,,\,2} \;=\; S^{(\Pi)}_{t\,,\,2}\, \sum_{\tau=1}^{t} R_{\tau}\, \left(S^{(\Pi)}_{\tau\,,\,2}\right)^{-1}$$

Then,

$$X^{(2)}_t \;=\; S^{(\Pi)}_{t\,,\,2}\, c^{(\gamma)}_0 \;+\; S^{(Q)}_{t\,,\,2}. \tag{6.61}$$

Let also

$$C^{(\Pi)}_t \;=\; S^{(\Pi)}_{t\,,\,2} \;-\; E\left[\,\frac{M_{t+1}}{M_t}\, S^{(\Pi)}_{t+1\,,\,2} \,|\, \mathcal{G}_t\right]\,,$$
$$C^{(Q)}_t \;=\; S^{(Q)}_{t\,,\,2} \;-\; E\left[\,\frac{M_{t+1}}{M_t}\, S^{(Q)}_{t+1\,,\,2} \,|\, \mathcal{G}_t\right]\,.$$

and

$$c_0^{(2)} \;=\; -\,\frac{\sum_{t=1}^{T} e^{-\rho\,t}\, E\left[\left(c_t^{(1)}\right)^{-\gamma} C_t^{(Q)}\, \chi_{s_t}\right]}{(c_0^{(1)})^{-\gamma} \;+\; \sum_{t=1}^{T} E\left[\left(c_t^{(1)}\right)^{-\gamma} C_t^{(\Pi)}\, \chi_{s_t}\right]}\,. \qquad (6.62)$$

Lemma 6.8 *Let* $(X_t^{(1)})$ *and* $(c_t^{(1)})$ *be the processes, constructed in Lemma 6.5. If* $\gamma\,>\,2\,,$ *then*

$$X_t(W) \;=\; Y_t^{\max} \;+\; X_t^{(1)}\,W \;+\; X_t^{(2)}\,W^2 \;+\; O(W^{\min\{3\,,\gamma\}})\,. \qquad (6.63)$$

Consequently,

$$c_t \;=\; c_t(0) \;+\; c_t^{(1)}\,W \;+\; c_t^{(2)}\,W^2 \;+\; O(W^{\min\{3\,,\gamma\}}) \qquad (6.64)$$

with

$$c_t^{(2)} \;=\; X_t^{(2)} \;-\; E\left[\frac{M_{t+1}}{M_t}\,X_{t+1}^{(2)}\,\Big|\,\mathcal{G}_t\right] \qquad (6.65)$$

for $t\,\geq\,1\,.$

Now, with the expansion for the optimal consumption stream on our hands, we can calculate the expansion for the premium.

We will need the following important identity (see, Corollary 5.4).

Lemma 6.9 *We have*

$$\frac{\partial \pi_0(W\,,\,\mathbf{Y})}{\partial W} \;=\; -1 \;+\; \left(\frac{c_0(W)}{W}\right)^{\gamma} Z_0^{\gamma} \qquad (6.66)$$

for $\gamma\,>\,1\,,$ *and*

$$\frac{\partial \pi_0(W\,,\,\mathbf{Y})}{\partial W} \;=\; -1 \;+\; \frac{c_0(W)}{W}\,(1\,+\,\langle\,\mathfrak{d}\,,\,\mathbf{1}\rangle) \qquad (6.67)$$

for $\gamma\;=\;1\,.$

Proof. By (5.16),

$$c_0^{-\gamma}\,\frac{\partial \pi_0(\mathbf{Y})}{\partial \mathbf{Y}}(\mathbf{Y}) \;=\; \langle\,\mathbf{Y}\,,\,\mathfrak{d}\,\mathbf{c}^{-\gamma}\,\rangle\,. \qquad (6.68)$$

By (4.5), (5.2) and (5.10),

$$\begin{aligned} 0 \;&=\; (c_0\;-\;\pi_0\;-\;W)\,c_0^{-\gamma} \;+\; \langle\,\mathbf{c}\;+\;\mathbf{Y}\,,\,\mathfrak{d}\,\mathbf{c}^{-\gamma}\,\rangle \\ &=\; c_0^{1-\gamma} \;+\; \langle\,\mathbf{c}\,,\,\mathfrak{d}\,\mathbf{c}^{-\gamma}\,\rangle \;-\; (\pi_0\,+\,W)\,c_0^{-\gamma} \;+\; \langle\,\mathbf{Y}\,,\,\mathfrak{d}\,\mathbf{c}^{-\gamma}\,\rangle \\ &=\; W^{1-\gamma}\,Z_0^{\gamma} \;-\; (\pi_0\,+\,W)\,c_0^{-\gamma} \;+\; \langle\,\mathbf{Y}\,,\,\mathfrak{d}\,\mathbf{c}^{-\gamma}\,\rangle\,. \end{aligned}$$

Using formula (5.16) for the first derivative of π_0, we get

$$c_0^{-\gamma} \frac{\partial \pi_0(\mathbf{Y})}{\partial \mathbf{Y}}(\mathbf{Y}) = \langle \mathbf{Y}, \mathfrak{d}\, \mathbf{c}^{-\gamma} \rangle = (W + \pi_0)\, c_0^{-\gamma} - W^{1-\gamma} Z_0^{\gamma} \tag{6.69}$$

Identity (6.66) follows now from (5.30). If $\gamma = 1$, identity (4.5) takes the form

$$0 = (c_0 - \pi_0 - W)\, c_0^{-1} + \langle \mathbf{c} + \mathbf{Y}, \mathfrak{d}\, \mathbf{c}^{-1} \rangle$$

That is, by (5.16),

$$c_0^{-1} \frac{\partial \pi_0(\mathbf{Y})}{\partial \mathbf{Y}}(\mathbf{Y}) = \langle \mathbf{Y}, \mathfrak{d}\, \mathbf{c}^{-1} \rangle = (\pi_0 + W)\, c_0^{-1} - (1 + \langle \mathfrak{d}, \mathbf{1} \rangle)$$

and (6.67) follows from (5.30). □

THEOREM 6.10 *We have*

$$\pi - \mathbf{Y}^u = \begin{cases} -W + B_1(\mathbf{Y})\, W^{\alpha(\mathbf{Y})} + o(W^{\alpha(\mathbf{Y})}), & \gamma = 1 \\ (-1 + A(\mathbf{Y}))\, W + B_2(\mathbf{Y})\, W^{\gamma} + O(W^2), & \gamma \in (1,2) \\ (-1 + A(\mathbf{Y}))\, W + B_3(\mathbf{Y})\, W^2 + O(W^3), & \gamma = 2 \\ (-1 + A(\mathbf{Y}))\, W + B_4(\mathbf{Y})\, W^2 + O(W^{\min\{3,\gamma\}}), & \gamma > 2 \end{cases} \tag{6.70}$$

Here, $\alpha(\mathbf{Y})$ *is given by* (6.32),

$$B_1(\mathbf{Y}) = \alpha^{-1} c_0^{(\alpha)} (1 + \langle \mathfrak{d}, \mathbf{1} \rangle) \tag{6.71}$$

and

$$A(\mathbf{Y}) = \left(c_0^{(1)}\right)^{\gamma} Z_0^{\gamma} \tag{6.72}$$

and

$$B_2(\mathbf{Y}) = c_0^{(\gamma)} \left(c_0^{(1)}\right)^{\gamma-1} Z_0^{\gamma} \tag{6.73}$$

and

$$B_4(\mathbf{Y}) = \frac{\gamma}{2} c_0^{(2)} \left(c_0^{(1)}\right)^{\gamma-1} Z_0^{\gamma} \tag{6.74}$$

The coefficient $B_3(\mathbf{Y})$ *also satisfies* (6.74), *but the coefficient* $c_0^{(2)}$ *is a little bit different from the one, calculated in Lemma 6.8.*

Proof. Let first $\gamma = 1$. By Lemma 6.6, $c_0(W) = c_0^{(\alpha)} W^{\alpha} + o(W^{\alpha})$. Integrating (6.67) with respect to W, we get

$$\pi_0(W, \mathbf{Y}) = \mathbf{Y}^u - W + \alpha^{-1} c_0^{(\alpha)} (1 + \langle \mathfrak{d}, \mathbf{1} \rangle)\, W^{\alpha} + o(W^{\alpha}) \tag{6.75}$$

which is what had to be proved.

Let now $\gamma \in (1,2)$. Then,

$$\frac{c_0(W)}{W} = c_0^{(1)} + W^{\gamma-1} c_0^{(1)} + O(W)$$

and therefore

$$\left(\frac{c_0(W)}{W}\right)^{\gamma} = \left(c_0^{(1)}\right)^{\gamma} + \gamma \left(c_0^{(1)}\right)^{\gamma-1} W^{\gamma-1} c_0^{(1)} + O(W).$$

Integrating (6.66) with respect to W, we get the required.

Other expansions follow in the same manner from Lemma 6.8 and (6.66).

□

7 Non perturbative bounds for the indifference price

We will need the following result, which is also of independent interest.

PROPOSITION 7.1 *Let* $\gamma > 1$ *. Then, the quotient* $W^{-1} c_0(W + \pi_0(W , \mathbf{Y}) , \mathbf{Y})$ *is monotone increasing in* W *and*

$$Z_0^{-1} = \lim_{W \to +\infty} \frac{c_0(W)}{W} \geq \frac{c_0(W)}{W} \geq \lim_{W \to 0} \frac{c_0(W)}{W} = c_0^{(1)}(\mathbf{Y}), \tag{7.1}$$

where Z_0 *is defined by* (5.4) *and* (5.7).

Proof. By Corollary 5.7, π_0 is convex in W . Therefore, $\partial\pi_0/\partial W$ is monotone increasing in W and, by (6.66), so is $c_0(W)/W$. The limit on the right hand side of (7.1) follows from Lemma 6.5.

Using homogeneity, we get

$$\begin{aligned} c_0(W)/W &= c_0(W + \pi(W , \mathbf{Y}) , -\mathbf{Y})\, W^{-1} \\ &= c_0\Big(W^{-1}\,(W + \pi(W , \mathbf{Y}))\, , -\mathbf{Y}\, W^{-1}\Big) \\ &= c_0\Big(1 + \pi(1 , \mathbf{Y}\, W^{-1}))\, , -\mathbf{Y}\, W^{-1}\Big) \end{aligned}$$

Therefore, by (5.7),

$$\lim_{W \to +\infty} \frac{c_0(W)}{W} = c_0(1 , 0) = Z_0^{-1}. \tag{7.2}$$

□

Combining Lemma 6.9 with Proposition 7.1, we immediately get

PROPOSITION 7.2 *We have*

$$\lim_{W \to 0} \frac{\partial\pi_0(W , \mathbf{Y})}{\partial W} = -1 + \left(c_0^{(1)}(\mathbf{Y})\, Z_0\right)^{\gamma}. \tag{7.3}$$

Consequently, the following inequality always holds:

$$\mathbf{Y}_0^u \geq \pi_0(W , \mathbf{Y}) \geq \mathbf{Y}_0^u - W\left(1 - \left(c_0^{(1)}(\mathbf{Y})\, Z_0\right)^{\gamma}\right). \tag{7.4}$$

Asymptotic expansion of Theorem 6.10 is a local result that only holds when W is small. It turns out that it is possible to prove sharp, global bounds for the premium using some interesting convexity properties of the function $c_0(W)/W$. The proof of this result is based on surprising algebraic identities for the derivatives of $c_0(W)$, leading to numerous cancellations.

PROPOSITION 7.3 *Let $\gamma > 1$. Fix $\mathbf{Y}$ and let (as in the proof of Theorem 5.5)*

$$k(W) = k(W, \mathbf{Y}) = c_0(W + \pi_0(W, \mathbf{Y}), \mathbf{Y}). \tag{7.5}$$

Let further

$$g(v) = \left(\frac{k\left(v^{1/(\gamma-1)}\right)}{v^{1/(\gamma-1)}}\right)^{1-\gamma}. \tag{7.6}$$

Then, the function $g(v)$ is convex and satisfies

$$g(0) = \left(c_0^{(1)}\right)^{1-\gamma} \quad , \quad g'(0) = (1-\gamma)\left(c_0^{(1)}\right)^{-\gamma} c_0^{(\gamma)}. \tag{7.7}$$

Proof. By abuse of notation, we will use c_0 to denote the value of $k(W)$ when we do not have to differentiate. By Lemma 3.3,

$$k^{1-\gamma} + \langle \mathbf{c}^{1-\gamma}(k, -\mathbf{Y}), \mathfrak{d}\rangle = W^{1-\gamma} Z_0^\gamma = c_0^{1-\gamma} + \|\mathbf{c}\|_c^2. \tag{7.8}$$

Identity (4.23) implies that

$$\frac{\partial \mathbf{c}(k, -\mathbf{Y})}{\partial k} = k^{-1}(I - \mathbf{P_c})\,\mathbf{c}. \tag{7.9}$$

Differentiating (7.8) with respect to W, we get

$$k' = \frac{k\,W^{-\gamma} Z_0^\gamma}{k^{1-\gamma} + \|\mathbf{b}\|_c^2} = c_0\,W^{-1}(c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\,N^{-1} \tag{7.10}$$

with

$$N = k^{1-\gamma} + \|\mathbf{b}\|_c^2.$$

Differentiating (7.10) with respect to W, we get

$$\begin{aligned} k'' &= N^{-2}\Big((W^{-\gamma} Z_0^\gamma\, k' - \gamma\, k\, W^{-\gamma-1} Z_0^\gamma)\, N \\ &- k\, W^{-\gamma} Z_0^\gamma\, ((1-\gamma)k^{-\gamma}\, k' + (\|\mathbf{b}\|_c^2)')\Big) \\ &= c_0\, W^{-1}(c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-2}\Big(\Big(W^{-1}(c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1} - \gamma\, W^{-1}\Big)\, N \\ &- \Big((1-\gamma)\, W^{-1} c_0^{1-\gamma}(c_0^{1-\gamma} + \|\mathbf{c}\|_c^2)\, N^{-1} + (\|\mathbf{b}\|_c^2)'\Big)\Big). \end{aligned}$$

By (3.64), (3.66) and (4.3),

$$\frac{\partial \mathbf{c}}{\partial c_0} = c_0^{-1}\,\mathbf{b} \quad \text{and} \quad \frac{\partial^2 \mathbf{c}}{\partial c_0^2} = c_0^{-2}(1+\gamma)(I - \mathbf{P_c})\,\mathbf{c}^{-1}(\mathbf{P_c}\,\mathbf{c})^2$$

Therefore, using the fact that $\mathbf{P_c}$ is an orthogonal projection with respect to $\langle \cdot\,,\,\cdot \rangle_c$, we get

$$\begin{aligned}(\|\mathbf{b}\|^2)' &= \frac{\partial}{\partial W}\,\langle\, k\,\frac{\partial \mathbf{c}}{\partial c_0}\,,\, \mathfrak{d}\,\mathbf{c}^{-\gamma}\,\rangle \\ &= k'\,k^{-1}\,\|\mathbf{b}\|_c^2 \;+\; k\,\langle\, \frac{\partial^2 \mathbf{c}}{\partial c_0^2}\,k',\, \mathfrak{d}\,\mathbf{c}^{-\gamma}\,\rangle - \;\gamma\,\langle\, k\,\frac{\partial \mathbf{c}}{\partial c_0}\,,\, \mathfrak{d}\,\mathbf{c}^{-\gamma-1}\,\frac{\partial \mathbf{c}}{\partial c_0}\,k'\rangle \\ &= k'\,k^{-1}\,\|\mathbf{b}\|_c^2 \;+\; k'\,k^{-1}\,(1+\gamma)\,\langle\, \mathbf{c}^{-1}\,(\mathbf{P_c c})^2\,,\,\mathbf{b}\rangle_c - \;\gamma\,k'\,k^{-1}\,\|\mathbf{b}\|_c^2 \\ &= (1-\gamma)\,k'\,k^{-1}\,\|\mathbf{b}\|_c^2 \;+\; k'\,k^{-1}\,(1+\gamma)\,\langle\, \mathbf{c}^{-1}\,(\mathbf{P_c c})^2\,,\,\mathbf{b}\rangle_c \\ &= W^{-1}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1}\,\Big(\,(1-\gamma)\,\|\mathbf{b}\|_c^2 \;+\; (1+\gamma)\,\langle\, \mathbf{c}^{-1}\,(\mathbf{P_c c})^2\,,\,\mathbf{b}\rangle_c\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}&(1-\gamma)\,W^{-1}\,c_0^{1-\gamma}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1} \;+\; (\|\mathbf{b}\|_c^2)' \\ &\quad= \; W^{-1}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1}\,\Big(\,(1-\gamma)\,N \;+\; (1+\gamma)\,\langle\, \mathbf{c}^{-1}\,(\mathbf{P_c c})^2\,,\,\mathbf{b}\rangle_c\Big)\,,\end{aligned} \tag{7.11}$$

and thus

$$\begin{aligned}k'' &= c_0\,W^{-1}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-2}\,\Big(\Big(W^{-1}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1} \;-\gamma\,W^{-1}\Big)\,N \\ &- \;\Big(W^{-1}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1}\,\Big(\,(1-\gamma)\,N \;+\; (1+\gamma)\,\langle\, \mathbf{c}^{-1}\,(\mathbf{P_c c})^2\,,\,\mathbf{b}\rangle_c\Big)\,\Big)\Big) \\ &= c_0\,W^{-2}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-2}\,\Big(\gamma\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2) \;-\gamma\,N \\ &- \;(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1}\,(1+\gamma)\,\langle\, \mathbf{c}^{-1}\,(\mathbf{P_c c})^2\,,\,\mathbf{b}\rangle_c\Big) \\ &= c_0\,W^{-2}\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-2}\,\Big(\gamma\,\|\mathbf{P_c c}\|_c^2 \\ &- \;(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1}\,(1+\gamma)\,\langle\, \mathbf{c}^{-1}\,(\mathbf{P_c c})^2\,,\,\mathbf{b}\rangle_c\Big)\,.\end{aligned}$$

Now,

$$(k/W)' \;=\; \frac{k'\,W \;-\; k}{W^2} \;=\; W^{-2}\,c_0\,\|\mathbf{P_c}\,\mathbf{c}\|_c^2\,N^{-1} \tag{7.12}$$

and

$$
\begin{aligned}
W^3\,(k/W)'' &= k''W^2 \;-\; 2\,k'W \;+\; 2\,k \\
&= c_0\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-2}\Big(\gamma\,\|\mathbf{P_c}\mathbf{c}\|_c^2 \\
&- \;(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{c})^2\,,\,\mathbf{b}\rangle_c\Big) \\
&- \;2\,c_0\,(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,N^{-1} \;+\; 2\,c_0 \\
&= c_0\,N^{-2}\Big(-\; (c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)^2\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{c})^2\,,\,\mathbf{b}\rangle_c \\
&+ \;(c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2)\,\gamma\,\|\mathbf{P_c}\mathbf{c}\|_c^2 \;-\; 2\,N\,(N \;+\; \|\mathbf{P_c}\mathbf{c}\|_c^2) \;+\; 2\,N^2\Big) \\
&= c_0\,N^{-2}\Big(-\; U^2\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{c})^2\,,\,\mathbf{b}\rangle_c \\
&+ \;U\gamma\,\|\mathbf{P_c}\mathbf{c}\|_c^2 \;-\; 2\,N\,(N \;+\; \|\mathbf{P_c}\mathbf{c}\|_c^2) \;+\; 2\,N^2\Big) \\
&= c_0\,N^{-2}\Big(-\; U^2\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{c})^2\,,\,\mathbf{b}\rangle_c \\
&+ \;U(\gamma \;-\; 2)\,\|\mathbf{P_c}\mathbf{c}\|_c^2 \;+\; 2\,\|\mathbf{P_c}\mathbf{c}\|_c^4\Big)\,,
\end{aligned}
$$

where

$$
U \;=\; c_0^{1-\gamma} \;+\; \|\mathbf{c}\|_c^2 \;=\; Z_0^{\gamma}\,W^{1-\gamma} \;=\; \|\mathbf{P_c}\,\mathbf{c}\|_c^2 \;+\; N\,. \tag{7.13}
$$

Now, for any function $f(x)$,

$$
\Big(\,(f(x^{1/(\gamma-1)}))^{1-\gamma}\Big)' \;=\; (1-\gamma)\,f^{-\gamma}\,f'\,(\gamma-1)^{-1}\,x^{\frac{2-\gamma}{\gamma-1}} \;=\; -\,f^{-\gamma}\,f'\,x^{\frac{2-\gamma}{\gamma-1}}, \tag{7.14}
$$

and

$$
\begin{aligned}
&\Big(\,(f(x^{1/(\gamma-1)}))^{1-\gamma}\Big)'' \\
&= \gamma\,(\gamma-1)^{-1}\,f^{-\gamma-1}\,(f')^2\,x^{2\,\frac{2-\gamma}{\gamma-1}} - (\gamma-1)^{-1}\,f^{-\gamma}\,f''\,x^{2\,\frac{2-\gamma}{\gamma-1}} - \frac{2-\gamma}{\gamma-1}\,f^{-\gamma}\,f'\,x^{\frac{3-2\,\gamma}{\gamma-1}} \\
&= x^{\frac{3-2\,\gamma}{\gamma-1}}\,(\gamma-1)^{-1}\,f^{-\gamma-1}\,\Big(\gamma\,(f')^2\,x^{1/(\gamma-1)} \;-\; f\,f''\,x^{1/(\gamma-1)} \;-\; (2-\gamma)\,f\,f'\Big)\,.
\end{aligned} \tag{7.15}
$$

Thus, it remains to show that

$$
\gamma\,(f'(W))^2\,W \;-\; f(W)\,f''(W)\,W \;-\; (2-\gamma)\,f(W)\,f'(W) \;\geq\; 0 \tag{7.16}
$$

for $f(W) \;=\; k(W)/W$ and $W \;=\; V^{1/(\gamma-1)}$. By the above (see, also (7.13)),

$$
\begin{aligned}
\gamma\;(f'(W))^2\,W \;-\; f(W)\,f''(W)\,W \;-\; (2-\gamma)\,f(W)\,f'(W)& \\
&= \gamma\,(W^{-2}\,c_0\,\|\mathbf{P_c}\,\mathbf{c}\|_c^2\,N^{-1})^2\,W \\
&- c_0\,W^{-1}\,c_0\,N^{-2}\Big(-\; U^2\,N^{-1}\,(1+\gamma)\,\langle\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{c})^2\,,\,\mathbf{b}\rangle_c \\
&+ \;U(\gamma \;-\; 2)\,\|\mathbf{P_c}\mathbf{c}\|_c^2 \;+\; 2\,\|\mathbf{P_c}\mathbf{c}\|_c^4\Big)\,W^{-2} \\
&- \;(2-\gamma)\,c_0\,W^{-1}\,W^{-2}\,c_0\,\|\mathbf{P_c}\,\mathbf{c}\|_c^2\,N^{-1} \\
&= c_0^2\,N^{-3}\,W^{-3}\,U^2\,(1 \;+\; \gamma)\,\langle\,\mathbf{c}^{-1}\,(\mathbf{P_c}\mathbf{c})^2\,,\,\mathbf{b}\rangle_c \;\geq\; 0\,,
\end{aligned}
$$

because $\mathbf{b} \geq 0$ by Lemma 4.7. Identity (7.7) follows from Lemma 6.7. $\square$

Now we are ready to prove the main result of this section

THEOREM 7.4 *Let* $\gamma > 1$ *and* $A(\mathbf{Y}) = \left(c_0^{(1)}(\mathbf{Y})\, Z_0\right)^{\gamma}$. *Then,*

$$\begin{aligned}
&\mathbf{Y}_0^u - W\,(1 - A(\mathbf{Y})) \leq \pi_0(W\,,\mathbf{Y}) \\
&\leq \mathbf{Y}_0^u - W\left(1 - A(\mathbf{Y})\left(1 - (\gamma - 1)\left(c_0^{(\gamma)}/c_0^{(1)}\right) W^{\gamma-1}\right)^{\frac{1}{1-\gamma}}\right).
\end{aligned}$$

Proof. By Proposition 7.3, $g(v)$ is convex and therefore

$$g(v) \geq g(0) + g'(0)\, v \tag{7.17}$$

for all $v \geq 0$. That is, by definition of $g(v)$,

$$\begin{aligned}
\left(\frac{c_0}{W}\right)^{1-\gamma} &\geq \left(c_0^{(1)}\right)^{1-\gamma} + (1-\gamma)\left(c_0^{(1)}\right)^{-\gamma} c_0^{(\gamma)}\, W^{\gamma-1} \\
&= \left(c_0^{(1)}\right)^{1-\gamma} - B(\mathbf{Y})\, W^{\gamma-1}.
\end{aligned}$$

with $B(\mathbf{Y}) = (1-\gamma)\left(c_0^{(1)}\right)^{-\gamma} c_0^{(\gamma)}$. Consequently, by Lemma 6.9,

$$\begin{aligned}
\frac{\partial \pi_0(W\,,\mathbf{Y})}{\partial W} &= -1 + \left(\frac{c_0(W)}{W}\right)^{\gamma} Z_0^{\gamma} \\
&\leq -1 + \left(\left(c_0^{(1)}\right)^{1-\gamma} - B(\mathbf{Y})\, W^{\gamma-1}\right)^{\frac{\gamma}{1-\gamma}} Z_0^{\gamma}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\pi_0(W\,,\mathbf{Y}) - \mathbf{Y}_0^u = \pi_0(W\,,\mathbf{Y}) - \pi_0(0\,,\mathbf{Y}) \\
&\leq -W + Z_0^{\gamma} \int_0^W \left(\left(c_0^{(1)}\right)^{1-\gamma} - B(\mathbf{Y})\, w^{\gamma-1}\right)^{\frac{\gamma}{1-\gamma}} dw \\
&= -W + W\left(c_0^{(1)}\right)^{\gamma}\left(1 - \left(c_0^{(1)}\right)^{\gamma-1} B(\mathbf{Y})\, W^{\gamma-1}\right)^{\frac{1}{1-\gamma}} Z_0^{\gamma}.
\end{aligned}$$

$\square$

8 Small claims / capital ratio

In this section we study the asymptotic behavior of the premium π when the size of the claims $\mathbf{Y}$ is small relative to the capital of the company. Since we have an explicit formula for the consumption stream when there are no claims (see, Proposition 5.3), the derivative $\mathbf{P_c}$ can be calculated explicitly by a recursive procedure. But, the expression is rather complicated. For the readers convenience, we perform the calculation for the so-called idiosyncratically incomplete markets. It is characterized in the following definition.

Definition 8.1 *A market* $(\mathcal{M}, \mathcal{G})$ *is idiosyncratically incomplete if*

(1) *There exists a subfiltration* $\mathcal{F} = (\mathcal{F}_t)$ *of* $\mathcal{G}$ *with* $\mathcal{F}_t \subset \mathcal{G}_t$ *for all* $t = 0, \ldots, T$ *such that the price and dividend process of any asset in the market is adapted to* $\mathcal{F}$;

(2) *The market* $\mathcal{M}$ *is complete with respect to* $\mathcal{F}$ *(but not with respect to* $\mathcal{G}$*). That is, any* $\mathcal{F}$*-adapted process can be replicated by an* $\mathcal{F}$*-adapted portfolio strategy;*

(3) *Filtration* $\mathcal{G}$ *does not contain any additional information about events in* $\mathcal{F}$. *Formally,*

$$E[X \,|\, \mathcal{F}_t] \;=\; E[X \,|\, \mathcal{G}_t] \tag{8.1}$$

for any $\mathcal{F}_{t+1}$*-measurable variable* X.

It is easy to see that and idiosyncratically incomplete market belongs to the class $\mathfrak{C}$, the hedgeable algebra is given by $\mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{G}_{t-1})$, the minimal algebra, containing $\mathcal{F}_t$ and $\mathcal{G}_{t-1}$, and the aggregate state price density process $\mathbf{M}$ is, in fact, $\mathcal{F}$-adapted.

Definition 8.1 means that, without a random endowment, an agent faces a complete market and his optimal consumption stream is given by the standard, complete market formula (see, (3.2) and (3.3))

$$c_t \;=\; e^{-\rho t \gamma^{-1}} M_t^{-\gamma^{-1}} c_0 \tag{8.2}$$

and

$$c_0 \;=\; \frac{W}{Z_0} \;=\; \frac{W}{1 \,+\, \sum_{t=1}^{T} e^{-\rho t \gamma^{-1}} E[M_t^{1-\gamma^{-1}}]}. \tag{8.3}$$

Furthermore, the process Z_t takes the following simple form,

$$Z_t \;=\; \mathrm{P}_{\mathcal{F}_t} \sum_{\tau=t}^{T} e^{-\rho t \gamma^{-1}} M_\tau^{1-\gamma^{-1}}.$$

See, Proposition 5.3.

Of course, the market becomes incomplete as soon as the agent has a random endowment (w_t), that is $\mathcal{G}$-adapted, but not $\mathcal{F}$-adapted. Interestingly enough, almost all models for indifference prices, considered in the literature are idio

Lemma 8.2 *Let* $\mathbf{Y} = 0$ *and*

$$\mathbf{Z} \;=\; \mathrm{diag}(Z_t)_{t=1}^{T} \tag{8.4}$$

be the multiplication operator by the process Z_t. *Then,*

$$\mathbf{P_c}|_{\mathbf{Y}\,=\,0} \;=\; B \;=\; c_0^{-1}\, \mathbf{cm}\, J\, Z^{-1}\, \mathbf{Q}\, J^*\, \mathbf{M}. \tag{8.5}$$

Proof. By (3.35) and (3.36), we only have to prove that

$$D(F) \;=\; -c_0\,\gamma^{-1}\mathbf{Q}\,J^*\,\mathbf{M}\;\mathbf{cm}\,J \;=\; -\,\gamma^{-1}\,Z. \tag{8.6}$$

Note that, by Definition 8.1, for any $\mathcal{F}_T$-measurable variable X,

$$E[X\,|\,\mathcal{F}_t] \;=\; E[X\,|\,\mathcal{G}_t] \;=\; E[X\,|\,\mathcal{H}_t] \tag{8.7}$$

and $Q_{t_1}\,Q_{t_2} \;=\; 0$ for any $t_1 \;\neq\; t_2$. By direct calculation,

$$\begin{aligned}
\Big(D(F)((x_\theta)\Big)_t \;&=\; Q_t\sum_{\tau=t}^{T} M_\tau\,\mathbf{cm}_\tau \sum_{\theta=1}^{\tau} x_\theta \\
&=\; Q_t\sum_{\tau=1}^{T} x_\tau \sum_{\theta=\max\{\tau\,,\,t\}}^{T} M_\theta\,\mathbf{cm}_\theta \\
&=\; Q_t\sum_{\tau=1}^{t} x_\tau\sum_{\theta=t}^{T} M_\theta\,\mathbf{cm}_\theta \;+\; Q_t\sum_{\tau=t+1}^{T} x_\tau \sum_{\theta=\tau}^{T} M_\theta\,\mathbf{cm}_\theta \\
&=\; c_0\,Z_t\,Q_t\sum_{\tau=1}^{t} x_\tau \;+\; Q_t\sum_{\tau=t+1}^{T} x_\tau\,\mathrm{P}_{\mathcal{G}_\tau}\sum_{\theta=\tau}^{T} M_\theta\,\mathbf{cm}_\theta \\
&=\; c_0\,x_t\,Z_t \;+\; c_0\,Q_t\sum_{\tau=t+1}^{T} x_\tau\,Z_\tau \;=\; c_0\,x_t\,Z_t.
\end{aligned} \tag{8.8}$$

Here we have used that $x_\tau \;\in\; Q_\tau\,L_2(\mathcal{G}_\tau)$ implies $Q_t\,x_\tau \;=\; 0$ or $t \;\neq\; \tau$ and $M_\tau\,\mathbf{cm}_\tau$ is $\mathcal{F}_\tau$-measurable for any τ. □

A direct consequence of Lemma 8.2 and Theorem 5.5 is

Lemma 8.3 *We have*

$$\frac{\partial\pi_0}{\partial\mathbf{Y}}|_{\mathbf{Y}\;=\;0} \;=\; \mathbf{M} \tag{8.9}$$

and

$$\frac{\partial^2\pi_0}{\partial\mathbf{Y}^2}(\mathbf{y}\,,\,\mathbf{y})|_{\mathbf{Y}\;=\;0} \;=\; \gamma\,\|B\,\mathbf{y}\|^2_{\mathrm{cm}}. \tag{8.10}$$

Definition 8.4 *For each $t \;=\; 1\,,\,\ldots\,,\,T$, let*

$$\mathrm{I}_t\,(\mathbf{y}\,,\,\mathbf{M}) \;=\; E\left[\sum_{\tau=t}^{T} y_\tau\,M_\tau\,\Big|\,\mathcal{G}_t\right] \;-\; E\left[\sum_{\tau=t}^{T} y_\tau\,M_\tau\,\Big|\,\mathcal{H}_t\right]. \tag{8.11}$$

We can now calculate the second order approximation to the indifference price the ratio $\mathbf{Y}/W$ of claims to capital is small.

Theorem 8.5 *We have*

$$\pi_0(W\,,\,\mathbf{Y}) \;=\; W\,\pi_0(1\,,\mathbf{Y}/W) \tag{8.12}$$

and therefore, when $\mathbf{Y}/W$ is small,

$$\pi_0(W\,,\,\mathbf{Y}) \;=\; \langle\,\mathbf{Y}\,,\,\mathbf{M}\rangle + W\,Z_0\sum_{t=1}^{T} E\left[\frac{\mathrm{Var}_{\mathcal{F}_t}(\mathrm{I}_t(\mathbf{Y})\,W^{-1}\,)}{Z_t}\right] + W\,O((\mathbf{Y}/W)^3) \tag{8.13}$$

Proof. We have

$$(B\,\mathbf{y})_t \;=\; c_0^{-1}\,\mathrm{cm}_t \sum_{\tau=1}^{t} \frac{\mathrm{I}_\tau(\mathbf{y}\,,\,\mathbf{M})}{Z_\tau} \tag{8.14}$$

and, consequently,

$$\mathrm{cm}_t^{-2}\,\mathrm{P}_{\mathcal{F}_t}\,(B\,\mathbf{y})_t^2 \;=\; \mathrm{P}_{\mathcal{F}_t}\left(\sum_{\tau=1}^{t} \frac{\mathrm{I}_\tau}{Z_\tau}\right)^2 \;=\; \mathrm{Var}_{\mathcal{F}_t}\left(\sum_{\tau=1}^{t} \frac{\mathrm{I}_\tau}{Z_\tau}\right) \tag{8.15}$$

since

$$\mathrm{P}_{\mathcal{F}_t}\frac{\mathrm{I}_\tau}{Z_\tau} \;=\; \frac{\mathrm{P}_{\mathcal{F}_\tau}\mathrm{I}_\tau}{Z_\tau} \;=\; 0.$$

Consequently,

$$\mathrm{Cov}_{\mathcal{F}_t}\left(\frac{\mathrm{I}_{\tau_1}}{Z_{\tau_1}}\,,\,\frac{\mathrm{I}_{\tau_2}}{Z_{\tau_2}}\right) \;=\; 0$$

since $I_\tau \in Q_\tau\, L_2(\mathcal{G}_\tau)$ for each τ. Hence,

$$\mathrm{cm}_t^{-2}\,\mathrm{P}_{\mathcal{F}_t}\,(B\,\mathbf{y})_t^2 \;=\; c_0^{-2}\sum_{\tau=1}^{t}\frac{\mathrm{Var}_{\mathcal{F}_\tau}(\mathrm{I}_\tau)}{(Z_\tau)^2}\,.$$

Let

$$V_\tau \;=\; \frac{\mathrm{Var}_{\mathcal{F}_\tau}(\mathrm{I}_\tau)}{(Z_\tau)^2}\,.$$

Then,

$$\begin{aligned}
c_0^{\gamma}\,\|B\,\mathbf{y}\|_{\mathrm{cm}}^2 \;&=\; c_0\,E\left[\sum_{\tau=1}^{T} e^{-\rho\,t\,\gamma^{-1}}\,M_t^{1-\gamma^{-1}}\,\mathrm{cm}_t^{-2}\,(B\,\mathbf{y})_t^2\right] \\
&=\; E\left[\sum_{\tau=1}^{T} e^{-\rho\,t\,\gamma^{-1}}\,M_t^{1-\gamma^{-1}}\,\mathrm{cm}_t^{-2}\,E\left[(B\,\mathbf{y})_t^2\,|\,\mathcal{F}_t\right]\right] \\
&=\; W^{-1}\,Z_0\,E\left[\sum_{\tau=1}^{T} e^{-\rho\,t\,\gamma^{-1}}\,M_t^{1-\gamma^{-1}}\sum_{\tau=1}^{t} V_\tau\right] \\
&=\; W^{-1}\,Z_0\sum_{t=1}^{T} E\left[Z_t\,V_t\right] \;=\; W^{-1}\,Z_0\sum_{t=1}^{T} E\left[\frac{\mathrm{Var}_{\mathcal{F}_t}(\mathrm{I}_t)}{Z_t}\right].
\end{aligned} \tag{8.16}$$

□

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