Price Impact and Portfolio Impact [*]

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Abstract

We study survival, price impact and portfolio impact in heterogeneous economies. We show that, under the equilibrium risk-neutral measure, long-run price impact is in fact equivalent to survival, whereas long-run portfolio impact is equivalent to survival under an agent-specific, *wealth-forward measure.* These results allow us to show that price impact and portfolio impact are two independent concepts: a non-surviving agent with no long-run price impact can have a significant long-run impact on other agents' optimal portfolios.

Keywords: survival, price impact, equilibrium, heterogeneous agents, optimal portfolios.

JEL Classification. D53, G11, G12

1   Introduction

The principal motivation for studying survival of irrational traders and their long-run price impact comes from the theory of efficient financial markets. If irrational traders do have long-run impact on asset prices, there will be persistent market inefficiencies, and prices will constantly deviate from fundamental values and give rise to inefficient allocations.

Starting with Friedman (1953), it has long been argued that irrational traders cannot survive in a competitive market, as they will constantly lose money betting on the realization of very unlikely states of the economy. Basing on this intuition, Friedman argued that irrational traders cannot influence long-run asset prices. In a recent seminal contribution, Kogan, Ross, Wang and Westerfield (2006) (henceforth, KRWW (2006)) demonstrated that survival and price impact are two independent concepts: even if irrational traders do not survive, they can still have a substantial long-run impact on asset prices. They also show that irrational traders portfolio policies can deviate significantly from what the asymptotic moments of stock returns suggest. KRWW (2006) suggest the following intuitive explanation of these surprising phenomena: "Under incorrect beliefs, irrational traders express their views by taking positions (bets) on extremely unlikely states of the economy. As a result, the state prices of these extreme states can be significantly affected by the beliefs of the irrational traders, even with negligible wealth. In turn, these states, even though highly unlikely, can have a large contribution to current asset prices." This intuition naturally gives rise to the following questions: what, precisely, are the extremely unlikely states responsible for the price impact, and what is the exact economic mechanism by which these states generate price impact? In this paper, we provide detailed answers to these questions.

We show that Friedman's original intuition is in fact correct, but with a small modification: price impact is indeed equivalent to survival, but under the risk-neutral rather than the physical measure. Namely, an agent has a long-run price impact if, and only if, the long-run share of the aggregate

wealth that he owns has a nonnegligible market value.

A similar result holds for portfolio impact. We show that the long-run impact of agent j on the portfolio of agent i is equivalent to the survival of agent j, but under agent i's *wealth-forward measure.* This measure has a density proportional to that of the risk-neutral measure, but it is multiplied with agent i's wealth. This is very intuitive: the agent, j, who bets on the realization of states in which agent i's wealth is the largest, will have the most significant impact on state prices in those states and, consequently, on agent i's optimal portfolio.

We consider an economy populated by an arbitrary number of agents with arbitrary heterogeneous risk aversions and beliefs, maximizing utility from terminal consumption. We derive closed-form asymptotic expressions for equilibrium quantities and study how price impact, portfolio impact and survival depend on the cross-sectional distribution of agents' characteristics. We show that allowing for more than two agents can lead to new, surprising phenomena that cannot occur in two-agent economies. In particular, we show that even a nonsurviving agent with no price impact may have long-run impact on other agent's equilibrium optimal portfolios. In contrast to the findings of KRWW (2006), we show that nonsurviving irrational agents can have both price and portfolio impact even if they are optimistic,[1] as long as the preferences are heterogeneous across agents.[2] As we show, the presence of such overoptimistic irrational traders is crucial for generating the empirically observed[3] U-shaped pattern for the equilibrium state price density. Finally, when the economy is populated by a large number of agents, both price impact and portfolio impact can be *permanent:* they may not

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[1]KRWW (2006) show that, in a two-agent economy with identical risk aversions, a nonsurviving irrational trader can have price impact if and only if he is pessimistic. Because a large proportion of stock traders seem to be overoptimistic (see Dimson, Marsh and Staunton (2003)), it is important to recognize that this phenomenon can also occur with optimistic traders.

[2]This phenomenon is related to findings of Yan (2008), who shows that, when preferences are heterogeneous, incorrect beliefs may have an opposite effect on the economy's behavior.

[3]See Jackwerth (2000).

vanish even for periods arbitrarily close to the terminal time horizon. This behavior is completely different from that in a two-agent economy in which both price and portfolio impact always vanish for periods sufficiently close to the terminal time horizon.

We also study price impact and survival in the presence of intermediate consumption. As in the case of only terminal consumption, the long-run price impact is always equivalent to survival under a particular measure. As an illustration, we show that an agent affects long-run bond volatilities if and only if he survives under the T-forward measure. Similarly, an agent affects long-run stock forward volatility if and only if he survives under the dividend forward measure. In the case of a large market populated by a continuum of agents, these results allow us to establish surprising closed-form expressions for long-run volatilities and to determine their dependence on the cross-sectional correlation of beliefs and risk aversions in the economy.

We now discuss the related literature.

Many papers study long-run survival of irrational agents in equilibrium models with intermediate consumption. Blume and Easley (2006) show that, when the aggregate endowment is bounded from zero and infinity, risk preferences do not matter for survival, and irrational agents do not survive.

In a recent seminal contribution, Yan (2008) considers an economy populated by many CRRA agents with heterogeneous risk aversions, discount factors and beliefs, and an aggregate endowment following a geometric Brownian motion. Yan shows that survival can be characterized by a single number, the survival index, and that only the agent with the lowest index survives in the long run. In stark contrast to the findings of Blume and Easley (2006), Yan finds that survival does depend on risk preferences. However, if two agents differ only in their beliefs, the more irrational one will become extinct in the long run. Yan shows that the state prices and, consequently, the market price of risk converge to those determined by the single surviving agent. Cvitanić, Jouini, Malamud and Napp (2009) consider the same model as Yan's and show also that the stock price volatility and optimal portfolios converge to those determined by the single surviving agent. However, they

show that the returns on long term bonds, as well as long run cumulative stock returns are impacted by non-surviving agents. Kogan, Ross, Wang and Westerfield (2010) (henceforth, KRWW (2010)) also study the link between survival and price impact in the presence of intermediate consumption but allow for general utilities with unbounded relative risk aversion and a general dividend process. They show that in order to have nonsurviving agents who impact the long-run equilibrium state prices, it is necessary to assume utilities with an unbounded relative risk aversion that grows sufficiently fast at infinity. In contrast to KRWW (2010), we show that, even with CRRA utilities, non-surviving agents can have a long-run impact on the prices of assets with long maturities, and this price impact is equivalent to survival under an asset-specific measure. Thus, the mechanism, relating price impact with survival under different measures is universal for both models with and without intermediate consumption.

There is a large literature on equilibrium risk sharing with heterogeneous risk attitudes and beliefs. Dumas (1989) analyzes numerically a continuous-time production economy with two agents having different risk aversions. Wang (1996) studies the equilibrium yield curve in a continuous-time economy with two agents for special values of risk aversions. Basak and Cuoco (1998) consider a restricted participation model with two agents having different risk preferences. Bhamra and Uppal (2009) consider the same two-agent model as Wang (1996) and derive conditions for equilibrium excess volatility. Many papers analyze models with heterogeneity of beliefs only (see, e.g., Basak (2000, 2005), KRWW (2006), Jouini and Napp (2007), Berrada (2009), and Xiong and Yan (2009)). Dumas, Kurshev and Uppal (2009) derive closed-form expressions for optimal portfolios in an economy with two investor types with different beliefs but identical risk aversions. None of these papers analyzes economies with more than two heterogeneous CRRA agents. Chan and Kogan (2002) and Xiouros and Zapatero (2009) study equilibrium asset prices in economies populated by a large number (a continuum) of agents with heterogeneous risk aversions and "Catching up with the Joneses" preferences. Basak and Yan (2009) study equilibrium asset prices in the presence of money illusion. Yan (2009) studies economies

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with a large number of irrational (noise) traders and shows that, in general, noise trading is not cancelled out by aggregation and can lead to equilibrium price overshooting and negative expected returns. However, these papers do not analyze equilibrium optimal portfolios. Cvitanić and Malamud (2009b) consider a general continuous time CCAPM with the aggregate dividend following an arbitrary Markov diffusion and multiple agents with arbitrary heterogeneous utilities. They obtain general representations for drift, volatility and optimal portfolios in terms of aggregate quantities, and they use these representations to make empirical predictions and establish general bounds for those quantities. However, they do not derive closed-form expressions for the optimal portfolios. To the best of our knowledge, our paper is the first one that obtains closed-form asymptotic expressions for optimal portfolios in equilibrium with more than two CRRA agents differing in their levels of risk aversions.

The paper is organized as follows. Section 2 provides the model setup. Section 3 studies survival and price impact. Section 4 analyzes portfolio impact and its relation to price impact and survival. Section 5 studies the model with intermediate consumption. Section 6 contains detailed, explicit calculations for case of large markets with a continuum of agents. Section 7 concludes. All proofs and technical results are found in the appendices.

2   Setup and Notation

2.1   The Model

We consider a standard setting similar to that of Wang (1996) and KRWW (2006). The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion B_t, $t \in [0, T]$ on a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\mathcal{F}$ is the augmented filtration generated by B_t. There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend

$$D_T = e^{\mu T + \sigma B_T}.$$

We also assume that a zero coupon bond with instantaneous constant risk-free rate $r = 0$ is available in zero net supply.[4]

There are K (types of) agents indexed by $i = 1, ..., K$. Agents have different expectations about the future of the economy. More precisely, agents disagree about the mean growth rate. We denote by μ_i the mean growth rate anticipated by agent i. Letting

$$\delta_i = \frac{\mu_i - \mu}{\sigma^2}$$

denote agent i's error in her perception of the growth of the economy normalized by its risk,[5] we introduce the probability measure P^i, which is equivalent to P and defined by its density

$$Z_{iT} = e^{\delta_i \sigma B_T - \frac{1}{2} \sigma^2 \delta_i^2 T}.$$

From agent i's point of view, the terminal dividend is given by

$$D_T = e^{(\mu + \delta_i \sigma^2) T + \sigma B_T^i}$$

where, by the Girsanov Theorem,

$$B_t^i \equiv B_t - \delta_i \sigma t$$

is a Brownian motion with respect to P^i.

Agent i chooses an admissible portfolio strategy π_{it}, the portfolio weight in the risky asset, so as to maximize the expected utility

$$E^{P^i} \left[\frac{W_{iT}^{1-\gamma_i} - 1}{1 - \gamma_i} \right]$$

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[4]As remarked below, because the agents maximize utility only from terminal wealth, interest rates can be taken to be exogenous. The assumption of zero interest rate is made for simplicity of exposition. The analysis directly extends to the case of nonzero r and, under some technical conditions, also to the case of stochastic interest rates.

[5]The parameter δ_i also represents the difference between agent i's perceived Sharpe ratio and the true one.

of his final wealth W_{iT}, where the wealth W_{it} of agent i evolves as

$$dW_{it} = W_{it} \pi_{it} S_t^{-1} dS_t$$

and E^{P^i} denotes the expectation under agent i's subjective beliefs.

In this equation, S_t is the stock price at time t. The instantaneous drift and volatility of the stock price S_t are denoted by μ_t^S and σ_t^S respectively, so that

$$\frac{dS_t}{S_t} = \mu_t^S dt + \sigma_t^S dB_t.$$

The market price of risk (MPR) κ_t is given by

$$\kappa_t = \frac{\mu_t^S}{\sigma_t^S}.$$

Agent k is initially endowed with $\psi_k > 0$ shares of stock, and the total supply of the stock is normalized to one,

$$\sum_{k=1}^{K} \psi_k = 1.$$

By definition, agent i is rational if his beliefs about the true probability measure are correct (that is, $\delta_i = 0$,) and is irrational otherwise. We say that an irrational agent i is pessimistic (optimistic) if $\delta_i < 0 \, (> 0)$. Risk aversions $\gamma_i > 0$ and beliefs $\delta_i \in \mathbb{R}$ are arbitrary and heterogeneous across agents.

2.2 Equilibrium

Definition 2.1. *We say that the market is in equilibrium if the agents behave optimally and both the risky asset market and the risk-free market clear.*

It is well known that the above financial market is complete if the volatility process σ_t^S of the stock price is almost everywhere strictly positive.[6]

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[6]This can be verified under some technical regularity conditions on the model primitives. See Hugonnier, Malamud and Trubowitz (2009). In our model, it is possible to show that $\sigma_t^S \geq \sigma$. See Proposition 3.7.

When the market is complete, there exists a unique state price density process $\xi = (\xi_t)$ such that the stock price is given by

$$S_t = \frac{E_t[\xi_T D_T]}{\xi_t} \tag{2.1}$$

where ξ_t is the density process of the equivalent martingale measure Q,

$$\left(\frac{dQ}{dP}\right)_t = \xi_t = E_t[\xi_T].$$

Thus, we can rewrite (2.1) in the form

$$S_t = E_t^Q[D_T]. \tag{2.2}$$

Because of the market completeness, any equilibrium allocation is Pareto efficient and can be characterized as an Arrow–Debreu equilibrium (see, e.g., Duffie and Huang (1986), Wang (1996)).[7]

It is well known (see, e.g., Duffie (2001)) that in this complete market setting, the optimal terminal wealth for agent i is of the form $W_{iT} = (y_i Z_{iT}^{-1} \xi_T)^{-b_i}$, where $b_i = \gamma_i^{-1}$ is the relative risk tolerance of agent i, and y_i is determined via the budget constraint $E[(y_i Z_{iT}^{-1} \xi_T)^{-b} \xi_T] = W_{i0} = \psi_i S_0 = \psi_i E[D_T \xi_T]$. That is,

$$W_{iT} = \frac{\psi_i E[D_T \xi_T]}{E[Z_{iT}^{b_i} \xi_T^{1-b_i}]} Z_{iT}^{b_i} \xi_T^{-b_i}.$$

Because in equilibrium the final wealth amounts of all the agents have to sum up to the aggregate dividend, equilibrium stochastic discount factor (SDF) ξ_T needs to solve the equation

$$\sum_i \frac{\psi_i E[D_T \xi_T]}{E[Z_{iT}^{b_i} \xi_T^{1-b_i}]} Z_{iT}^{b_i} \xi_T^{-b_i} = \sum_i W_{iT} = D_T. \tag{2.3}$$

As in KRWW (2006), we will write $X_t \sim Y_t$ if $\lim_{t\to\infty} X_t/Y_t = 1$.

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[7]Existence and uniqueness of equilibrium can be shown using the same arguments as those in Dana (1995), Dana (2001) and Malamud (2008).

3 Survival and Price Impact

As we shall see below, both survival and price impact of agent i can be completely characterized by a single characteristic

$$\tilde{\gamma}_i \stackrel{def}{=} \gamma_i - \delta_i.$$

We shall refer to $\tilde{\gamma}_i$ as the effective risk aversion of agent i. This quantity has a very clear economic interpretation. If the agent is irrationally optimistic, then he is willing to take riskier positions in the stock than a rational agent with the same risk aversion γ_i. Similarly, a pessimistic agent behaves as if his risk aversion were higher, and the size $-\delta_i$ of his pessimism precisely determines the gap between his true and effective risk aversion. Note also that, in a single-agent economy populated only by agent i, the equilibrium state price density ξ_T is proportional to $D_T^{-\tilde{\gamma}_i}$, so $\tilde{\gamma}_i$ determines the elasticity of the state price density with respect to the aggregate consumption.

Because in our economy the markets are complete, the prices coincide with those in an artificial economy populated by a single, representative agent with a utility function U (see Duffie (2001)). The equilibrium state price density equals the marginal utility of the representative agent, evaluated at the aggregate endowment,

$$\xi_T = U'(D_T). \tag{3.1}$$

The identity

$$Z_{iT} = e^{-0.5\,\delta_i^2\sigma^2 + \delta_i\,\sigma\,B_T} = e^{(-0.5\delta_i^2\sigma^2 - \delta_i\mu)\,T}\,D_T^{\delta_i}$$

and the equilibrium market clearing (2.3) imply that the marginal utility $U'(x)$ is uniquely determined by the equation

$$\sum_i \left(y_i\,e^{(0.5\delta_i^2\sigma^2 + \delta_i\mu)\,T}\,x^{-\delta_i}\,U'(x)\right)^{-b_i} = x. \tag{3.2}$$

We shall refer to

$$\tilde{\gamma}^U(x) \;=\; -\frac{x\,U''(x)}{U'(x)}$$

as the aggregate *effective risk aversion*. Because of beliefs heterogeneity, the representative agent's utility need not be concave and, consequently, $\tilde{\gamma}^U(x)$ may be negative for some values of x, in which case the representative agent exhibits risk-loving behavior.[8] The following proposition is true.

Proposition 3.1. *The aggregate effective risk aversion is given by*

$$\tilde{\gamma}^U(D_T) \;=\; \sum_i \tilde{\gamma}_i\,\lambda_{iT} \qquad (3.3)$$

where

$$\lambda_{iT} \;=\; \frac{b_i\,W_{iT}}{\sum_j\,b_j\,W_{jT}}\,. \qquad (3.4)$$

It is monotone decreasing in D_T and satisfies

$$\max_i \tilde{\gamma}_i \;=\; \lim_{x\to+0}\tilde{\gamma}^U(x) \;\geq\; \tilde{\gamma}^U(x) \;\geq\; \lim_{x\to\infty}\tilde{\gamma}^U(x) \;=\; \min_i \tilde{\gamma}_i\,. \qquad (3.5)$$

Proposition 3.1 shows that the aggregate effective risk aversion is a convex combination of individual effective risk aversions $\tilde{\gamma}_i$. The weight of agent i in this convex combination is proportional to the product of the agent's risk tolerance b_i and wealth W_{iT}. This is intuitively clear: a wealthy agent will have a larger impact on the equilibrium state prices. However, if an agent's risk tolerance is low, he will invest most of his money into bonds, and his terminal wealth will be almost state independent and will have little impact on the curvature of the representative agent's utility. The fact that the effective relative risk aversion is decreasing is very natural: agents with low levels of effective risk aversions take bets on high levels of D_T, dominate in those states and drive the aggregate risk aversion down. Similarly, agents

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[8]If risk aversion γ is constant across agents, then it is natural to say that the representative agent has risk aversion γ and beliefs that are aggregated from individual agents' beliefs. See Jouini and Napp (2007) and Yan (2009). However, when risk aversion is heterogeneous, disentangling risk aversion from beliefs is no longer possible. For this reason, we refer to $\tilde{\gamma}^U$ as the effective risk aversion, even though it aggregates both risk aversion and beliefs.

with high effective risk aversion gain in states with low D_T and drive risk aversion up.

Formula (3.5) has important implications for the state prices "smile effect". Namely, empirical evidence (see, e.g., Jackwerth (2000)) suggests that the state prices, estimated from European call option prices, are not monotone decreasing in the market portfolio, as they would be if the representative agent had a standard, concave utility. Rather, they are U-shaped. Proposition 3.1 implies that the following proposition is true.

Proposition 3.2. *In our model, state prices smile if and only if there exists an agent i who is so optimistic that his effective risk aversion is negative; that is, $\delta_i > \gamma_i$.*

Indeed, in this case, $\min_i \tilde{\gamma}_i < 0$ and, by (3.5) and monotonicity, there exists a critical value x_*, such that $\tilde{\gamma}^U(x) > 0$ for $x < x_*$ and $\tilde{\gamma}^U(x) < 0$ for $x > x_*$. Therefore, $U''(D_T) = -\tilde{\gamma}^U(D_T)\xi_T D_T^{-1}$ is positive (negative) for $D_T > x_* (< x_*)$. This precisely means that the state prices are U-shaped. The reason is that, if an agent i is so extremely optimistic, the marginal value of an additional stock share will be increasing in the level D_t of the dividend and will generate an upward-sloping marginal utility.

The aggregate effective risk aversion $\tilde{\gamma}^U$ plays a crucial role in the representations for all equilibrium quantities. The following proposition is true.

Proposition 3.3. *The market price of risk κ_t is equal to the market price of the aggregate effective risk aversion times the dividend volatility,*

$$\kappa_t = \sigma E_t^Q[\tilde{\gamma}^U(D_T)] \tag{3.6}$$

whereas the stock price volatility is given by

$$\sigma_t^S = \sigma \left(1 - S_t^{-1} \operatorname{Cov}_t^Q(\tilde{\gamma}^U(D_T), D_T)\right). \tag{3.7}$$

Formula (3.6) has a very clear and intuitive interpretation. In the case in which all agents have the same effective risk aversion $\tilde{\gamma}$, the market price of

risk is constant and is given by

$$\kappa \;=\; \sigma \, \tilde{\gamma}.$$

That is, in equilibrium, the risk premium the agents require for holding the stock is given by the product of the stock riskiness σ and the agent's effective risk aversion $\tilde{\gamma}$. When the agents are heterogeneous, the effective risk aversion $\tilde{\gamma}^U$ determines the aggregate risk attitude of the economy and the market prices it under the risk-neutral measure.

Formula (3.7) is more subtle. It shows that the spread between the stock price volatility and the fundamental volatility is given by the covariance of the aggregate risk aversion and the aggregate dividend. The reason is that, when the market price of risk is stochastic, the nonmyopic agents increase or decrease stock investment depending on the expected future fluctuations of the market price of risk. The equilibrium hedging demand raises or decreases the total equilibrium demand for stocks and therefore drives the equilibrium stock price up or down. Because D_t is the single state variable in our model, $S_t \;=\; S(t, D_t)$ is a smooth function[9] of D_t, and therefore, by Ito's formula, we get

$$\sigma_t^S \;=\; \sigma \, \frac{\partial \log S(t, D_t)}{\partial \log D_t} \,. \tag{3.8}$$

Thus, stock price volatility is nothing but the sensitivity of the stock price to the changes in the dividend. Because equilibrium optimal portfolios respond to changes in D_t in a nonmyopic way, driven by the future fluctuations in the market price of risk, so does the equilibrium stock price. By (3.7), future cyclical fluctuations of the market price of risk are captured in the stock volatility via the covariance of the future market price of risk with the aggregate dividend.

We now give the definitions of the long-run price impact and of survival.

Definition 3.1. *We say that an agent i has a long-run market price of risk (volatility) impact for $t = \lambda T$ with some $\lambda \in (0,1)$ if the asymptotic*

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[9]Under some technical conditions. See Hugonnier, Malamud and Trubowitz (2009).

behavior of $\kappa_{\lambda T}$ ($\sigma_{\lambda T}^S$) as $T \to \infty$ depends on (at least one of) the agent's characteristics γ_i and δ_i.

Given a probability measure O, we say that an agent i survives with respect to O for $t = \lambda T$ if the price under the measure O of the share of the total wealth that the agent i owns is nonnegligible in the long run. That is,

$$\limsup_{T \to \infty} E_{\lambda T}^O[W_{iT} D_T^{-1}] > 0 \tag{3.9}$$

with positive physical probability.

When the measure O coincides with P, the physical probability measure, this definition of survival is essentially equivalent to the standard one (see KRWW (2006) and Yan (2008)): an agent i survives if, in the long run, he still owns a positive share of the total wealth with positive probability.

When O is a different measure, the expectation, or price of the share $W_{iT} D_T^{-1}$ is calculated under the O-measure, but it should still be nonnegligible with *positive P-probability*. This definition is natural, as can be most easily seen in the case in which O coincides with the equilibrium risk-neutral measure Q. Indeed, in that case, $E_t^Q[W_{iT} D_T^{-1}]$ is the market price of the share of the total wealth that agent i owns. Intuitively, we need this price to be nonzero with positive physical probability in order for an agent i to have a long-run price impact. This intuition is justified by the following proposition.

Proposition 3.4. *An agent i has a long-run impact on the market price of risk for $t = \lambda T$ if and only if he survives with respect to the equilibrium risk-neutral measure Q. If agent i is the only one who survives with respect to Q for $t = \lambda T$ then*

$$\kappa_t \sim \sigma \, \tilde{\gamma}_i \, .$$

Instead of relegating it to the Appendix, we now present the proof of Proposition 3.4, because it is very intuitive. It follows directly from (3.6) and Proposition 3.1 that

$$\kappa_t = E_t^Q[\tilde{\gamma}^U(D_T)] = \sum_i \tilde{\gamma}_i \, E_t^Q[\lambda_{iT}] \, . \tag{3.10}$$

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Therefore, agent i has a nonnegligible impact on the market price of risk if and only if the market price $E_t^Q[\lambda_{iT}]$ of the weight λ_{iT} is nonzero with positive P-probability. Market clearing condition (2.3) implies that the denominator of the weight λ_{iT} (see (3.4)) can be bounded via

$$\min_j b_j D_T \leq \sum_j b_j W_{jT} \leq \max_j b_j D_T \qquad (3.11)$$

and therefore

$$\frac{b_i}{\max_j b_j} E_t^Q[W_{iT} D_T^{-1}] \leq E_t^Q[\lambda_{iT}] \leq \frac{b_i}{\min_j b_j} E_t^Q[W_{iT} D_T^{-1}]. \qquad (3.12)$$

The first statement of Proposition 3.4 now follows directly from Definition 3.1, and the second statement follows from (3.10) and (3.12). The bound (3.12) has a very clear intuitive meaning: the strength of the price impact of agent i is determined by the size of the share of the total wealth that he owns.

In order to state an analogous result for the impact on the volatility, we shall need the following definition.

Definition 3.2. *The probability measure*

$$dQ^D \stackrel{def}{=} \frac{D_T}{E^Q[D_T]} dQ$$

will be referred to as the dividend-forward measure.

The dividend-forward measure "discounts" future cash flows with the value of terminal dividends.[10] Thus, it puts a lot of weight on those states in which D_T is high and little weight on the states in which D_T is low. Consequently, this measure is cyclical: given two payoffs with identical distributions under Q, the dividend-forward measure will assign a higher price to the more cyclical (i.e., positively covarying with D_T) payoff. Using this

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[10]The name "dividend-forward measure" comes from an analogy with the T-forward measure used in the pricing of interest rate derivatives.

new measure and recalling that $S_t = E_t^Q[D_T]$, we can rewrite representation (3.7) in the form

$$\sigma_t^S = \sigma \left(1 - E_t^{Q^D}[\tilde{\gamma}^U(D_T)] + E_t^Q[\tilde{\gamma}^U(D_T)] \right). \qquad (3.13)$$

Then, essentially the same argument as that in the proof of Proposition 3.4 implies that the following proposition is true.

Proposition 3.5. *An agent i has a long-run volatility impact for $t = \lambda T$ if and only if he survives either with respect to the equilibrium risk-neutral measure or with respect to the dividend-forward measure.*[11]

Why does the dividend-forward measure matter for long-run volatility impact and not for the market price of risk impact? The reason is that the market price of risk is determined by the *level* of the aggregate effective risk aversion, whereas the stock price volatility is determined by its *fluctuations*.[12] The dividend-forward measure captures precisely these fluctuations and therefore influences the long-run volatility impact.

Now, given the general results of Propositions 3.4 and 3.5, we need to determine which agents do survive under a particular measure. In order to state the results, we shall need several definitions.

Denote by 0 the agent whose effective risk aversion is the closest to 1. That is,

$$|\tilde{\gamma}_0 - 1| = \min_i |\tilde{\gamma}_i - 1|. \qquad (3.14)$$

Without loss of generality, we shall assume that such a unique agent exists.[13] Intuitively, agent 0 has preferences that are closest to those of the rational logarithmic agent. Because the logarithmic agent has the highest wealth growth rate, it is natural to expect that agent 0 will also have the highest wealth growth rate among all other agents in the economy and, consequently,

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[11]Note that, if agent i is the only one surviving with respect to both measures, the effects will cancel each other out and σ_t^S will coincide with the dividend volatility σ. In this case, we still say that agent i has volatility impact.

[12]See (3.6)–(3.7) and the discussion after Proposition 3.3.

[13]This is true for generic values of risk aversions and beliefs.

will own the whole economy in the long run. This is indeed true, as shown in the following proposition.

Proposition 3.6. *For any $\lambda \in (0,1)$ and $t = \lambda T$, agent 0 is the only one who survives with respect to the physical measure, as well as the only one who survives with respect to the dividend-forward measure. In particular, for $t = \lambda T$ and any $\lambda \in (0,1)$,*

$$E_t^{Q^D}[\,\tilde{\gamma}^U(D_T)\,] \;\sim\; \tilde{\gamma}_0\,.$$

Furthermore, agent 0 owns the whole economy in the long run; that is,

$$\lim_{T \to \infty} W_{0\,\lambda\,T}\, S_{\lambda T}^{-1} \;=\; 1 \tag{3.15}$$

P-almost surely for any $\lambda \in (0,1]$.[14]

It is instructive to compare the result for survival, under the physical measure, with an analogous result in the presence of intermediate consumption. In a beautiful recent paper, Yan (2008) considers the same model as we do but with intermediate consumption. He shows that an agent i, with risk aversion γ_i, beliefs δ_i and discount rate ρ_i, survives in the long run if and only if his *survival index*[15]

$$I_i \;=\; 0.5\,\delta_i^2\,\sigma^2 \,+\, \rho_i \,+\, \gamma_i\,\mu$$

is the lowest among all the agents. In particular, because the endowment is unbounded, the elasticity of intertemporal substitution (EIS)[16] matters for survival.[17] In particular, if preferences vary across investors, even slightly, it becomes possible for an irrational investor to dominate the market in the

---

[14]Note that $S_T = D_T$.

[15]Yan uses δ_i to denote the agent's beliefs about the market price of risk, whereas our δ_i characterizes beliefs about the Sharpe ratio. For this reason, our δ_i coincides with Yan's δ_i, divided by σ.

[16]Because preferences are time additive, EIS is the reciprocal of the risk aversion.

[17]This result stands in stark contrast to previous results (see Blume and Easley (2006)), which only consider bounded endowments and thus cannot cover the benchmark case of geometric Brownian motion.

long run. However, if both time and risk preferences are held constant across agents, then those with incorrect beliefs cannot survive in the long run. The general result of Proposition 3.6 is similar to that of Yan (2008): the agent with the lowest survival index $|1 - \tilde{\gamma}_i|$ dominates in the long run, and this survival index also depends on the agent's risk aversion. However, as in KRWW (2006), irrational traders may survive even when they have the same preferences as rational agents. As Yan (2008) observes, what matters for survival is whether investors optimize over savings decisions. In models where agents optimize only over portfolio choice but not savings, wrong beliefs might lead to higher wealth growth rate and thus be beneficial for survival. However, in models where investors maximize over both portfolios and savings, an incorrect belief is always a disadvantage for survival.

In contrast to survival under the physical measure, the fact that agent 0 also survives under the dividend-forward measure is not obvious. The reason is that, even though the measures P, Q and Q^D are equivalent for every finite T, they will typically be singularly continuous with respect to each other at $T = \infty$. This is precisely the fact giving rise to most of the phenomena discussed in both our paper and that of KRWW (2006). To illustrate this, let us consider the case of a homogeneous economy, populated solely by agent 0. In this case, the densities of Q and Q^D are given by $e^{-\tilde{\gamma}_0 \sigma B_T - 0.5 \tilde{\gamma}_0^2 \sigma^2 T}$ and $e^{(1-\tilde{\gamma}_0) \sigma B_T - 0.5 (1-\tilde{\gamma}_0)^2 \sigma^2 T}$ respectively. By the Girsanov Theorem, B_t will have a drift $-\tilde{\gamma}_0$ under Q and $1 - \tilde{\gamma}_0$ under Q^D. By the strong law of large numbers for Brownian motion, we have, as $t \to \infty$:

$$
\begin{array}{rcll}
B_t/t & \to & 0 & P - a.s. \quad\quad (3.16) \\
B_t/t & \to & -\tilde{\gamma}_0 & Q - a.s. \quad\quad (3.17) \\
B_t/t & \to & 1 - \tilde{\gamma}_0 & Q^D - a.s. \quad\quad (3.18)
\end{array}
$$

This means that the measure P is supported on those paths of B_t that stay equal to 0 on average, Q is supported on those paths of B_t that decrease as $-\tilde{\gamma}_0 t$ on average, and Q^D is supported on those paths of B_t that grow/decrease as $(1-\tilde{\gamma}_0) t$ on average. Thus, the agent surviving with respect

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to Q is the one who puts the largest weight on the paths of B_t that decrease as $-\tilde{\gamma}_0 t$ on average, and the agent surviving with respect to Q^D is the one who puts the largest weight on those paths of B_t that grow/decrease as $(1 - \tilde{\gamma}_0) t$ on average. This calculation explains why the agent 0 survives under the dividend-forward measure: as agent i starts owning a large part of the economy in some states for large t, the density of the dividend-forward measure starts converging to $e^{(1-\tilde{\gamma}_i) \sigma B_T - 0.5 (1-\tilde{\gamma}_i)^2 \sigma^2 T}$. In those states, the absolute drift of B_t under Q^D will be close to $|1 - \tilde{\gamma}_i|$. Because agent 0 survives under P, he puts the largest weight on the paths with zero and/or very small absolute drift. Because the absolute drift $|1 - \tilde{\gamma}_0|$ is the smallest for agent 0, he will also put the largest weight on the paths with the drift $1 - \tilde{\gamma}_0$ and, consequently, will survive with respect to Q^D.

The situation with the risk-neutral measure Q is completely different. As we will see below, the agents that do not survive with respect to the physical measure P may still survive with respect to the risk-neutral measure Q and therefore, by Propositions 3.4 and 3.5, may have a long-run price impact.

From now on, we will assume that the agents are ordered in such a way that their effective risk aversions are increasing:

$$\tilde{\gamma}_{-k} < \cdots < \tilde{\gamma}_{-1} < \tilde{\gamma}_0 < \tilde{\gamma}_1 < \cdots < \gamma_l \tag{3.19}$$

where $k + l + 1 = K$. By the definition of $\tilde{\gamma}_0$, we must have $\tilde{\gamma}_1 > 1 > \tilde{\gamma}_{-1}$ and $\tilde{\gamma}_0 - 1 > 1 - \tilde{\gamma}_1$. Therefore, $2/(\tilde{\gamma}_0 + \tilde{\gamma}_1) < 1$.

Define the intervals

$$
\begin{aligned}
\Pi_l &= \left(0 , \; \frac{2}{\tilde{\gamma}_{l-1} + \tilde{\gamma}_l} \right) , \\
\Pi_0 &= \left(\frac{2}{\tilde{\gamma}_0 + \tilde{\gamma}_1} , \; 1 \right) \qquad \text{and} \\
\Pi_i &= \left(\frac{2}{\tilde{\gamma}_i + \tilde{\gamma}_{i+1}} , \; \frac{2}{\tilde{\gamma}_{i-1} + \tilde{\gamma}_i} \right)
\end{aligned}
\tag{3.20}
$$

for $i = 1, \cdots, l - 1$.

The main result of this section is the following theorem.

Theorem 3.1. *Agent i survives with respect to the risk-neutral measure for $t = \lambda T$ if and only if $i \geq 0$ and $\lambda \in \Pi_i$. For $\lambda \in \Pi_i$ and $t = \lambda T$, we have*

$$\kappa_t \; \sim \; \sigma \, \tilde{\gamma}_i \qquad , \qquad \sigma_t^S \; \sim \; \sigma \left(1 + \tilde{\gamma}_i - \tilde{\gamma}_0 \right). \qquad (3.21)$$

Theorem 3.1 shows that extinction of nonsurviving agents is a gradual process. When $t = \lambda T$ is small, even agents with high values of effective risk aversion have price impact. However, as t increases and approaches T, the price impact of nonsurviving agents gradually decreases, and finally, for $\lambda \in \Pi_0$, price impact vanishes completely, and both the market price of risk and volatility converge to their respective values in the single-agent economy populated solely by the surviving agent 0.

The fact that, except for the surviving agent 0, only agents with effective risk aversion higher than 1 have price impact is surprising. Intuitively, given the result of Proposition 3.6, we might expect that the strength of an agent's price impact depends on the distance of his effective risk aversion from one and thus that it should be symmetric for risk aversions both above and below one. Theorem 3.1 implies that this is not true. The reason is that agents with effective risk aversions below 1 take very risky positions in the stock, betting on the realizations of states with very high values of the aggregate dividend D_T. Because the probability of such states occurring is very low, these agents become extinct so quickly that they do not have any long-run price impact. The behavior of agents with effective risk aversion above 1 is very different: Rather than betting on the realizations of states with extreme values of D_T, they form balanced portfolios to achieve a smooth distribution of wealth across states. In particular, in the extreme case of his risk aversion increasing to infinity, an agent will only invest in bonds and will achieve a nonrandom terminal wealth equally distributed across states. This stable, cautious behavior allows those more risk-averse agents to have a long-run price impact.

In the case of a two-agent economy populated by a rational trader r and an irrational trader n, both with risk aversion γ, the result of Theorem 3.1 has also been shown by KRWW (2006). Because risk aversion is constant across

traders, equilibrium state price density can be found in closed form, and this substantially facilitates the analysis. When risk aversion is heterogeneous, state price density cannot be calculated in closed form, and the techniques of KRWW cannot be applied. Our proof of Theorem 3.1 is based on new, powerful techniques that allow us to study the case of an arbitrary number of agents with arbitrary heterogeneous risk aversions and beliefs.

KRWW (2006) show that, when $\gamma \geq 1$, the irrational trader survives only if he is moderately optimistic. The reason is that, when δ_n is positive, $\tilde{\gamma}_n = \gamma - \delta_n < \tilde{\gamma}_r$ and therefore, when δ_n is not too large, $\tilde{\gamma}_n$ will be closer to 1 than γ. Consequently, by (3.14), the irrational agent will be agent 0. When the irrational agent is pessimistic, $\tilde{\gamma}_n > \gamma = \tilde{\gamma}_r \geq 1$ and Theorem 3.1 implies that agent n does not survive, but does have a price impact. If $\gamma < 1$, then the irrational agent will not survive, but he will have a price impact if and only if he is sufficiently pessimistic. Thus, in the framework of KRWW (2006), a nonsurviving irrational agent can have price impact only if he is pessimistic. Because most stock market participants seem to be overoptimistic[18] and because, by Proposition 3.2, the presence of optimistic traders is crucial for generating the empirically observed state price smile effect, it is important to know whether nonsurviving optimistic traders can have long-run price impact.[19]

Theorem 3.1 shows that this is indeed possible when risk aversions are heterogeneous. In particular, if the rational trader is less risk averse than the irrational one, i.e., $\gamma_r < \gamma_n$ and the irrational trader is not too optimistic, so that $\tilde{\gamma}_n = \gamma_n - \delta_n > \gamma_r$, then the irrational trader will not survive, but he will have price impact.

As KRWW (2006) note, the fact that nonsurviving agents can have long-run price impact can be explained by the fact that irrational traders bet on the realizations of highly improbable extreme states of the economy.

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[18]Dimson, Marsh and Staunton (2003).

[19]If overoptimistic traders do not survive, they will only generate smile in the finite horizon state price density, and this smile will vanish in the long run. In particular, investigating how the empirically observed smile effect depends on the time horizon would give insight into how quickly the price impact of overoptimistic traders vanishes.

Consequently, the state prices of these extreme states are significantly affected by these irrational traders, even when their share of total wealth is small. If these extreme states correspond to very low levels of aggregate dividend, the corresponding state prices will be very high and will therefore have a significant impact on long-run asset prices. This important insight naturally leads to the question: "what are the states through which irrational traders impact asset prices?" Theorem 3.1 and Propositions 3.4, 3.5 provide a complete answer to this question. By Theorem 3.1, agent i has long-run price impact for $\lambda \in \Pi_i$ because he is the only one who survives under the risk-neutral measure. Because, by Theorem 3.1, the market price of risk is approximately given by $\sigma \tilde{\gamma}_i$, the same argument as that in (3.16)-(3.18) implies that the risk-neutral measure is supported on the paths of B_t satisfying

$$B_t \sim -\sigma \tilde{\gamma}_i t. \tag{3.22}$$

Consequently, for $\lambda \in \Pi_i$, the agent i has price impact because he places the largest weight on the paths of B_t satisfying (3.22). This is precisely the set of states that drives the long-run price impact at time $t = \lambda T$.

The nature of these "price impact sets" is quite intuitive. The above-mentioned paths of B_t correspond to the states of very low aggregate dividend D_t. The marginal utilities in these states are high, and therefore the corresponding state prices are also high and have a large impact on current prices. When λ is small, there are still many paths of B_t that can decay very fast, as $-\sigma \tilde{\gamma}_l t$ (here, $\tilde{\gamma}_l$ is the largest effective risk aversion in the economy). As the most risk-averse agent, l has the highest marginal utility in those states and therefore uses a large part of his wealth to hedge against those low-wealth states and has an impact on the stock price. As λ increases, the probability of such a fast decay of B_t gets smaller, and the agents with high risk aversion overhedge against those extreme states and gradually lose their price impact. Finally, when $\lambda \in \Pi_0$, the paths that decay faster than $-\sigma \tilde{\gamma}_0 t$ become highly unlikely, and agent 0 takes the best bets and is the only one with an impact on the price.

As Yan (2008, Appendix B.2) observes, there is an important difference

between the survival/price impact results of KRWW (and, of course, of those in our paper) and the analogous results in infinite horizon models with intermediate consumption, such as those of Blume and Easley (2006) and Yan (2008). When we change the horizon of the economy in our model, we get a different model with different portfolio policies and different equilibrium price dynamics. In contrast, in Blume and Easley (2006) and Yan (2008), the horizon T is infinite from the very beginning, and therefore the equilibrium dynamics are fixed. The way to interpret the survival and price impact results in our paper is to look at the explicit upper bounds on the rates of convergence, derived in the appendix. If the convergence takes place at a rate ρ, then, effectively, when the horizon T is larger than ρ^{-1}, the probability that the variable in question (e.g., the market price of risk) significantly deviates from the limit will be very small.

The discussion in the previous paragraph naturally leads to the question of whether nonsurviving agents can have price impact in models with intermediate consumption. As Yan (2008) shows, equilibrium state prices, stock prices and interest rates always converge to those in the homogeneous economy populated by the single surviving agent. A modification of Yan's argument implies that the market price of risk is also determined solely by the single surviving agent.[20] KRWW (2010) prove that this is a general fact for models with CRRA preferences. They show that in order to have nonsurviving agents who impact the long-run equilibrium state prices, it is necessary to assume utilities with an unbounded relative risk aversion that grows sufficiently fast at infinity. An important insight of KRWW (2010) is that, in a model with intermediate consumption and bounded risk aversion, the volatility of the consumption share of a nonsurviving agent always vanishes relative to the volatility of the endowment growth. By contrast, when risk aversion is unbounded and grows sufficiently fast at infinity, it may happen that this volatility does not vanish, which, in turn, may lead to long-run price impact. Below, we show that the link between price impact and survival under different measures is also present in models

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[20]Cvitanić, Jouini, Malamud and Napp (2009) show that the same is true for stock volatility and optimal portfolios.

with intermediate consumption when we consider assets with long maturities.

We complete this section with an interesting observation about the equilibrium of stock volatility. It follows directly from Theorem 3.1 that, for all $\lambda \in (0,1) \setminus \Pi_0$, the asymptotic stock price volatility is strictly larger than the dividend volatility. This is in fact true also for finite values of T, as is shown by the following proposition.

Proposition 3.7. *The stock price volatility is always higher than the dividend volatility.*

By Proposition 3.1, aggregate effective risk aversion $\tilde{\gamma}^U(D_T)$ is monotone decreasing in D_T. Therefore the covariance of $\tilde{\gamma}^U$ with D_T is negative, and Proposition 3.7 follows directly from (3.7). The intuition behind this "excess volatility" is as follows: because the aggregate risk aversion is decreasing, the market price of risk is low in good states and high in bad states. This makes the price go up in good states and, by similar arguments, go down in bad states, driving price volatility up. Put differently, the tension between the movements in the future dividend and the market price of risk creates excess volatility, because the larger the dividend, the larger the demand for the stock, while at the same time, the lower the market price of risk, the lower the demand.[21]

4 Survival and Portfolio Impact

4.1 General Results

The starting point for our analysis is the following representation for equilibrium optimal portfolios.

Proposition 4.1. *The equilibrium optimal portfolio weight of agent i is given by*

$$\pi_{it} = \pi_{it}^{\text{myopic}} + \pi_{it}^{\text{hedging}}$$

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[21]The fact that heterogeneity often leads to excess volatility is also shown by Dumas, Kurshev and Uppal (2009) and Bhamra and Uppal (2009).

where the myopic component of the optimal portfolio is given by

$$\pi_{it}^{\mathrm{myopic}} \;=\; \frac{\kappa_t \,+\, \delta_i\,\sigma}{\gamma_i\,\sigma_t^S} \tag{4.1}$$

and the hedging component is given by

$$\pi_{it}^{\mathrm{hedging}} \;=\; (\gamma_i^{-1}\,-\,1)\,(\sigma_t^S\,W_{it})^{-1}\,\sigma\,\mathrm{Cov}_t^Q\big(\tilde{\gamma}^U(D_T)\,,\,W_{iT}\big)\,. \tag{4.2}$$

By the definition of δ_i, agent i believes that the market price of risk equals $\kappa_t\,+\,\delta_i\,\sigma$. Therefore, $\pi_{it}^{\mathrm{myopic}}$ is the standard Merton's instantaneous mean-variance efficient optimal portfolio. This is the portfolio that agent i would hold if he ignored future fluctuations in the investment opportunity set and assumed that the market price of risk does not fluctuate.

The second, nonmyopic part of the optimal portfolio is responsible for hedging against future fluctuations of the market price of risk. By (4.2), it is proportional to the covariance of the future market price of risk[22] with the future agent's wealth and vanishes identically when $\gamma_i\,=\,1$; that is, when the agent is completely myopic. To understand the intuition behind this representation, let us consider the case in which risk aversion γ_i is sufficiently high (above one) and the future market price of risk negatively covaries with the agent's future wealth. Because the agent's risk aversion is high, his marginal utility in the states with low wealth level is high and so he wants to hedge against those states. Because, by assumption, the market price of risk negatively covaries with the agent's wealth, it is high in low-wealth states. This makes the stock a highly attractive instrument for hedging against these low-wealth states and gives rise to a positive hedging demand. Similar arguments apply when risk aversion is below one and/or the market price of risk positively covaries with wealth.

In order to formulate the results for portfolio impact, we will need the following definition.

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[22]By Proposition 3.3, $\kappa_T\,=\,\sigma\,\tilde{\gamma}_T^U\,.$

Definition 4.1. *The probability measure*

$$dQ^{W_i} \stackrel{def}{=} \frac{W_{iT}}{E^Q[W_{iT}]} dQ$$

will be referred to as the agent i's wealth-forward measure.

The measure dQ^{W_i} places large weight on the states in which agent i's wealth level is high and small weight on the states in which agent's wealth is low. It is the agent-specific analog of the dividend-forward measure.[23] Using this measure, we can rewrite representation (4.2) for the hedging portfolio in the form

$$\pi_{it}^{\text{hedging}} = (\gamma_i^{-1} - 1)(\sigma_t^S)^{-1} \sigma \left(E_t^{Q^{W_i}}[\tilde{\gamma}^U(D_T)] - E_t^Q[\tilde{\gamma}^U(D_T)] \right). \quad (4.3)$$

Note that this representation is very similar to the analogous representation (3.13) for stock volatility.[24] Therefore, the same argument as in the proof of Proposition 3.4 implies that the following is true.

Proposition 4.2. *An agent j has a long-run impact on the agent i's optimal portfolio for $t = \lambda T$ if and only if he survives either with respect to the agent i's wealth-forward measure, or with respect to the equilibrium risk-neutral measure.*[25]

Naturally, an agent j surviving under the risk-neutral measure has impact on other agents' optimal portfolios because, by Theorem 3.1, agent j has impact on the long-run market price of risk and volatility. In particular, it follows directly from Theorem 3.1 that, for $\lambda \in \Pi_j$ and $t = \lambda T$,

$$\pi_{it}^{\text{myopic}} \sim \frac{\tilde{\gamma}_j + \delta_i}{\gamma_i(1 + \tilde{\gamma}_j - \tilde{\gamma}_0)}.$$

---

[23]We can view the aggregate dividend as the terminal wealth of the representative agent.

[24]The reason is that, by the Ito formula, $\pi_{it}\sigma_t^S$ is the volatility of agent i's wealth process W_{it}, whereas σ_t^S is the volatility of the representative agent's wealth process S_t.

[25]As in Proposition 3.5, if agent i is the only one surviving with respect to both measures, the effects will cancel each other in (4.3), and σ_t^S will coincide with the dividend volatility σ. In this case, we still say that agent j has impact on agent i's optimal portfolio.

By contrast, the fact that an agent surviving under the agent i's wealth-forward measure can also have impact on the agent i's optimal portfolio, even if he does not have any price impact, is surprising. The economic mechanism responsible for this phenomenon comes from the equilibrium hedging behavior. In contrast to the myopic part of the optimal portfolio, the hedging demand is driven not by the size of the market price of risk but rather by its future fluctuations. Formula (4.2) shows that the agent hedges against these fluctuations depending of the comovement of the market price of risk with his wealth. In the long run, what matters for the agent is the behavior of the market price of risk in the extreme states, in which his wealth is either very low or very high. The agent i's wealth-forward measure precisely captures this effect.

In order to determine which agents survive with respect to the agent i's wealth-forward measure, we will need several definitions. For simplicity, we will only consider the case of $\gamma_i > 1$. The results in the case $\gamma_i < 1$ are similar.

Recall that, by definition (see (3.19)), we have

$$i > 0 \iff \tilde{\gamma}_i > \max\{\tilde{\gamma}_0\,,\, 1\}\,.$$

Fix an agent i with risk aversion $\gamma_i > 1$, and define intervals $\Theta_1^i, \cdots, \Theta_i^i$ as follows.

- If $i \geq 1$, we define

$$\Theta_0^i \;=\; \left(\frac{\tilde{\gamma}_i - 0.5\,(\tilde{\gamma}_0 + \tilde{\gamma}_1)}{0.5(\tilde{\gamma}_0 + \tilde{\gamma}_1)(\gamma_i - 1) - \delta_i}\,,\, 1 \right)$$

and, for $j \in \{1, \ldots, i-1\}$

$$\Theta_j^i \;=\; \left(\frac{\tilde{\gamma}_i - 0.5\,(\tilde{\gamma}_j + \tilde{\gamma}_{j+1})}{0.5(\tilde{\gamma}_j + \tilde{\gamma}_{j+1})(\gamma_i - 1) - \delta_i}\,,\, \frac{\tilde{\gamma}_i - 0.5\,(\tilde{\gamma}_{j-1} + \tilde{\gamma}_j)}{0.5(\tilde{\gamma}_{j-1} + \tilde{\gamma}_j)(\gamma_i - 1) - \delta_i} \right)$$

and, finally,

$$\Theta_i^i \;=\; \left(0\,,\, \frac{\tilde{\gamma}_i - 0.5\,(\tilde{\gamma}_{i-1} + \tilde{\gamma}_i)}{0.5(\tilde{\gamma}_{i-1} + \tilde{\gamma}_i)(\gamma_i - 1) - \delta_i} \right).$$

- If $i \leq -1$, we define

$$\Theta_0^i = \left(\frac{0.5\left(\tilde{\gamma}_0 + \tilde{\gamma}_{-1}\right) - \tilde{\gamma}_i}{\delta_i - 0.5(\tilde{\gamma}_0 + \tilde{\gamma}_{-1})(\gamma_i - 1)}, 1 \right)$$

and, for $j \in \{i+1, \cdots, -1\}$

$$\Theta_j^i = \left(\frac{0.5\left(\tilde{\gamma}_j + \tilde{\gamma}_{j+1}\right) - \tilde{\gamma}_i}{\delta_i - 0.5(\tilde{\gamma}_j + \tilde{\gamma}_{j+1})(\gamma_i - 1)}, \frac{0.5\left(\tilde{\gamma}_{j-1} + \tilde{\gamma}_j\right) - \tilde{\gamma}_i}{\delta_i - 0.5(\tilde{\gamma}_{j-1} + \tilde{\gamma}_j)(\gamma_i - 1)} \right)$$

and, finally,

$$\Theta_i^i = \left(0, \frac{0.5\left(\tilde{\gamma}_{i+1} + \tilde{\gamma}_i\right) - \tilde{\gamma}_i}{\delta_i - 0.5(\tilde{\gamma}_{i+1} + \tilde{\gamma}_i)(\gamma_i - 1)} \right).$$

Now we are ready to state the following theorem.

Theorem 4.1. *The following is true.*

(1) *if $i \geq 1$, then an agent j survives with respect to the agent i's wealth-forward measure for $t = \lambda T$ if and only if $\tilde{\gamma}_0 \leq \tilde{\gamma}_j \leq \tilde{\gamma}_i$ and $\lambda \in \Theta_j^i$;*

(2) *if $i \leq -1$, then an agent j survives with respect to the agent i's wealth-forward measure for $t = \lambda T$ if and only if $\tilde{\gamma}_0 \geq \tilde{\gamma}_j \geq \tilde{\gamma}_i$ and $\lambda \in \Theta_j^i$;*

(3) *if $i = 0$, then only agent 0 survives with respect to his own wealth-forward measure.*

Consequently,[26] for any $i \neq 0$, $\lambda \in \Theta_j^i \cap \Pi_m$ and $t = \lambda T$, we have

$$\pi_{it}^{\text{hedging}} \sim (\gamma_i^{-1} - 1) \frac{\tilde{\gamma}_j - \tilde{\gamma}_m}{1 + \tilde{\gamma}_m - \tilde{\gamma}_0}. \tag{4.4}$$

For $i = 0$, $\lambda \in \Pi_m$ and $t = \lambda T$, we have

$$\pi_{0t}^{\text{hedging}} \sim 1 - \pi_{0t}^{\text{myopic}} \sim (\gamma_0^{-1} - 1) \frac{\tilde{\gamma}_0 - \tilde{\gamma}_m}{1 + \tilde{\gamma}_m - \tilde{\gamma}_0}.$$

---

[26]Formula (4.4) follows directly from Theorem 3.1, Proposition 4.2, (4.3) and items (1)-(2) above.

The result of Theorem 4.1 is quite surprising. Because, by Theorem 3.1, both the market price of risk and the stock volatility are deterministic functions of $\lambda = t/T$ for large T, it is natural to expect that the asymptotic optimal portfolios are myopic. Theorem 4.1 shows that this is not true and that the hedging components of the optimal portfolios may stay significantly different from zero for very large fractions of the economy's horizon. As already noted by KRWW (2006), in the special case of two agents with identical risk aversion, the nature of this phenomenon is similar to that of the price impact of nonsurviving agents. Even though, for large T, both the market price of risk and volatility are almost deterministic, there are still extreme states in which they deviate from their asymptotic values. These deviations may have a significant effect on the agents' hedging demands and may give rise to nontrivial hedging portfolios.

As in the case of price impact discussed in the previous section, it is important to know which states are responsible for portfolio impact and how precisely this portfolio impact is generated. Our results provide complete answers to these questions.

By Proposition 4.2, these "portfolio impact states" constitute the support of the agent i's wealth-forward measure Q^{W_i}. By the same argument as that in (3.16)-(3.18), Theorem 4.1 implies that for $\lambda \in \Theta_j^i$, the measure Q^{W_i} is supported on the paths of B_t satisfying

$$B_t \sim -\sigma \, \tilde{\gamma}_j \, T \,. \tag{4.5}$$

These are the paths determining the "portfolio impact states." Clearly, they correspond to states with very low values of D_t and, consequently, very high marginal utilities. When λ is small, the Q^{W_i}-probability of such extreme paths is sufficiently large, and agent j puts a large weight on these states and has a significant price impact. However, as λ increases, Q^{W_i}-probability of these states decreases and eventually becomes negligible, so that, for $\lambda \in \Theta_0^i$, only the surviving agent 0 impacts the agent i's optimal portfolio.

Theorem 4.1 implies that only agents with relatively similar risk aversion and belief profiles survive under agent i's wealth-forward measure. That is,

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only agents that are closer to the surviving agent 0 and are on the same side of 1 survive under Q^{W_i}. This is very intuitive: As these agents are closer to the agent 0, they survive longer than agent i, and will therefore have a longer impact on his portfolio. In contrast, agents with effective risk aversion on the other side of 1 take hedging positions in a different direction. Therefore, the states in which their wealth shares are large will be very different from the states in which agent i's wealth share is large. Consequently, they will not survive under Q^{W_i}. Finally, because agent 0 owns the whole economy in the long run, his total portfolio weight should converge to 1 as $T \to \infty$. Therefore, nonsurviving agents have impact on his hedging portfolio only if they have impact on the market price of risk and volatility. The fact that nonsurviving agents cannot survive under agent 0's wealth-forward measure also follows from Proposition 3.6: For large T, $W_{0T} \approx D_T$ and therefore, surviving under the agent 0's wealth-forward measure is equivalent to surviving under the dividend-forward measure.

As time goes by, more and more nonsurviving agents lose their price impact, and the instantaneous moments of the stock price converge to their asymptotic constant values, determined by the surviving agent 0. Therefore, intuitively, one might expect that the equilibrium hedging demand decreases with time. Theorem 4.1 shows that this is not the case. In fact, the dependence (4.4) of the hedging demand on $\lambda = t/T$ may exhibit nontrivial patterns, and we will discuss this in detail in the "Examples" section below. Here, we only take a quick look at the effectively most risk-averse agent l. It follows from Theorem 4.1 that this agent is the only one whose hedging demand is zero for small values of t/T. Indeed, by (4.3), $\pi_{it}^{\text{hedging}} = 0$ for $t/T = \lambda \in \Theta_i^i \cap \Pi_l$ if and only if $\tilde{\gamma}_i = \tilde{\gamma}_l$. The reason is that, for very low values of λ, agent l survives under both the risk-neutral and his own wealth-forward measures. Therefore, the long-run fluctuations of the market price of risk do not matter for him for small t, and he does not hedge against them. By contrast, agents with lower effective risk aversions will take advantage of the very high values ($\sim \sigma \tilde{\gamma}_l$) that the equilibrium market price of risk will take in the states in which agent l survives; therefore, they hold significant

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positive hedging portfolios.

We now discuss the relationship between price impact and portfolio impact. Propositions 3.4 and 4.2 imply that price impact and portfolio impact are two independent concepts because they are equivalent to survival under different measures. However, it is not, a priori, obvious whether it is possible for a nonsurviving agent with no price impact to have a long-run impact on the portfolios of other agents. The following result shows that this is impossible in two-agent economies.

Proposition 4.3. *Let $K = 2$. Then, a nonsurviving agent with no price impact cannot have portfolio impact.*

Indeed, by Theorem 3.1, an agent j has no price impact if and only if $\tilde{\gamma}_j < 1$. If he does not survive, Theorem 4.1 implies that he can only have impact on agent i's optimal portfolio if $\tilde{\gamma}_i < \tilde{\gamma}_j$. However, this means that agent j survives and not agent i, which is a contradiction.

Thus, in two-agent economies, price impact and portfolio impact are equivalent. This was also shown by KRWW (2006) in the case of identical risk aversions. Theorem 4.1 implies that this is no longer the case for economies with more than two agents. Indeed, suppose that $k > 1$;[27] that is, there exist at least two nonsurviving agents with effective risk aversion lower than one. Then, by Theorem 3.1, agent -1, whose effective risk aversion is the highest among those below 1, will neither survive nor have price impact. Nevertheless, by Theorem 4.1, he will have long-run impact on the optimal portfolios of all agents i with $i < -1$. We discuss this example in greater detail below.

4.2 Examples

Example 1. As an illustration of the portfolio impact phenomenon discussed above, consider the case in which there are three agents with

$$\tilde{\gamma}_{-2} < \tilde{\gamma}_{-1} < \tilde{\gamma}_0 < 1 .$$

_____

[27]See (3.19).

By Theorem 3.1, neither one of agents -1 and -2 has price impact, and therefore

$$\kappa_t \sim \sigma \tilde{\gamma}_0 \ , \ \sigma_t \sim \sigma$$

for all $t = \lambda T$, $\lambda \in (0,1)$. Therefore, the portfolio of agent 0 is completely myopic and equals 1 because, in the long run, he owns the whole economy. However, agents -1 and -2 hold nontrivial hedging portfolios for a significant fraction of the time horizon. For agent -1, Theorem 4.1 implies that the hedging portfolio

$$\pi_{-1\,t}^{\text{hedging}} \ = \ (\gamma_{-1}^{-1} - 1) \begin{cases} \tilde{\gamma}_{-1} - \tilde{\gamma}_0 \ , & \lambda < \frac{0.5(\tilde{\gamma}_0 - \tilde{\gamma}_{-1})}{\delta_{-1} - 0.5(\tilde{\gamma}_0 + \tilde{\gamma}_{-1})(\gamma_{-1} - 1)} \\ 0 \ , & \lambda > \frac{0.5(\tilde{\gamma}_0 - \tilde{\gamma}_{-1})}{\delta_{-1} - 0.5(\tilde{\gamma}_0 + \tilde{\gamma}_{-1})(\gamma_{-1} - 1)} \end{cases}$$

is impacted by his own preferences and, of course, by the single surviving agent 0, who determines the long-run asset prices.

By Theorem 4.1, the hedging portfolio of agent -2 is given by

$$\begin{aligned} \pi_{-2\,t}^{\text{hedging}} \ &\sim \ (\gamma_{-2}^{-1} - 1) \\ &\times \begin{cases} \tilde{\gamma}_{-2} - \tilde{\gamma}_0 \ , & \lambda < \lambda_1 \\ \tilde{\gamma}_{-1} - \tilde{\gamma}_0 \ , & \lambda_1 < \lambda < \lambda_2 \\ 0 \ , & \lambda_2 < \lambda \, . \end{cases} \end{aligned} \tag{4.6}$$

where

$$\lambda_1 \ = \ \frac{0.5(\tilde{\gamma}_{-1} - \tilde{\gamma}_{-2})}{\delta_{-2} - 0.5(\tilde{\gamma}_{-1} + \tilde{\gamma}_{-2})(\gamma_{-2} - 1)} \ , \ \lambda_2 \ = \ \frac{0.5(\tilde{\gamma}_{-1} + \tilde{\gamma}_0) - \tilde{\gamma}_{-2}}{\delta_{-2} - 0.5(\tilde{\gamma}_{-1} + \tilde{\gamma}_0)(\gamma_{-2} - 1)} \, .$$

Thus, even though agent -1 neither survives nor has any price impact, he has a significant long-run impact on the portfolio of agent -2. As we explain above, this is a new phenomenon that can only occur in models with at least three agents. The impact is the strongest when the effective risk aversion $\tilde{\gamma}_{-1}$ of agent -1 is sufficiently close to that of -2. This is very intuitive: because agent -1 has preferences similar to those of agent -2, he will place the largest weight on similar states of the world. As a consequence, he will survive with respect to agent -2's wealth-forward measure and will have a

long-run impact on his optimal portfolio.

Example 2. Let $K = 2$ and a be the surviving agent (i.e., agent 0) and b the nonsurviving agent. If $\tilde{\gamma}_b > 1$, then, by Theorem 3.1, we get, for $t = \lambda T$,

$$\kappa_t \sim \sigma \begin{cases} \tilde{\gamma}_b \,, & \lambda < \lambda_S \\ \tilde{\gamma}_a \,, & \lambda > \lambda_S \end{cases}$$

and

$$\sigma_t^S \sim \sigma \begin{cases} 1 + \tilde{\gamma}_b - \tilde{\gamma}_a, & \lambda < \lambda_S \\ 1 \,, & \lambda > \lambda_S \end{cases}$$

with

$$\lambda_S \overset{def}{=} \frac{2}{\tilde{\gamma}_a + \tilde{\gamma}_b} \,.$$

However, if $\tilde{\gamma}_b < 1$, then

$$\kappa_t \sim \sigma \tilde{\gamma}_a \,, \ \sigma_t^S \sim \sigma$$

for all $\lambda \in (0, 1)$.

A special case of this example has been analyzed by KRWW (2006). Their economy is populated by two agents, a rational agent r and an irrational agent n with $\tilde{\gamma}_r = \gamma$, $\tilde{\gamma}_n = \gamma - \eta$. Thus, the threshold is $\lambda_S = 2/(2\gamma - \eta)$, and the irrational agent survives if and only if he is moderately optimistic:

$$|1 - \tilde{\gamma}_n| < |1 - \tilde{\gamma}_r| \ \Leftrightarrow \ 0 < \eta < \eta^* := 2(\gamma - 1) \,.$$

He does not survive, but still has price impact when he is pessimistic because

$$\eta < 0 \ \Leftrightarrow \ \tilde{\gamma}_n > \tilde{\gamma}_r$$

and he neither survives nor has price impact if he is too optimistic, because

$$\eta > \eta^* \ \Leftrightarrow \ 1 - \tilde{\gamma}_n > \gamma - 1.$$

Theorem 3.1 and the concept of effective risk aversion provide a unified

picture explaining how pessimism and optimism determine price impact and survival.

For the portfolio impact, we must define the corresponding thresholds. Because, for the surviving agent a, we have $\pi_{at} \sim 1$, we must have

$$\pi_{at}^{\text{hedging}} \sim 1 - \pi_{at}^{\text{myopic}}.$$

For the nonsurviving agent b, we need to define the threshold

$$\lambda_b = \frac{0.5(\tilde{\gamma}_b - \tilde{\gamma}_a)}{0.5(\tilde{\gamma}_a + \tilde{\gamma}_b)(\gamma_b - 1) - \delta_b}.$$

If $\tilde{\gamma}_b < 1$ then agent b has no price impact and therefore, by Theorem 4.1,

$$\pi_{bt}^{\text{hedging}} \sim (\gamma_b^{-1} - 1) \begin{cases} \tilde{\gamma}_b - \tilde{\gamma}_a, & \lambda < \lambda_b \\ 0, & \lambda > \lambda_b. \end{cases}$$

When $\tilde{\gamma}_b > 1$, agent b has portfolio impact, and therefore we need to consider, separately, the cases $\lambda_b > \lambda_S$ and $\lambda_b < \lambda_S$.

If $\lambda_b < \lambda_S$, Theorem 4.1 implies that

$$\pi_{bt}^{\text{hedging}} \sim (\gamma_b^{-1} - 1) \begin{cases} 0, & \lambda < \lambda_b \\ \frac{\tilde{\gamma}_a - \tilde{\gamma}_b}{1 + \tilde{\gamma}_b - \tilde{\gamma}_a}, & \lambda_b < \lambda < \lambda_S \\ 0, & \lambda > \lambda_S. \end{cases} \qquad (4.7)$$

If $\lambda_b > \lambda_S$, Theorem 4.1 implies that

$$\pi_{bt}^{\text{hedging}} \sim (\gamma_b^{-1} - 1) \begin{cases} 0, & \lambda < \lambda_S \\ \tilde{\gamma}_b - \tilde{\gamma}_a, & \lambda_S < \lambda < \lambda_b \\ 0, & \lambda > \lambda_b. \end{cases} \qquad (4.8)$$

As in Example 3 above, we obtain the counterintuitive result that the hedging demand is not monotone decreasing in λ. In fact, the hedging demand of the nonsurviving agent exhibits either a hump-shaped or a U-shaped pattern.

Our results provide a clear explanation for this result: for small λ, agent b survives under both the risk-neutral measure and agent b's wealth-forward measure, and therefore, from his point of view, the fluctuations of the market price of risk are irrelevant and do not need to be hedged against. For the intermediate periods, the agents surviving under the risk- neutral and the agent b's wealth-forward measures, respectively, are different[28] and, for this reason, the fluctuations of the market price of risk are significant for agent b and force him to hold a nontrivial hedging portfolio. Finally, for very large λ, only agent a survives under both measures, and the market price of risk fluctuations are again irrelevant for agent b, and his hedging demand vanishes.

5 Intermediate Consumption

In this section, we analyze a model in which agents derive utility from intermediate consumption. Namely, agent i maximizes

$$
E \left[\int_0^\infty Z_{it} \, e^{-\rho_i t} \, \frac{c_{it}^{1-\gamma_i} - 1}{1 - \gamma_i} \right],
$$

where $Z_{it} = e^{\delta_i \sigma B_t - 0.5 \sigma^2 \delta_i^2 t}$ is the density of agent i's subjective beliefs and ρ_i is agent i's time-preference rate. By contrast to the case of only terminal consumption, we assume here that the horizon T is infinite.[29]

The risky asset (the stock) is a claim on the dividend stream D_t and the stock price is given by

$$
S_t = E_t \left[\int_t^\infty \frac{\xi_\tau}{\xi_t} D_\tau \, d\tau \right].
$$

Here, $(\xi_\tau, \tau \geq 0)$ is the equilibrium state price density process. Standard results (see, e.g., Yan (2008)) imply that the optimal consumption stream of

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[28]That is, only one of the two agents a, b survives with respect to the risk neutral measure, whereas the other one survives with respect to the wealth-forward measure of agent b.

[29]In Appendix A we analyze the equilibrium behavior in the same model for the case when the horizon T is small.

agent i is given by

$$c_{it} = e^{-\rho_i b_i t} Z_{it}^{b_i} \xi_t^{-b_i} c_{i0},$$

with $b_i = \gamma_i^{-1}$. Initial consumption rate c_{i0} is determined by the budget constraint

$$E\left[\int_0^\infty \xi_t (c_{it} - \psi_i D_t) dt\right] = 0 \iff c_{i0} = \frac{\psi_i S_0}{E\left[\int_0^\infty e^{-\rho_i b_i t} Z_{it}^{b_i} \xi_t^{1-b_i} dt\right]}.$$

The equilibrium state price density process $(\xi_t, t \geq 0)$ is uniquely determined by the market clearing condition

$$\sum_i c_{it} = D_t.$$

This model has been studied by Yan (2008) and Cvitanić, Jouini, Malamud and Napp (2009). Following Yan (2008), we say that agent i survives in the long run if his fraction of aggregate consumption is nonzero in the long run; that is,

$$\limsup_{t\to\infty} c_{it} D_t^{-1} > 0,$$

with positive P-probability.

As we mentioned above,[30] Yan (2008) showed that only agent A with the lowest survival index $I_A = \min_i I_i$ with

$$I_i = 0.5\,\delta_i^2\,\sigma^2 + \rho_i + \gamma_i\,\mu \tag{5.1}$$

survives in the long run, and the equilibrium-state price density converges to that corresponding to agent A.

As in the terminal consumption case, we can define the representative agent whose marginal (time-dependent) utility $U_x(t,x)$ solves (see (3.2))

$$\sum_i c_{0i} \left(e^{(\rho_i + 0.5\delta_i^2\sigma^2 + \delta_i\mu)\,t}\, x^{-\delta_i}\, U_x(t,x)\right)^{-b_i} = x. \tag{5.2}$$

---

[30]See the discussion after Proposition 3.6.

Similarly,

$$\tilde{\gamma}^U(t, x) \;=\; -\,\frac{x\,U_{xx}(t, x)}{U_x(t, x)}$$

is the aggregate effective risk aversion. Consistent with Proposition 3.1, we have

$$\tilde{\gamma}^U(t, x) \;=\; \sum_i \tilde{\gamma}_i\,\lambda_{it} \qquad,\qquad \lambda_{it} \;=\; \frac{b_i\,c_{it}}{\sum_j b_i\,c_{jt}}\,. \qquad (5.3)$$

By contrast to models with only terminal consumption.[31] The market price of risk κ_t is uniquely determined by aggregate market risk aversion $\tilde{\gamma}_t$ at time t and aggregate risk σ. Namely,

$$\kappa_t \;=\; \tilde{\gamma}^U(t, D_t)\,\sigma\,.$$

The equilibrium interest rate r_t is determined by the instantaneous marginal rate of intertemporal substitution and coincides with the negative of the drift of the equilibrium state price density,

$$\xi_t^{-1}\,d\xi_t \;=\; -\,r_t\,dt \;-\; \kappa_t\,dB_t\,.$$

It follows directly from (5.3) and the argument in (3.11) that an agent has a long-run impact on the market price of risk if and only if he survives in the long run *under the physical measure*. Thus, in contrast to the terminal consumption case (see Proposition 3.6), survival under the physical measure is equivalent to the long-run market price of risk impact in the presence of intermediate consumption.

Denote by ξ_{it}, r_i and Y_i the state price density, the risk-free rate, and the dividend yield in the economy populated by a single agent i

$$\xi_{it} \;\stackrel{def}{=}\; e^{-(\rho_i + 0.5\,\sigma^2\delta^2 + \gamma_i\,\mu)\,t \,-\, \tilde{\gamma}_i\,\sigma\,B_t}\;,\quad r_i \;\stackrel{def}{=}\; \rho_i \;+\; \gamma_i\,\mu_i \;-\; 0.5\,\gamma_i^2\,\sigma^2$$

and

$$Y_i \;\stackrel{def}{=}\; \rho_i \;+\; (\gamma_i - 1)\,\mu_i \;-\; 0.5\,(\gamma_i - 1)^2\,\sigma^2\,.$$

---

[31]In the case of only terminal consumption, risk preferences only reveal an agent's attitude toward consumption risk at time T, and therefore the market price of risk κ_t is driven by the expectations of future aggregate risk and risk aversion; see Proposition 3.3.

Here, $\mu_i = \mu + \delta_i \sigma^2$ is the agent i's perceived dividend growth rate. As in Yan (2008), we assume that

$$Y_i > 0 \qquad \text{for all } i. \tag{5.4}$$

This condition guarantees that the equilibrium stock price is finite.

The equilibrium interest rate r_t is given by [32]

$$
\begin{aligned}
r_t &= \sum_i \lambda_{it} r_i - \sigma^2 \left(\sum_i \tilde{\gamma}_i \lambda_{it} \right) \left(\sum_i \tilde{\gamma}_i (1 - b_i) \lambda_{it} \right) \\
&+ 0.5 \sigma^2 \left(\sum_i (1 - b_i) \lambda_{it} \right) \left(\sum_i \tilde{\gamma}_i \lambda_{it} \right)^2 + 0.5 \sigma^2 \sum_i (1 - b_i) \tilde{\gamma}_i^2 \lambda_{it}.
\end{aligned}
\tag{5.5}
$$

Thus, similarly to the market price of risk, the size of agent i's impact on the interest rate r_t is determined by the weight λ_{it}. Because, by (5.3) and the equilibrium market clearing

$$
\lambda_{it} \in \left[\frac{b_i c_{it} D_t^{-1}}{\max_j b_j}, \frac{b_i c_{it} D_t^{-1}}{\max_j b_j} \right],
$$

we get that survival and long-run interest rate impact are also equivalent.

It is also possible to show that the stock price dividend ratio, the stock price volatility and the agents' optimal portfolios also converge to the values corresponding to the single surviving agent.[33] The reason is that the stock price, its volatility, and the agent's wealth are obtained by *integrating discounted future cash flows* over time. Thus, even though these future cash flows can deviate substantially from their almost sure asymptotic values, these deviations will have a negligible impact on the present values of integrated cash flows because of discounting. As an illustration, suppose that

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[32]Equation (5.5) follows from Ito's formula by differentiating (5.2) and because r_t is the negative drift of ξ_t.

[33]In particular, portfolio impact, market price of risk impact and stock price volatility impact are all equivalent to survival under the physical measure; see Cvitanić, Jouini, Malamud and Napp (2009).

the interest rate r is constant. Then

$$\frac{S_t}{D_t} \; = \; E_t^Q \left[\int_t^{t+T} e^{-r\,(\tau-t)} \frac{D_\tau}{D_t} \, d\tau \right] \; + \; e^{-r\,T} E_t^Q \left[\int_{t+T}^\infty e^{-r\,(\tau-(t+T))} \frac{D_\tau}{D_t} \, d\tau \right].$$
$$(5.6)$$

Under assumption (5.4), $E_t^Q \left[\int_{t+T}^\infty e^{-r\,(\tau-(t+T))} \frac{D_\tau}{D_t} \, d\tau \right]$ is uniformly bounded. Therefore, we can make its contribution to (5.6) arbitrarily small by making T sufficiently large. The result follows because for each finite T, as $t \to \infty$, the term $E_t^Q \left[\int_t^{t+T} e^{-r\,(\tau-t)} \frac{D_\tau}{D_t} \, d\tau \right]$ converges to that corresponding to the single surviving agent.[34]

Similarly, due to discounting, assuming that T is infinite from the beginning is equivalent to first taking a finite T and then sending it to infinity. Indeed, let P_t^T be the stock price when the horizon is equal to T. Then, the price–dividend ratios satisfy

$$\lim_{T \to \infty} D_{\lambda T}^{-1} S_{\lambda T}^T \; = \; \lim_{T \to \infty} D_{\lambda T}^{-1} S_{\lambda T}^\infty \; = \; \lim_{t \to \infty} D_t^{-1} S_t^\infty$$

because, due to discounting, the contribution of the dividends paid out for $t > T$ will be negligible for large T (see equation (5.6)).

We conclude that price impact and survival are always equivalent when we consider assets continuously paying dividends. In order to generate price impact effects similar to those in the model with only terminal consumption, we need to consider assets with long maturities and lump-sum payoffs. Here, we only examine two classes of such assets: long-maturity bonds and forward contracts.[35]

Let

$$\beta(t\,,\,T) \;\overset{def}{=}\; E_t \left[\frac{\xi_T}{\xi_t} \right] \;=\; E_t^Q \left[e^{-\int_t^T r_s \, ds} \right],$$

be the time t price of a zero coupon bond that pays one unit of consumption

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[34]The convergence of optimal portfolios and stock price volatility can be proved by a similar argument, using Proposition A.1 in Appendix.

[35]It is possible to extend the analysis to the case of futures, options and other derivatives with lump-sum payments at maturity.

at maturity T. Under the risk-neutral measure Q, we have

$$(\beta(t,T))^{-1} d\beta(t,T) \;=\; r_t\, dt \;+\; \sigma^B(t\,,\,T)\, dB_t^Q\,.$$

We will need the following.

Definition 5.1. *The T-forward measure Q^T is defined by*

$$dQ^T \;=\; \frac{e^{-\int_0^T r_s ds}}{\beta(0\,,\,T)}\, dQ\,.$$

By definition, the T forward measure uses the T-bond as the numeraire. Given an $\mathcal{F}_T$-measurable random variable X, its expectation under Q^T can be calculated as

$$E_t^{Q^T}[X] \;=\; \frac{E_t^Q[X\,\xi_T]}{E_t^Q[\xi_T]}\,.$$

The next result shows how the T-forward measure can be used to calculate the equilibrium bond volatilities. [36]

Proposition 5.1. *We have*

$$\sigma^B(t\,,\,T) \;=\; \kappa_t \;-\; E_t^{Q^T}[\kappa_T]\,. \tag{5.7}$$

Note that the representation (5.7) for bond volatilities is very similar to the representation (3.6) for the equilibrium market price of risk in the case of no intermediate consumption. Consistent with Definition 3.1, we say that an agent i has bond volatility impact for $t = \lambda T$ if the asymptotic behavior of $\sigma^B(t\,,\,T)$ depends on (at least one of) the agent's characteristics $(\rho_i\,,\,\gamma_i\,,\,\delta_i)$.

By the above, $\kappa_t \sim \sigma\,\gamma_A$ as $t \to \infty$.[37] Therefore, the same argument as in the proof of Proposition 3.4 implies the following result.

Proposition 5.2. *Agent i has a long-run impact on the bond volatility for $t = \lambda T$ if and only if he survives with respect to the T-forward measure Q^T.*

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[36]The proof of Proposition 5.1 is completely analogous to that of Proposition 3.3.

[37]Here, A is the single surviving agent under P; see (5.1).

If agent i is the only one who survives with respect to Q^T for $t = \lambda T$ then

$$\sigma^B(t, T) \sim \sigma(\tilde{\gamma}_A - \tilde{\gamma}_i).$$

In order to determine which agents do survive with respect to the T-forward measure, let us denote, for each i

$$l_i(\lambda) = I_i - 0.5(1 - \lambda)(\tilde{\gamma}_i \sigma)^2 = (1 - \lambda) r_i + \lambda I_i. \qquad (5.8)$$

We have the following result.[38]

Proposition 5.3. *An agent i survives with respect to the T-forward measure for $t = \lambda T$ if and only if*

$$l_i(\lambda) = \min_j l_j(\lambda).$$

Thus, using terminology analogous to that of Yan (2008), $l_i(\lambda)$ is the survival index of agent i with respect to the T-forward measure. Because $(l_i(\lambda), \lambda \in [0, 1])$ is a family of line segments, there exist pairs of values $((A_j, \lambda_j), j = 1, \cdots, K)$ such that

$$\min_i l_i(\lambda) = l_{A_j}(\lambda) \text{ for all } \lambda \in (\lambda_j, \lambda_{j+1}),$$

where $\lambda_0 = 0$ and $\lambda_{K+1} = 1$. It follows from the definition of $l_i(\lambda)$ that A_0 is the agent with the lowest "single-agent" interest rate $r_{A_0} = \min_i r_i$, whereas $A_K = A$ is the single surviving agent with the lowest survival index.

Combining Propositions 5.2 and 5.3, we arrive at the following analog of Theorem 3.1.

Proposition 5.4. *For $t = \lambda T$ and $\lambda \in (\lambda_j, \lambda_{j+1})$, we have*

$$\sigma^B(t, T) \sim \sigma(\tilde{\gamma}_A - \tilde{\gamma}_{A_j}).$$

The intuition behind Proposition 5.4 is similar to that behind Theorem

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[38]The proof of Proposition 5.3 is completely analogous to that of Theorem 3.1. Details are available from the authors upon request.

3.1. For $\lambda \in (\lambda_j, \lambda_{j+1})$, the market price of risk behaves asymptotically as $\sigma \tilde{\gamma}_{A_j}$ under the measure Q^T, and agent A_j is the one that puts the largest weight on the extreme paths of B_t that decay as $-\sigma \tilde{\gamma}_{A_j} t$. When λ is sufficiently close to 1 (i.e., $\lambda > \lambda_K$), then the probability of such extreme paths becomes so small that only agent A survives and bond volatility vanishes identically. By contrast, for small values of λ (i.e., $\lambda < \lambda_1$), the probability of such extreme paths is still sufficiently high and even the agents with a very high survival index (but low interest rate r_i) will impact bond volatilities. Because $l_i(\lambda) = (1 - \lambda) r_i + \lambda I_i$, the weight of the survival index I_i gradually increases as $\lambda \uparrow 1$ and the agents with a low survival index gradually lose their bond volatility impact. Cvitanić, Jouini, Malamud and Napp (2009) showed that the bond prices $B(\lambda T, T)$ for $\lambda \in (\lambda_j, \lambda_{j+1})$ also behave asymptotically as do those in the single-agent economy populated by agent j. However, they do not establish the link between bond price impact and survival with respect to the T-forward measure. Without this important link established in Proposition 5.2, the precise economic mechanism behind the asymptotic results of Cvitanić, Jouini, Malamud and Napp is unclear.

As our second example, we consider the behavior of forward contracts on the stock.[39] The T-forward price of the stock at time t is given by

$$F(t,T) = \frac{E_t[\xi_T S_T]}{E_t[\xi_T]}$$

and the forward stock volatility $\sigma^F(t, T)$ is defined by

$$F(t,T)^{-1} dF(t,T) = \mu^F(t, T) dt + \sigma^F(t, T) dB_t.$$

We need the following definition.

---

[39]When interest rate volatility is small, the behavior of forward prices is similar to that of future prices. It is possible to extend our analysis for futures contracts, however, it would require several additional technical conditions and sophisticated estimates for the interest rate process. For this reason, we confine our analysis to the case of forward contracts.

Definition 5.2. *The dividend-forward measure Q^D is defined on $\mathcal{F}_T$ by*

$$dQ^D \;=\; \frac{D_T\,\xi_T}{E[D_T\xi_T]}\,dP\,.$$

We then have the following result.[40]

Proposition 5.5. *Let σ_t^S be the stock price volatility. We have*

$$\sigma^F(t\,,\,T) \;=\; \frac{E_t^{Q^D}[\sigma_T^S\,Y_T^{-1}]}{E_t^{Q^D}[Y_T^{-1}]} \;+\; \sigma\left(E_t^{Q^T}[\kappa_T] \;-\; \frac{E_t^{Q^D}[\kappa_T\,Y_T^{-1}]}{E_t^{Q^D}[Y_T^{-1}]}\right).$$

Proposition 5.5 is a direct analog of Proposition 3.3. Now, the same argument as in the proof of Proposition 3.4 implies that the following is true.

Proposition 5.6. *An agent i has a long-run impact on forward volatility for $t = \lambda T$ if and only if he survives either with respect to the T-forward measure Q^T or with respect to the dividend-forward measure Q^D.*

Consistent with (5.8), we can define

$$l_i^D(\lambda) \;\overset{def}{=}\; (I_i - \mu) \;-\; 0.5\,(1-\lambda)\,(1-\tilde{\gamma}_i)^2\,\sigma^2 \;=\; (1-\lambda)\,Y_i \;+\; \lambda\,(I_i - \mu)\,.$$

This is the analog of Yan's (2008) survival index for the dividend-forward measure. Because $\left(l_i^D\,(\lambda)\,,\lambda \in [0,1]\right)$ is a family of line segments, there exist pairs of values $((A_j^D\,,\,\lambda_j^D)\,,\,j = 1,\cdots,K)$ such that

$$\min_i l_i\,(\lambda) = l_{A_j^D}(\lambda) \text{ for all } \lambda \in (\lambda_j^D,\lambda_{j+1}^D),$$

where $\lambda_0^D = 0$ and $\lambda_{K+1}^D = 1$. It follows from the definition of $l_i(\lambda)$ that A_0^D is the agent with the lowest "single-agent" dividend yield $Y_{A_0} = \min_i Y_i$, whereas $A_K = A$ is the single surviving agent with the lowest survival index. We then have the following.[41]

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[40]The proof of Proposition 5.5 is completely analogous to that of Proposition 3.3.
[41]The proof of Proposition 5.7 is completely analogous to that of Theorem 3.1. Details are available from the authors upon request.

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Proposition 5.7. *An agent i survives with respect to the dividend-forward measure for $t = \lambda T$ if and only if*

$$l_i(\lambda) \; = \; \min_j l_j(\lambda) \,.$$

Therefore, for $\lambda \in (\lambda_j^D, \lambda_{j+1}^D) \cap (\lambda_k, \lambda_{k+1})$, we have

$$\sigma^F(t, T) \; \sim \; \sigma \left(1 + \tilde{\gamma}_{A_k} - \tilde{\gamma}_{A_j^D} \right) .$$

Proposition 5.7 is a direct analog of Theorem 3.1. As in Theorem 3.1, the long-run fundamental forward volatility σ is independent of λ for $\lambda > \max\{\lambda_K, \lambda_K^D\}$ because the stock price volatility in a single-agent economy is constant and coincides with the dividend volatility σ. The excess volatility component $\sigma \left(\tilde{\gamma}_{A_k} - \tilde{\gamma}_{A_j} \right)$ captures the cyclical fluctuations in the market price of risk κ_T, measured by the difference in expectations of κ_T under the T-forward measure and the more cyclical dividend-forward measure.

Example. Let $K = 2$ and suppose that agent 1 is the only one who survives in the long run, that is

$$I_1 < I_2 \;\Leftrightarrow\; \rho_1 - \rho_2 \;<\; 0.5 \, (\delta_2^2 - \delta_1^2) \, \sigma^2 + (\gamma_2 - \gamma_1) \, \mu \,.$$

If $r_1 < r_2$, that is

$$\rho_1 - \rho_2 \;<\; (\gamma_2 \, \mu_2 - \gamma_1 \mu_1) + 0.5 \, (\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2)\sigma^2 \,,$$

then the asymptotic bond volatility is zero

$$\sigma^B(t, T) \;\sim\; 0$$

for all $t = \lambda T$. However, if $r_2 < r_1$, that is

$$(\gamma_2 \, \mu_2 - \gamma_1 \mu_1) + 0.5 \, (\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2)\sigma^2 \;<\; \rho_1 - \rho_2 \;<\; 0.5 \, (\delta_2^2 - \delta_1^2) \, \sigma^2 + (\gamma_2 - \gamma_1) \, \mu,$$

then

$$\sigma^B(t\,,\,T) \ \sim \ \sigma \begin{cases} \tilde{\gamma}_1 - \tilde{\gamma}_2 \,, & \lambda \in (0\,,\,\lambda_1) \\ 0\,, & \lambda \in (\lambda_1\,,\,1)\,, \end{cases}$$

where

$$\lambda_1 \ = \ \frac{r_1 - r_2}{(r_1 - r_2) - (I_1 - I_2)} \ = \ \frac{r_1 - r_2}{0.5\,\sigma^2\,(\tilde{\gamma}_2^2 - \tilde{\gamma}_1^2)}\,.$$

For the forward stock volatilities, we have to consider several cases. If $Y_1 < Y_2$, then $A_k = 1$ always, and therefore

$$\sigma^B(t\,,\,T) \ \sim \ \sigma \begin{cases} 1 + \tilde{\gamma}_1 - \tilde{\gamma}_2 \,, & \lambda \in (0\,,\,\lambda_1) \\ 1\,, & \lambda \in (\lambda_1\,,\,1)\,, \end{cases}$$

that is $\sigma^B(t\,,\,T) - \sigma^F(t\,,\,T) \ \sim \ \sigma$ for all $t = \lambda T$. Here, we set $\lambda_1 = 0$ if $r_1 < r_2$.

However, if $Y_2 < Y_1$, we need to separately consider the cases when λ_1^D is smaller or larger than λ_1, where

$$\lambda_1^D \ = \ \frac{Y_1 - Y_2}{(Y_1 - I_1) - (Y_2 - I_2)} \ = \ \frac{Y_1 - Y_2}{0.5\sigma^2\,((\tilde{\gamma}_2 - 1)^2 - (\tilde{\gamma}_1 - 1)^2)}.$$

If $\lambda_1^D < \lambda_1$ then

$$\sigma^F(t\,,\,T) \ \sim \ \sigma \begin{cases} 1\,, & \lambda \in (0\,,\,\lambda_1^D) \\ 1 + \tilde{\gamma}_2 - \tilde{\gamma}_1 \,, & \lambda \in (\lambda_1^D\,,\,\lambda_1) \\ 1\,, & \lambda \in (\lambda_1\,,\,1)\,. \end{cases}$$

However, if $\lambda_1^D > \lambda_1$, then

$$\sigma^F(t\,,\,T) \ \sim \ \sigma \begin{cases} 1\,, & \lambda \in (0\,,\,\lambda_1) \\ 1 + \tilde{\gamma}_1 - \tilde{\gamma}_2 \,, & \lambda \in (\lambda_1\,,\,\lambda_1^D) \\ 1\,, & \lambda \in (\lambda_1^D\,,\,1)\,. \end{cases}$$

6 Large Markets

This section examines equilibrium behavior when the economy is large so that the agents' effective risk aversions densely cover the interval $\left[\underset{\sim}{\Gamma}, \widetilde{\Gamma}\right]$. This assumption is quite natural for developed markets with a large number of participants. Indeed, with such a continuum of agents, many asymptotic expressions simplify and take a particularly elegant form.

6.1 Large Markets with Terminal Consumption

If $\widetilde{\Gamma} \leq 1$,[42] then $\widetilde{\Gamma} = \tilde{\gamma}_0$, and none of the nonsurviving agents has a price impact, so that the long-run stock price is only impacted by the least optimistic (and the most realistic) trader with effective risk aversion $\tilde{\gamma}_0$. In this case, $(\kappa_t, \sigma_t^S) \sim (\sigma \tilde{\gamma}_0, \sigma)$ for $t = \lambda T$, $\lambda \in (0, 1)$.

Because each agent in our economy is uniquely characterized by his risk aversion γ and beliefs δ, we will use $\pi_{(\gamma,\delta)t}$ to denote his optimal portfolio. A direct calculation based on Theorem 4.1 implies that

$$
\pi_{(\gamma,\delta)t}^{\text{hedging}} \sim \begin{cases} \frac{(\lambda-1)\delta + \gamma}{1 + \lambda(\gamma-1)} - \tilde{\gamma}_0, & \lambda < \frac{\tilde{\gamma}_0 - \gamma + \delta}{\delta - \tilde{\gamma}_0(\gamma-1)} \\ 0, & \lambda \geq \frac{\tilde{\gamma}_0 - \gamma + \delta}{\delta - \tilde{\gamma}_0(\gamma-1)}. \end{cases}
$$

Thus, even though none of the agents except agent 0 has any price impact, they all have significant long-run impact on the optimal portfolios of other agents in the economy. As time goes by, nonsurviving agents gradually lose their portfolio impact, and therefore the hedging demand is monotone decreasing in $\lambda = t/T$. However, if $\tilde{\gamma}_0 = 1$, we have $\frac{\tilde{\gamma}_0 - \gamma + \delta}{\delta - \tilde{\gamma}_0(\gamma-1)} = 1$, and therefore the portfolio impact of nonsurviving agents is *permanent*: agents hold nontrivial hedging portfolios even for periods arbitrarily close to the terminal time horizon.

Consider now the same example as the previous one, except we assume

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[42]For example, if the economy is populated by very optimistic traders. According to Dimson, Marsh and Staunton (2003) this was indeed the case for a long time. The presence of very optimistic traders is also important for replicating the empirically observed state price patterns. See Proposition 3.2.

that $\widetilde{\Gamma} \geq 1$. As the risk aversions become more and more dense in $\left[\underline{\Gamma}, \widetilde{\Gamma}\right]$, the interval Π_i converges to a single point $\tilde{\gamma}_i^{-1}$. Furthermore, in this case, $\tilde{\gamma}_0 = \max\{1, \underline{\Gamma}\}$ and, by Theorem 3.1, the instantaneous moments of the stock price converge to

$$
(\kappa_t, \sigma_t) \sim \sigma
\begin{cases}
(\tilde{\gamma}_0, 1), & \tilde{\gamma}_0^{-1} < \lambda \\
(\lambda^{-1}, 1 + \lambda^{-1} - \tilde{\gamma}_0), & \widetilde{\Gamma}^{-1} < \lambda \leq \tilde{\gamma}_0^{-1} \\
(\widetilde{\Gamma}, 1 + \widetilde{\Gamma} - \tilde{\gamma}_0), & \lambda \leq \widetilde{\Gamma}^{-1}.
\end{cases}
\tag{6.1}
$$

Thus, for $\lambda \in (0, \widetilde{\Gamma}^{-1})$, only the effectively most risk-averse agent has an impact on both the market price of risk and volatility. Then, for $\lambda \in \left(\widetilde{\Gamma}^{-1}, \tilde{\gamma}_0^{-1}\right)$, agents with effective risk aversion λ^{-1} impact the instantaneous moments of the stock price. Finally, for $\lambda \in \left(\tilde{\gamma}_0^{-1}, 1\right)$, the price impact of nonsurviving agents vanishes completely, and the instantaneous moments converge to those determined by the surviving agent 0.

As for the optimal portfolio, we again use the notation $\pi_{(\gamma,\delta)t}$ to denote an agent's optimal portfolio. Then, for λ satisfying

$$
\lambda \in \left[\widetilde{\Gamma}^{-1}, \min\left\{\tilde{\gamma}_0^{-1}, \frac{\tilde{\gamma}_0 - \tilde{\gamma}_i}{\delta_i - \tilde{\gamma}_0(\gamma_i - 1)}\right\}\right] \overset{def}{=} [\underline{\Lambda}, \overline{\Lambda}],
\tag{6.2}
$$

Theorem 4.1 yields that

$$
\pi_{(\gamma,\delta)t}^{\text{hedging}} = \frac{(1 - \gamma^{-1})(1 - \lambda)(\lambda\delta + 1)}{(1 + \lambda(\gamma - 1))(\lambda(1 - \tilde{\gamma}_0) + 1)}, \quad \lambda \in [\underline{\Lambda}, \overline{\Lambda}].
\tag{6.3}
$$

An appealing property of formula (6.3) is that it is model independent. Given an *arbitrary* pair (γ, δ) of risk aversion and beliefs, we can substitute it into this formula and obtain the required optimal hedging portfolio. Interestingly enough, it is a linear function of the agent's beliefs δ but is highly nonlinear in γ and may exhibit varying behavior. We summarize these somewhat unexpected patterns in the following proposition.

Proposition 6.1. *We have the following.*[43]

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[43]As above, we only consider the case $\gamma \geq 1$. However, the effective risk aversion $\gamma - \delta$

(1) *Fix risk aversion γ and a fraction $\lambda = t/T$ of time horizon. Then,*

    (i) $\pi^{\text{hedging}}_{(\gamma,\delta)\,t}$ *is monotone increasing in the beliefs δ;*

    (ii) *it is always positive if $\delta > -1$, that is, when the agent is either optimistic or moderately pessimistic;*

    (iii) *If $\delta < -1$, then the hedging demand is positive for $\lambda < -\delta^{-1}$ and negative for $\lambda > -\delta^{-1}$.*

(2) *Fix beliefs δ and a fraction $\lambda = t/T$ of the time horizon. Then,*

    (i) *if $\delta\,\lambda + 1 > 0$, then the hedging demand $\pi^{\text{hedging}}_{(\gamma,\delta)\,t}$ is monotone increasing in the risk aversion γ for $\gamma < 1 + \lambda^{-1/2}$ and is monotone decreasing in γ for $\gamma > 1 + \lambda^{-1/2}$;*

    (ii) *if $\delta\,\lambda + 1 < 0$, then the direction of monotonicity reverses.*

The result of Item (1.i) is intuitive. The more optimistic a trader is, the more the stock is attractive to him. However, surprisingly, even an extremely pessimistic trader will still hold a positive hedging portfolio for $t/T < -\delta^{-1}$ if $-\delta^{-1} > (\widetilde{\Gamma})^{-1}$. The reason is that, if there are agents in the economy whose effective risk aversion is higher than $-\delta$, they will have price impact for small values of λ and will drive the market price of risk up so much that it will be optimal even for a very pessimistic trader to capitalize on these attractive high returns and to hold a positive hedging portfolio. However, as λ increases, the price impact of these highly risk-averse agents vanishes, the market price of risk goes down and a pessimistic agent starts selling the stock and ends up holding a negative hedging portfolio.

The result of Item (2) is surprising at first sight. Intuitively, one would expect that the hedging demand is monotone decreasing in risk aversion and that the more risk-averse agents will hold less stock. This is indeed true for the total portfolio holding in stock: a calculation shows that $\pi_{(\gamma,\delta)\,t}$ is indeed monotone decreasing in risk aversion. However, the behavior of the hedging component is more subtle. For small values of λ, there is a large uncertainty about the future values of the market price of risk: almost all

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can take any real values.

agents still have price impact, and the market price of risk fluctuates greatly. These fluctuations force the more risk-averse agents to use a significant part of their wealth to hedge against them. However, as λ increases, the behavior of the market price of risk becomes more and more stable, being influenced essentially only by the few surviving agents with effective risk aversions close to $\tilde{\gamma}_0$. This drives the hedging demand of the highly risk-averse agents down. By contrast, the agents with low risk aversion see stocks as an attractive low-risk investment and increase their hedging demand. The behavior of the hedging portfolios is shown in Figures 1 and 2 below.

In the short-time and long-time regimes, by Theorem 4.1 and (6.1), the hedging portfolio is given by

$$\pi^{\text{hedging}}_{(\gamma,\delta)\,t} = \frac{\left(1 - \gamma^{-1}\right)\left((1-\lambda)\,(\delta + \widetilde{\Gamma}) + (\widetilde{\Gamma}\,\lambda - 1)\,\gamma\right)}{\left(1 + \lambda\,(\gamma - 1)\right)\left(1 + \widetilde{\Gamma} - \tilde{\gamma}_0\right)} \;,\; \lambda < \widetilde{\Gamma}^{-1}$$

and

$$\pi^{\text{hedging}}_{(\gamma,\delta)\,t} = 0 \;,\; \lambda > \min\left\{ \tilde{\gamma}_0^{-1} \,,\, \frac{\tilde{\gamma}_0 - \tilde{\gamma}_i}{\delta_i - \tilde{\gamma}_0\,(\gamma_i - 1)} \right\}$$

respectively.[44] Thus, as in Example 2 above, the portfolio impact is permanent if and only if $\tilde{\gamma}_0 = 1$.

6.2   Large Markets with Intermediate Consumption

In the case of intermediate consumption, it is more efficient to parameterize the agents according to their beliefs. Thus, we assume that the beliefs parameter takes values of $\delta \in [\delta_{\min}\,,\,\delta_{\max}]$. We also assume that the dividend growth rate μ is positive. In general, the ordering of the agents in terms of their price impact will depend in a nontrivial way on the joint cross-sectional distribution of risk aversion, discount rates and beliefs. In order to illustrate the effects of the cross-sectional correlation of beliefs and risk aversion, we

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[44]The limit of the portfolio policy for values of $\lambda \in \left(\min\left\{\tilde{\gamma}_0^{-1}, \frac{\tilde{\gamma}_0 - \tilde{\gamma}_i}{\delta_i - \tilde{\gamma}_0\,(\gamma_i - 1)}\right\}, \max\left\{\tilde{\gamma}_0^{-1}, \frac{\tilde{\gamma}_0 - \tilde{\gamma}_i}{\delta_i - \tilde{\gamma}_0\,(\gamma_i - 1)}\right\}\right)$ can be characterized explicitly as well, but the results depend on the ordering between the two thresholds, which in turn is determined by the values of the model parameters. We omit these results to simplify the exposition.
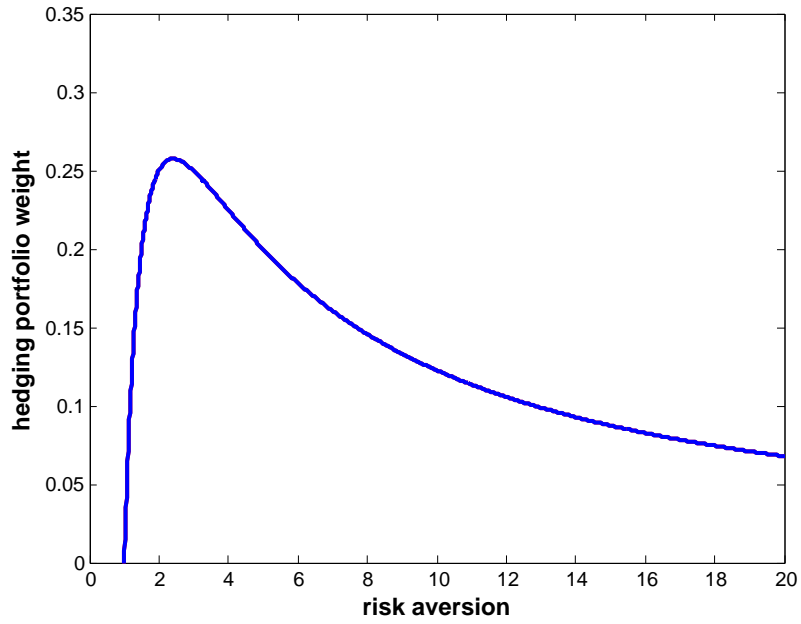
Figure 1: Hedging Portfolio weight as a function of risk aversion. Parameters: $\delta = 1$, $\tilde{\gamma}_0 = 1$ and $\lambda = 0.5$.
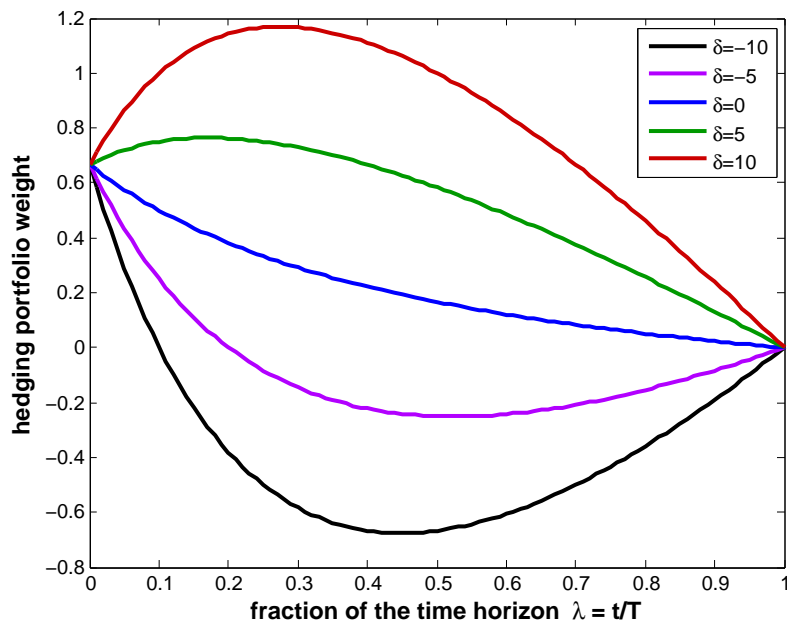

Figure 2: Hedging Portfolio weight as a function of time horizon, for beliefs taking values $\delta = -10$, -5, 0, 5, 10. Parameters: $\gamma = 3$, $\tilde{\gamma}_0 = 1$ and $\widetilde{\Gamma}$ is large.

consider the case where the risk aversion is a linear function of beliefs and ρ is constant across agents [45]

$$\gamma = f\,\delta + g, \qquad (6.4)$$

for some numbers f, $g \in \mathbb{R}$. For simplicity, we also assume that $|f-1| < 1$, that is $f \in [0,2]$. In particular, we assume that risk aversion and beliefs are positively correlated. Then, a direct calculation shows that $\min_\delta l_\delta(\lambda)$ is attained at[46]

$$\delta(\lambda) = \frac{(1-\lambda)\,\sigma^2\,(f-1)\,g - f\,\mu}{\sigma^2\,(1 - (1-\lambda)\,(f-1)^2)}, \qquad (6.5)$$

and we let $\tilde{\gamma}(\lambda)$ denote the corresponding effective risk aversion. To avoid considering the boundary cases, we assume that both

$$\delta(0) = \frac{\sigma^2\,(f-1)\,g - f\,\mu}{\sigma^2\,(1 - (f-1)^2)} \ \text{and}\ \delta(1) = -\frac{f\,\mu}{\sigma^2}$$

belong to the interval $[\delta_{\min}, \delta_{\max}]$ and we assume that $g \geq \sigma^{-2}\max\{f^2, f\}\,\mu$, so that γ defined in (6.4) is positive. Then, by, Proposition 5.2, we have

$$\sigma^B(t,\,T) \ \sim\ \tilde{\gamma}(1) - \tilde{\gamma}(\lambda), \qquad (6.6)$$

for $t = \lambda T$, where

$$\tilde{\gamma}(\lambda) = \frac{(1-\lambda)\,\sigma^2\,(f-1)^2\,g - f\,(f-1)\,\mu}{\sigma^2\,(1 - (1-\lambda)\,(f-1)^2)} + g. \qquad (6.7)$$

One interesting consequence of (6.7) is that very pessimistic agents (i.e., those with $\delta < \min\{\delta(0), \delta(1)\}$), as well as very optimistic ones (i.e., those with $\delta > \max\{\delta(0), \delta(1)\}$ will not have any bond volatility impact. This is intuitive: beliefs $\delta(0)$ and $\delta(1)$ impose bounds on the size of irrationality for

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[45]Of course, even if there is cross-sectional correlation between γ and δ, the true relation would be $\gamma = f\tilde{\gamma}_i + g +$noise. The results in the presence of noise would be similar, but the asymptotic expressions would get much messier because there would always be multiple agents with the same survival index. To avoid this, we assume that the noise component is absent.

[46]If $f \notin [0,2]$, the minimum will necessarily be attained by one of the extreme values $\delta_{\min}$, $\delta_{\max}$ for some values of λ and we impose $f \in [0,2]$ to avoid this boundary solution.

an agent to have a nonnegligible price impact.

When time t is large, we can consider (6.6) as an approximation to the long-run term structure $\{\sigma^B(t,\,t/\lambda)\,,\,\lambda\,\in\,(0\,,\,1)\,\}$ of bond volatilities. Formula (6.7) shows that this term structure can exhibit different patterns depending on the size of the slope coefficient f. The following result is a direct consequence of (6.6).

Proposition 6.2. *Under the above assumptions, the long-run term structure of bond volatilities is upward sloping if and only if $f > \bar{f}$, where*

$$\bar{f} \;=\; \frac{\mu + \sqrt{\mu^2 + 4\mu\sigma^2 g}}{2\mu} \;>\; 1. \qquad (6.8)$$

It follows directly from (6.5) that $\delta(\lambda)$ is monotone increasing in λ if and only if either $f > \bar{f}$ or $f < 1$. If this is the case, then the surviving agent with beliefs $\delta(1)$ will be the most optimistic one among those who have bond volatility impact. Furthermore, the most pessimistic agents will impact the low maturity bond volatilities and, as the maturity increases, more and more optimistic agents will come into play. If $f > \bar{f}$, then these very optimistic agents will drive bond volatility up at long maturities. However, if $f < 1$, even though the long end of the yield curve will still be dominated by the highly optimistic agents, the bond volatility term structure will be downward sloping. The reason is that, for $f < 1$, effective risk aversion $\tilde{\gamma}(\delta)$ is monotone decreasing in δ. Thus, for the more optimistic agents, determining the long end of the bond volatility term structure will have lower effective risk aversion and drive the bond volatilities down.

If the slope f satisfies $f \in (1\,,\,\bar{f})$, then both $\delta(\lambda)$ and $\tilde{\gamma}(\lambda)$ are monotone decreasing in λ and, consequently, $\sigma^B(t\,,\,T) < 0$. That is, the instantaneous correlation of bond returns with the dividends is negative.[47]

A similar analysis can be performed for forward stock volatilities. In this

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[47]There is indeed some empirical evidence suggesting that bond returns are counter-cyclical; see Campbell (1987).

case, belief $\delta^D(\lambda)$ minimizing $l^D_\delta(\lambda)$ is given by

$$\delta^D(\lambda) \;=\; \frac{(1-\lambda)\,\sigma^2\,(f-1)\,(g-1)\;-\;f\,\mu}{\sigma^2\,(1\,-\,(1-\lambda)\,(f-1)^2)}\,. \tag{6.9}$$

By Proposition 5.7, the long-run forward volatility is given by

$$\sigma^F(t\,,\,T) \;\sim\; \sigma\left(1\,+\,\tilde{\gamma}(\lambda)\,-\,\tilde{\gamma}^D(\lambda)\right), \tag{6.10}$$

where $\tilde{\gamma}^D(\lambda) \;=\; (f-1)\,\delta^D(\lambda)\,+\,g$. Using (6.5) and (6.9), we arrive at

$$\sigma^F(t\,,\,T) \;\sim\; \sigma\,\frac{1}{1\,-\,(1-\lambda)\,(f-1)^2}\,. \tag{6.11}$$

Formula (6.11) has several intriguing features: (i) in contrast to long-run bond volatilities (see (6.6) and (6.7)), long-run forward volatilities are independent of the growth rate μ and the "shift" g in risk aversion. The only quantities that matter are the dividend volatility and the slope f; (ii) forward volatility is always larger than the dividend volatility σ, is monotone decreasing in λ and converges to σ as $\lambda \uparrow 1$; (iii) long-run $\sigma^F(t\,,\,T)$ is largest when f is either 0 or 2, in which case it is given by σ/λ and coincides with the asymptotic stock volatility in the model with terminal consumption.[48] The intuition behind formula (6.11) is similar to that behind Theorem 3.1: as time goes by, both the agents with very low and very high levels of effective risk aversions vanish from the market and gradually lose their price impact. This drives down the variability of aggregate risk aversion and reduces forward volatility. The speed with which these nonsurviving agents vanish is decreasing in the absolute slope $|f-1|$ of $\tilde{\gamma}$ with respect to beliefs δ. Consequently, when the absolute slope $|f-1|$ is large, the agents with extreme values of $\tilde{\gamma}$ are present in the market for a long time and drive up the forward volatility.

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[48]See (6.1) for the case $\tilde{\gamma}_0 = 1$.

7 Conclusions

It has long been argued that nonsurviving agents cannot influence long-run asset prices. KRWW (2006) demonstrated that survival and price impact are two independent concepts, and that even agents owning a negligible proportion of aggregate wealth may have a significant long-run impact on equilibrium asset prices.

This paper shows that survival, when defined properly, is in fact equivalent to price impact. That is, an agent has long-run price impact if and only if the equilibrium market price of his share of aggregate wealth is nonnegligible. Another, related phenomenon studied in the paper is that of portfolio impact: the possibility for nonsurviving agents to have long-run impact on other agents' optimal portfolios. As in the case of price impact, this paper shows that portfolio impact is also equivalent to survival: the price of the share of the aggregate wealth must be nonnegligible. However, this price should be calculated under an agent-specific *wealth-forward measure*, as introduced in this paper. These general results imply that price impact and portfolio impact are two independent concepts: a nonsurviving agent with no price impact can have a significant impact on other agents' optimal portfolios.

These new phenomena are demonstrated in the framework of a general equilibrium model with an economy populated by an arbitrary number of agents with arbitrary heterogeneous risk aversions and beliefs. The exact economic mechanisms responsible for price and portfolio impact are identified through an explicit description of the set of paths through which the nonsurviving agents impact long run asset prices. The importance of the presence of optimistic agents for generating the empirically observed state prices smile effect, as well as the possibility for nonsurviving optimistic agents to have long-run price and portfolio impact, are demonstrated. We show that our methods can also be directly extended to models with intermediate consumption. In contrast to KRWW (2010), we show that, even with CRRA utilities, nonsurviving agents can have a long-run impact on the prices of assets with long maturities, and that this price impact is equivalent to survival under an asset-specific measure.

The techniques presented in this paper provide a general tool for analyzing survival, price impact and portfolio impact in heterogeneous economies, and allow for the derivation of closed-form asymptotic expressions for equilibrium optimal portfolios in economies populated by more than two agents, which, to the best of our knowledge, has never been done before in the literature.

Appendix

A Large Markets with Finite Horizon

One of the major difficulties in analyzing equilibrium with heterogeneous agents is the impossibility to derive an analytic expression for the equilibrium state price density. Recently, Xiouros and Zapatero (2009) showed that, with a continuum of agents, a careful choice of the distribution of the initial consumption c_0 may be able to resolve this issue. As is common in the modern asset pricing literature, they fix the welfare weights determining the equilibrium allocation;[49] see, e.g., Chan and Kogan (2002). The market clearing conditions take the form[50]

$$\int_{\mathbb{R}} c_0(i)\, e^{-t\,\rho(i)\,b(i)}\, Z_{i\,t}^{b(i)}\, \xi_t^{-b(i)}\, di = D_t,$$

where $b(i) = \gamma(i)^{-1}$ is agent i's risk tolerance. Xiouros and Zapatero assume[51] that the agents are rational, have a common discount rate ρ, and are parametrized by their risk tolerance b. Furthermore, the initial consumption is given by $c_0(b) = b^{\eta^{-1}-1} \frac{e^{-b/\theta}}{\theta^{\eta^{-1}}\Gamma(\eta^{-1})}$. In this case, equation (5.2) takes the form

$$\int_0^{+\infty} b^{\eta^{-1}-1} \frac{e^{-b/\theta}}{\theta^{\eta^{-1}}\,\Gamma(\eta^{-1})}\, e^{-z\,b}\, db = D_t. \qquad (A.1)$$

The left-hand side of (A.1) is the moment-generating function of a $\Gamma(\eta^{-1}, \theta)$-density and is given by $(1+\theta z)^{-\eta^{-1}}$. Therefore,[52]

$$U_x(t, x) = e^{-\rho t}\, e^{z(D_t)}\,,\ \ z(x) = \frac{x^{-\eta}-1}{\theta}\,,\ \ \tilde{\gamma}^U(x) = \frac{\eta}{\theta}\, x^{-\eta}\,. \qquad (A.2)$$

---

[49]Of course, in equilibrium the weights must be determined endogenously and depend on the horizon of the economy. This can be easily incorporated into our framework, however, it only leads to more complicated expressions without generating further economic insights.

[50]The welfare weights corresponding to the equilibrium allocation are given by $c_0(i)^{\gamma(i)}$.

[51]Xiouros and Zapatero also allow the agents to have "Catching up with the Joneses" preferences in addition to heterogeneous risk aversion.

[52]Because $\tilde{\gamma}^U$ is t-independent, we write $\tilde{\gamma}^U(x)$ instead of $\tilde{\gamma}^U(t, x)$.

Thus, the market price of risk and the instantaneous interest rate are given explicitly by

$$\kappa_t \;=\; \sigma\,\tilde{\gamma}^U(D_t)\,,\;\; r_t \;=\; \rho + \tilde{\gamma}^U(D_t)\left(\hat{\mu} - 0.5\,\sigma^2\left(\tilde{\gamma}^U(D_t) + \eta + 1\right)\right). \quad \text{(A.3)}$$

The idea of choosing a suitable initial consumption distribution is very general and can be applied, for example, to the case of heterogeneous beliefs. Indeed, suppose that the risk tolerance $b = \gamma^{-1}$ is constant across agents, who are parametrized by their beliefs δ, and the initial consumption is normally distributed, $c_0(\delta) = \frac{1}{\sqrt{2\pi\,\eta^2}}\,e^{-(\delta - \alpha)^2/(2\eta^2)}$ for some α, $\eta \in \mathbb{R}$. Then, equation (5.2) for the representative agent's marginal utility $U_x(t, x) = e^{-\rho t}\,e^{z(t,x)}$ takes the form

$$e^{-zb}\,\frac{1}{\sqrt{2\pi\,\eta^2}}\int_{\mathbb{R}} e^{-(\delta-\alpha)^2/(2\eta^2)}\,e^{-(0.5\delta^2\sigma^2 + \delta\mu)t\,b + \delta\,b\,\log x}\,d\delta \;=\; x\,.$$

Evaluating the integral, we get

$$z \;=\; -\gamma\left(\log x - \phi(t,x)\right), \quad\quad\quad\quad\quad \text{(A.4)}$$

where

$$\phi(t,x) \;=\; \frac{1}{2}\log\left(1 + \eta^2\,\sigma^2\,t\,b\right) - \frac{\alpha^2}{2\eta^2} + \frac{\left(\alpha - \mu\,t\,b\,\eta^2 + \eta^2 b\log x\right)^2}{2\,\eta^2\left(1 + \eta^2\,\sigma^2\,t\,b\right)}\,. \quad \text{(A.5)}$$

Formulae (A.4)-(A.5) provide an analytic expression for the representative agent in an economy with a continuum of agents and heterogeneous beliefs. To the best of our knowledge, such an analytic expression has never been derived in the literature before. In fact, Jouini and Napp (2007, p. 1156) write that: "Notice that it is not possible (except in the exponential case) to construct M and to obtain q^* as functions of the aggregate characteristics (e.g. consumption) of the economy and not of the individual ones." In contrast, (A.4) does express the equilibrium state price density as a function of aggregate characteristics. Consistent with Proposition 3.2, equilibrium

state prices are U-shaped. In fact, $\log \xi_t$ is convex in $\log D_t$. Furthermore,

$$\tilde{\gamma}^U(t,x) \;=\; \gamma \left(1 \,-\, \frac{b\,(\alpha - \mu\,t\,b\,\eta^2 \,+\, \eta^2 b \log x)}{1 + \eta^2\,\sigma^2\,t\,b} \right). \tag{A.6}$$

Therefore, the market price of risk and the interest rate are given explicitly by

$$\kappa_t \;=\; \sigma\,\tilde{\gamma}^U(t,D_t),$$
$$r_t \;=\; \rho \,+\, \tilde{\gamma}^U(t,D_t)\,\left(\hat{\mu} \,-\, 0.5\,\sigma^2\,\left(\tilde{\gamma}^U(t,D_t) \,+\, \gamma\,(1 \,+\, \phi_{xx}(t,D_t))) \right) \right), \tag{A.7}$$

with
$$\phi_{xx}(t,x) \;=\; \frac{b\,(\eta^2\,b(1-\log x) \,-\, \alpha + \mu\,t\,b\,\eta^2)}{1 \,+\, \eta^2\,\sigma^2\,t\,b}.$$

Even though the market price of risk and the interest rate are available in closed form for both models (A.2) and (A.7), equilibrium asset prices for finite values of t can only be computed numerically.[53] Nevertheless, it is possible to derive asymptotic expressions for equilibrium quantities when the horizon T is small.[54]

The following representations can be derived using the same Malliavin calculus techniques as in the proof of Proposition 3.3; see Cvitanić, Jouini, Malamud and Napp (2009).

Proposition A.1. *The equilibrium stock volatility is given by*

$$\sigma_t^S \;=\; \sigma \,+\, \frac{E_t\left[\int_t^T (\kappa_t \,-\, \kappa_\tau)\,\xi_\tau\,D_\tau\,d\tau \right]}{E_t\left[\int_t^T \xi_\tau\,D_\tau\,d\tau \right]},$$

and the optimal portfolio of agent i is given by

$$\sigma_t^S\,\pi_{it} \;=\; \kappa_t \,+\, \frac{E_t\left[\int_t^T (b_i\,\delta_i\,\sigma \,+\, (b_i - 1)\,\kappa_\tau)\xi_\tau c_{i\tau}d\tau \right]}{E_t\left[\int_t^T \xi_\tau c_{i\tau}d\tau \right]}.$$

---

[53]Formulae (A.4)-(A.7) are perfectly suited for such a numerical analysis. However, this goes beyond the main scope of this paper and we leave it for future research.

[54]Section 6.3 studies their asymptotic behavior for large values of T. This asymptotic behavior is independent of the distribution of c_0.

Note that the dividend process $D_t = e^{\mu t + \sigma B_t}$ satisfies $D_t^{-1} dD_t = \hat{\mu} dt + \sigma dB_t$ with $\hat{\mu} \stackrel{def}{=} \mu + 0.5\sigma^2$ and therefore the generator $\mathcal{L}$ of the process D_t is given by $\mathcal{L} = 0.5\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \hat{\mu} x \frac{\partial}{\partial x}$. Our calculations are based on the Lemma B.1 in the next section.

Let $\kappa_t^{(i)} \stackrel{def}{=} \kappa_t + \delta_i \sigma$ be the agent i's perceived market price of risk and let $\sigma_t^S \pi_{it}^{\mathrm{myopic}} = b(i) \kappa_t^{(i)}$ be his myopic portfolio. By definition,

$$\pi_{it}^{\mathrm{hedging}} \stackrel{def}{=} \pi_{it} - \pi_{it}^{\mathrm{myopic}}.$$

Applying Lemma B.1 to the representations of Propositions A.1 and 5.1 we arrive at the following result.[55]

Proposition A.2. *For small $T - t$, we have*

- *the stock price volatility is given by*

$$\sigma_t^S \approx \sigma + 0.5\,(T-t)\left(\sigma_S^{(1)}\frac{d}{dx}\tilde{\gamma}^U(t, D_t) + \sigma_S^{(2)}\frac{d^2}{dx^2}\tilde{\gamma}^U(t, D_t) + \frac{\partial}{\partial t}\tilde{\gamma}^U(t, D_t)\right),$$
$$(A.8)$$

  where $\sigma_S^{(1)} = -\sigma^2\left(\tilde{\gamma}^U(t, D_t) + D_t\right) - \hat{\mu} D_t$, $\sigma_S^{(2)} = -\frac{1}{2}\sigma^2 D_t^2$;

- *the bond volatility is given by*

$$\sigma^B(t,\,\tau) \approx (\tau - t)\left(\sigma_B^{(1)}\frac{d}{dx}\tilde{\gamma}^U(t, D_t) + \sigma_B^{(2)}\frac{d^2}{dx^2}\tilde{\gamma}^U(t, D_t) + \frac{\partial}{\partial t}\tilde{\gamma}^U(t, D_t)\right),$$
$$(A.9)$$

  where $\sigma_B^{(1)} = -\sigma^2\tilde{\gamma}^U(t, D_t) - \hat{\mu} D_t$, $\sigma_S^{(2)} = -\frac{1}{2}\sigma^2 D_t^2$;

- *the hedging portfolio weight is given by*

$$\sigma_t^S \pi_{it}^{\mathrm{hedging}} \approx \frac{1}{2}(1 - b_i)(T - t)$$
$$\times \left(\pi^{(1)}\frac{\partial}{\partial x}\tilde{\gamma}^U(t, D_t) + \pi^{(2)}\frac{\partial^2}{\partial x^2}\tilde{\gamma}^U(t, D_t) + \frac{\partial}{\partial t}(\tilde{\gamma}^U(t, D_t))\right),$$
$$(A.10)$$

  where $\pi^{(1)} = -\sigma\left(b_i \kappa_t^{(i)} D_t + \kappa_t\right) - \hat{\mu} D_t$, $\pi^{(2)} = -\frac{1}{2}\sigma^2 D_t^2$.

---

[55]The calculation is very lengthy and is available from the authors upon request.

Naturally, aggregate effective risk aversion is the key state variable determining the equilibrium dynamics. Therefore, its derivatives can be viewed as stochastic factors, driving the evolution of the economy. Formulae (A.8), (A.9) and (A.10) present approximate expansions of equilibrium volatilities and optimal portfolios in terms of such stochastic factors: the first- and second-order derivatives of aggregate risk aversion with respect to the aggregate dividend, as well as its derivative with respect to time. By analogy with the literature on the term structure of interest rates, we can call the first two factors respectively slope and convexity of the market price of risk. Clearly, the exposures $\sigma_{B,S}^{(2)}$ and $\pi^{(2)}$ to the convexity factor are always negative: convexity of risk aversion drives down both stock and bond volatility. The sign of the nonmyopic exposure of agent's portfolio to convexity depends on whether the risk aversion is above or below one. The exposure to the slope will be negative for both volatilities when the dividend growth is positive (i.e., $\hat{\mu} > 0$) and may change sign otherwise. Finally, for the optimal portfolios, the sign will depend on the level of the perceived market price of risk $\kappa_t^{(i)}$.

In the setting of Xiouros and Zapatero (2009), we can use (A.2) to rewrite the representations of Proposition A.2 in the following form:[56]

- the stock volatility is given by

$$\sigma_t^S \approx \sigma + 0.5\,(T - t)\,\tilde{\gamma}^U\,\eta\left(\hat{\mu} + 1 - 0.5\sigma^2\left(2\tilde{\gamma}^U + \eta + 1\right)\right),$$

- the bond volatility is given by

$$\sigma^B(t,\tau) \approx (\tau - t)\,\tilde{\gamma}^U\,\eta\left(\hat{\mu} - 0.5\,\sigma^2\left(2\tilde{\gamma}^U + \eta + 1\right)\right), \qquad \text{(A.11)}$$

- the optimal hedging portfolio weight of agent i is given by

$$\sigma_t^S\,\pi_{it}^{\text{hedging}} \approx \frac{\sigma^2\,\tilde{\gamma}^U}{2}\,(1 - b_i)\,(T - t)$$
$$\left((b_i\,D_t(\tilde{\gamma}^U + \delta_i) + \tilde{\gamma}^U)\eta\,D_t^{-1} - \frac{1}{2}\,\eta\,(\eta + 1)\right). \qquad \text{(A.12)}$$

---

[56]It is also possible to derive analogous formulae in the setting of (A.6). The results are available from the authors upon request.

There are several important consequences of these representations. First, the stock volatility will be smaller than the dividend volatility when D_t is sufficiently small. By Proposition 3.7, this can never occur in models with only terminal consumption. Second, bond volatility will be negative for small values of D_t. The size of these effects is controlled by the parameter η. Because, by assumption, the risk tolerance has a $\Gamma(\eta^{-1}, \theta)$ density, η is the quotient of the cross-sectional variance of the risk tolerance and its squared mean; therefore, it is a natural measure of the size of heterogeneity in the economy. The behavior of the nonmyopic portfolio is more subtle and depends on the exact values of the model parameters.

B   Technical Lemmas

Lemma B.1. *Let*

$$ f(t, x) \; = \; E_t \left[\int_t^T \psi(\tau, D_\tau) d\tau \,|\, D_t = x \right] , $$

and assume that ψ is sufficiently smooth. Then, for small $T - t$,

$$ f(t, x) \; = \; (T-t)\, \psi(T, x) \, + \, 0.5\, (T-t)^2 \left(\mathcal{L}\, \psi(T, x) - \psi_t(T, x) \right) \, + \, O((T-t)^3) . $$

Indeed, f satisfies the Black–Scholes PDE $f_t + \mathcal{L}f + \psi(t, x) = 0$, $f(T, x) = 0$. Therefore, $f_t|_{t=T} = -\psi(T, x)$, $f_{tt}|_{t=T} = \mathcal{L}\psi(T, x) - \psi_t(T, x)$ and the claim immediately follows from Taylor's formula.[57]

Lemma B.2. *We have*

$$ \max_i \left(D_T^{-\gamma_i} Z_{iT} \left(\frac{\psi_i\, E[D_T \xi_T]}{E[Z_{iT}^{b_i}\, \xi_T^{1-b_i}]} \right)^{\gamma_i} \right) $$
$$ \leq \; \xi_T \; \leq \; \max_i n^{\gamma_i} \max_i \left(D_T^{-\gamma_i} Z_{iT} \left(\frac{\psi_i\, E[D_T \xi_T]}{E[Z_{iT}^{b_i}\, \xi_T^{1-b_i}]} \right)^{\gamma_i} \right) . \tag{B.1} $$

---

[57]Repeating the same argument, it is possible to derive a higher-order Taylor expansion in the powers of $T - t$. However, the corresponding approximations for the equilibrium quantities contain numerous terms that become difficult to interpret. For this reason, we only study the first-order approximation.

Proof. Let $z_i = \psi_i D_T^{-\gamma_i} Z_{iT} \left(E[D_T \xi_T] / E[Z_{iT}^{b_i} \xi_T^{1-b_i}] \right)^{\gamma_i}$. Then, the equilibrium equation is $\sum_i z_i^{b_i} \xi_T^{-b_i} = 1$ and it suffices to show that $\max_i n^{\gamma_i} z_i \geq \xi_T \geq \max_i z_i$. Indeed, if $\xi_T < \max_i z_i = z_j$ then $1 = \sum_i z_i^{b_i} \xi_T^{-b_i} > z_j^{b_j} \xi_T^{-b_j} > 1$ which is a contradiction. Similarly, if $\max_i n^{\gamma_i} z_i < \xi_T$ then $1 = \sum_i z_i^{b_i} \xi_T^{-b_i} < \sum_i z_i^{b_i} (n^{\gamma_i} z_i)^{-b_i} = 1$ which is a contradiction. The proof is complete. Q.E.D.

The next lemma establishes bounds of the welfare weights.

Lemma B.3. *Let ξ_T be the equilibrium SDF. If $\gamma_i < 1$ then*

$$1 \leq \frac{E[D_T \xi_T]^{1-\gamma_i} E[Z_{iT}^{b_i} \xi_T^{1-b_i}]^{\gamma_i}}{E[Z_{iT} D_T^{1-\gamma_i}]} \leq \psi_i^{\gamma_i - 1}.$$

If $\gamma_i > 1$ then

$$\psi_i^{\gamma_i - 1} \leq \frac{E[D_T \xi_T]^{1-\gamma_i} E[Z_{iT}^{b_i} \xi_T^{1-b_i}]^{\gamma_i}}{E[Z_{iT} D_T^{1-\gamma_i}]} \leq 1.$$

Proof. The utility of agent i's optimal wealth is given by

$$
\begin{aligned}
\frac{1}{1-\gamma_i} E[Z_{iT} W_{iT}^{1-\gamma_i}] &= \frac{1}{1-\gamma_i} \psi_i^{1-\gamma_i} \left(\frac{E[D_T \xi_T]}{E[\xi_T^{1-b_i} Z_{iT}^{b_i}]} \right)^{1-\gamma_i} E[\xi_T^{1-b_i} Z_{iT}^{b_i}] \\
&= \frac{1}{1-\gamma_i} \psi_i^{1-\gamma_i} E[D_T \xi_T]^{1-\gamma_i} E[\xi_T^{1-b_i} Z_{iT}^{b_i}]^{\gamma_i}.
\end{aligned}
$$
(B.2)

The utility from just consuming his endowment (the terminal dividend of his initial portfolio) is

$$\frac{1}{1-\gamma_i} E[Z_{iT} (\psi_i D_T)^{1-\gamma_i}] = \frac{1}{1-\gamma_i} \psi_i^{1-\gamma_i} E[Z_{iT} D_T^{1-\gamma_i}].$$

Furthermore, by definition, in equilibrium we must have $W_{iT} \leq D_T$ and therefore

$$\frac{1}{1-\gamma_i} E[Z_{iT} (\psi_i D_T)^{1-\gamma_i}] \leq \frac{1}{1-\gamma_i} E[Z_{iT} W_{iT}^{1-\gamma_i}] \leq \frac{1}{1-\gamma_i} E[Z_{iT} D_T^{1-\gamma_i}].$$

Multiplying both sides by $1 - \gamma_i$ and using (B.2), we get the result. Q.E.D.

Combining Lemmas B.2 and B.3, we arrive at the next lemma.

Lemma B.4. *There exist constant $C_1 > C_2 > 0$ such that*

$$C_2 \sum_i \frac{Z_{iT} \, D_T^{-\gamma_i}}{E[Z_{iT} \, D_T^{1-\gamma_i}]} \;\leq\; \frac{\xi_T}{E[D_T \, \xi_T]} \;\leq\; C_1 \sum_i \frac{Z_{iT} \, D_T^{-\gamma_i}}{E[Z_{iT} \, D_T^{1-\gamma_i}]}.$$

Lemma B.5. *Let $\lambda \in \Pi_J$ and $t = \lambda T$. Then, for all $j \neq J$ and all i, we have*

$$\frac{E_t[W_{iT} \, D_T^{-1} \, Z_{jT} \, D_T^{-\gamma_j}] \, E[Z_{jT} \, D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{JT} \, D_T^{-\gamma_J}] \, E[Z_{JT} \, D_T^{1-\gamma_J}]^{-1}} \;\to\; 0$$

almost surely as $T \to \infty$.

Proof. Because, in equilibrium, $W_{iT} \, D_T^{-1} \leq 1$, we have

$$\frac{E_t[W_{iT} \, D_T^{-1} \, Z_{jT} \, D_T^{-\gamma_j}] \, E[Z_{jT} \, D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{JT} \, D_T^{-\gamma_J}] \, E[Z_{JT} \, D_T^{1-\gamma_J}]^{-1}} \;\leq\; \frac{E_t[Z_{jT} \, D_T^{-\gamma_j}] \, E[Z_{jT} \, D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{JT} \, D_T^{-\gamma_J}] \, E[Z_{JT} \, D_T^{1-\gamma_J}]^{-1}}$$

$$= \; e^{\sigma(\tilde{\gamma}_J - \tilde{\gamma}_j)B_t} \, e^{\frac{1}{2}\sigma^2 T((\tilde{\gamma}_j^2 - \tilde{\gamma}_J^2)(1-\lambda) + (1-\tilde{\gamma}_J)^2 - (1-\tilde{\gamma}_j)^2)}$$

$$\tag{B.3}$$

and the claim follows if

$$\tilde{\gamma}_j^2(1-\lambda) - (1 - \tilde{\gamma}_j)^2 \;<\; \tilde{\gamma}_J^2(1-\lambda) - (1 - \tilde{\gamma}_J)^2 \tag{B.4}$$

for all $j \neq J$. Now, the function

$$h(\gamma) \;=\; \gamma^2(1-\lambda) \;-\; (1-\gamma)^2 \;=\; -\lambda\gamma^2 \;+\; 2\gamma - 1$$

is a concave parabola, attaining maximum at the vertex $1/\lambda$. Therefore, (B.4) holds if $\tilde{\gamma}_J$ is the closest to $1/\lambda$ among all effective risk aversions. Let $0 < J < l$. Then,

$$\frac{\tilde{\gamma}_{J-1} + \tilde{\gamma}_J}{2} \;<\; \frac{1}{\lambda} \;<\; \frac{\tilde{\gamma}_J + \tilde{\gamma}_{J+1}}{2}$$

and therefore, because $\tilde{\gamma}_{J-1} < \tilde{\gamma}_J < \tilde{\gamma}_{J+1}$, the claim is immediate. The cases $J = 0$, l are analogous. Q.E.D.

Lemma B.6. *Let* $\lambda \in \Pi_J$ *and* $t = \lambda T$. *Then, for all* $i \neq J$,

$$\frac{E_t[\, W_{iT}\, D_T^{-1}\, Z_{JT}\, D_T^{-\gamma_J}\,]}{E_t[Z_{JT}\, D_T^{-\gamma_J}]} \;\to\; 0$$

almost surely as $T \to \infty$.

Proof. Applying Lemma B.3, we get

$$E[D_T\xi_T]^{1-\gamma_i} E[Z_{iT}^{b_i}\, \xi_T^{1-b_i}]^{\gamma_i} \;\geq\; K_2\, E[Z_{iT}\, D_T^{1-\gamma_i}]\,,$$

that is

$$\frac{E[D_T\xi_T]}{E[Z_{iT}^{b_i}\, \xi_T^{1-b_i}]} \;\leq\; \left(\frac{E[D_T\xi_T]}{K_2\, E[Z_{iT}\, D_T^{1-\gamma_i}]} \right)^{b_i}.$$

Similarly,

$$\left(\frac{E[D_T\xi_T]}{E[Z_{JT}^{b_J}\, \xi_T^{1-b_J}]} \right)^{\gamma_J} \;\geq\; K_3\, \frac{E[D_T\xi_T]}{E[Z_{JT}\, D_T^{1-\gamma_J}]}\,.$$

By Lemma B.2,

$$\xi_T \;\geq\; \left(\frac{\psi_J\, E[D_T\xi_T]}{E[Z_{JT}^{b_J}\, \xi_T^{1-b_J}]} \right)^{\gamma_J} Z_{JT}\, D_T^{-\gamma_J}\,,$$

and hence,

$$W_{iT} \;\leq\; K_4\, \frac{E[Z_{JT}\, D_T^{1-\gamma_J}]^{b_i}}{E[Z_{iT}\, D_T^{1-\gamma_i}]^{b_i}}\, Z_{iT}^{b_i}\, Z_{JT}^{-b_i}\, D_T^{\gamma_J\, b_i}\,. \tag{B.5}$$

Therefore,

$$W_{iT}D_T^{-1} \;\leq\; K_5\, e^{\frac{1}{2}\sigma^2 T b_i ((1-\tilde{\gamma}_J)^2 - (1-\tilde{\gamma}_i)^2)}\, e^{\sigma \left(b_i(\tilde{\gamma}_J + \delta_i) - 1 \right) B_T}\,. \tag{B.6}$$

Because, by the equilibrium market clearing, $W_{iT} D_T^{-1} \leq 1$, we get

$$
\frac{E_t[W_{iT} D_T^{-1} Z_{JT} D_T^{-\gamma_J}]}{E_t[Z_{JT} D_T^{-\gamma_J}]} = \frac{E_{\lambda T}[W_{iT} D_T^{-\tilde{\gamma}_J - 1}]}{E_{\lambda T}[D_T^{-\tilde{\gamma}_J}]}
$$

$$
\leq E_{\lambda T}\left[e^{-\tilde{\gamma}_J \sigma (B_T - B_{\lambda T}) - (\tilde{\gamma}_J)^2 \sigma^2 (1-\lambda)T} \right. \tag{B.7}
$$

$$
\left. \times \min\left\{ K_5 e^{\frac{1}{2}\sigma^2 T b_i((1-\tilde{\gamma}_J)^2 - (1-\tilde{\gamma}_i)^2)} e^{\sigma\left(b_i(\tilde{\gamma}_J + \delta_i) - 1\right) B_T} , 1 \right\} \right].
$$

Denote $\eta = \frac{1}{2}\sigma^2 b_i((1 - \tilde{\gamma}_J)^2 - (1 - \tilde{\gamma}_i)^2)$, $\zeta = \sigma b_i (\tilde{\gamma}_J - \tilde{\gamma}_i)$ and $C_t = \log K_5 + \zeta B_{\lambda T}$. We need to consider the cases $\zeta > 0$ and $\zeta < 0$ separately.
If $\zeta > 0$ that is $\tilde{\gamma}_J > \tilde{\gamma}_i \Leftrightarrow \tilde{\gamma}_{J-1} \geq \tilde{\gamma}_i$. Then,

$$
E_{\lambda T}\left[e^{-\tilde{\gamma}_J \sigma (B_T - B_{\lambda T}) - 0.5(\tilde{\gamma}_J)^2 \sigma^2 (1-\lambda)T} \right.
$$

$$
\left. \times \min\left\{ K_5 e^{\frac{1}{2}\sigma^2 T b_i((1-\tilde{\gamma}_J)^2 - (1-\tilde{\gamma}_i)^2)} e^{\sigma\left(b_i(\tilde{\gamma}_J + \delta_i) - 1\right) B_T} , 1 \right\} \right] \tag{B.8}
$$

$$
= e^{C_t + (\eta - 0.5\tilde{\gamma}_J^2 \sigma^2 (1-\lambda))T} \frac{1}{\sqrt{2\pi}}
$$

$$
\times \int_{-\infty}^{-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2}} e^{-x^2/2} e^{x(-\tilde{\gamma}_J \sigma + \zeta)((1-\lambda)T)^{1/2}} dx
$$

$$
+ e^{-0.5\tilde{\gamma}_J^2 \sigma^2 (1-\lambda)T} \frac{1}{\sqrt{2\pi}} \int_{-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2}}^{+\infty} e^{-x^2/2} e^{-\tilde{\gamma}_J \sigma x((1-\lambda)T)^{1/2}} dx
$$

$$
= e^{C_t + (\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + 0.5\zeta))T}
$$

$$
\times N\left(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} - (-\tilde{\gamma}_J \sigma + \zeta)((1-\lambda)T)^{1/2} \right)
$$

$$
+ \left(1 - N\left(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} + \tilde{\gamma}_J \sigma ((1-\lambda)T)^{1/2} \right) \right). \tag{B.9}
$$

The following lemma is well known.

Lemma B.7. *We have*

$$
\lim_{x \to -\infty} \frac{N(x)}{e^{-x^2/2}(-x)^{-1}} = \lim_{x \to +\infty} \frac{1 - N(x)}{e^{-x^2}x^{-1}} = \frac{1}{\sqrt{2\pi}}.
$$

Because, for generic parameter values, we will have exponential decay, the factor x^{-1} in the asymptotic of Lemma B.7 can be neglected. Similarly, by the strong law of large numbers for Brownian motion, $B_{\lambda T}/T \to 0$ almost surely, and therefore, in the expressions of the form $B_{\lambda T} + aT = T(a + B_{\lambda T}/T)$, the term with $B_{\lambda T}$ can be also ignored when calculating asymptotic behavior.

We first observe that the term

$$e^{(\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + 0.5\zeta))T}$$
$$\times \; N\big(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} - (-\tilde{\gamma}_J \sigma + \zeta)((1-\lambda)T)^{1/2}\big) \tag{B.10}$$

always converges to zero. Indeed, if $\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + 0.5\zeta) < 0$ then we are done. If $\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + 0.5\zeta) > 0$ we have

$$-\zeta^{-1}\eta T ((1-\lambda)T)^{-1/2} - (-\tilde{\gamma}_J \sigma + \zeta)((1-\lambda)T)^{1/2}$$
$$= -\zeta^{-1}((1-\lambda)T)^{-1/2} T (\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + \zeta))) \tag{B.11}$$
$$< -\zeta^{-1}((1-\lambda)T)^{-1/2} T (\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + 0.5\zeta)))$$

is negative and converges to $-\infty$, so that Lemma B.7 applies and we need to show that

$$\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + 0.5\zeta) - 0.5\zeta^{-2}(1-\lambda)^{-1}(\eta + (1-\lambda)\zeta(-\tilde{\gamma}_J \sigma + \zeta))^2 < 0,$$

which follows by direct calculation. Thus, we only need to show that the term

$$1 - N\big(-\zeta^{-1}(\eta T + C_t)((1-\lambda)T)^{-1/2} + \tilde{\gamma}_J \sigma ((1-\lambda)T)^{1/2}\big)$$

converges to zero. This happens precisely when $0 < -\zeta^{-1}\eta + \tilde{\gamma}_J \sigma (1-\lambda)$. That is, we need that $1 - \lambda > \frac{(\tilde{\gamma}_J + \tilde{\gamma}_i - 2)}{2\tilde{\gamma}_J}$. We have

$$1 - \lambda > 1 - \frac{2}{\tilde{\gamma}_J + \tilde{\gamma}_{J-1}} \geq 1 - \frac{2}{\tilde{\gamma}_J + \tilde{\gamma}_i} = \frac{\tilde{\gamma}_J + \tilde{\gamma}_i - 2}{\tilde{\gamma}_J + \tilde{\gamma}_i} > \frac{\tilde{\gamma}_J + \tilde{\gamma}_i - 2}{2\tilde{\gamma}_J}$$

and the claim follows. The case $\zeta < 0$ is analogous.

<div style="text-align: right">Q.E.D.</div>

To prove Proposition 3.6, we will need auxiliary lemmas analogous to Lemmas B.5 and B.6.

Lemma B.8. *Let $\lambda \in (0,1)$ and $t = \lambda T$. Then, for all $j \neq 0$ and all i we have*

$$\frac{E_t[W_{iT} D_T^{-1} Z_{jT} D_T^{1-\gamma_j}] E[Z_{jT} D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{0T} D_T^{1-\gamma_0}] E[Z_{0T} D_T^{1-\gamma_0}]^{-1}} \to 0$$

almost surely as $T \to \infty$.

Proof. The proof is analogous to that of Lemma B.5. We have

$$\frac{E_t[W_{iT} D_T^{-1} Z_{jT} D_T^{1-\gamma_j}] E[Z_{jT} D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{0T} D_T^{1-\gamma_0}] E[Z_{0T} D_T^{1-\gamma_0}]^{-1}} \leq \frac{E_t[Z_{jT} D_T^{-\gamma_j}] E[Z_{jT} D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{0T} D_T^{1-\gamma_0}] E[Z_{0T} D_T^{1-\gamma_0}]^{-1}}$$

$$= e^{\sigma(\tilde{\gamma}_0 - \tilde{\gamma}_j) B_{\lambda T}} e^{\frac{1}{2}\sigma^2(|1-\tilde{\gamma}_0|^2 - |1-\tilde{\gamma}_j|^2)(1-\lambda) T}$$

(B.12)

and the claim follows from the strong law of large numbers for B_T. Q.E.D.

Lemma B.9. *Let $\lambda \in (0,1)$ and $t = \lambda T$. Then, for all $i \neq J$,*

$$\frac{E_t[W_{iT} D_T^{-1} Z_{0T} D_T^{1-\gamma_0}]}{E_t[Z_{0T} D_T^{1-\gamma_0}]} \to 0$$

almost surely as $T \to \infty$.

Proof. Note that

$$\frac{E_t[W_{iT} D_T^{-1} Z_{0T} D_T^{1-\gamma_0}]}{E_t[Z_{0T} D_T^{1-\gamma_0}]} = \frac{E_t[W_{iT} D_T^{-1} D_T^{1-\tilde{\gamma}_0}]}{E_t[D_T^{1-\tilde{\gamma}_0}]}.$$

Therefore, the same argument as that in (B.7) and subsequent arguments in the proof of Lemma B.6 apply, with minor modifications, and using the bound (B.6) with $J = 0$. Defining

$$\eta = \frac{1}{2}\sigma^2 b_i((1-\tilde{\gamma}_0)^2 - (1-\tilde{\gamma}_i)^2) , \quad \zeta = \sigma b_i(\tilde{\gamma}_0 - \tilde{\gamma}_i)$$

we need to show that $-\zeta^{-1}\eta + (\tilde{\gamma}_0 - 1)\sigma(1-\lambda)$ is positive when $\tilde{\gamma}_i < \tilde{\gamma}_0$ and

is negative otherwise. Let $\tilde{\gamma}_0 > \tilde{\gamma}_i$. We have

$$-\zeta^{-1}\eta + (\tilde{\gamma}_0 - 1)\,\sigma(1 - \lambda) \;=\; \sigma\left(1 - 0.5\,(\tilde{\gamma}_0 + \tilde{\gamma}_i) + (\tilde{\gamma}_0 - 1)(1 - \lambda)\right).$$

Because this expression is linear in λ, it suffices to prove its positivity for $\lambda = 0$ and $\lambda = 1$. Both cases follow from the inequalities $\tilde{\gamma}_0 > \tilde{\gamma}_i$ and $\tilde{\gamma}_0 + \tilde{\gamma}_i < 2$. The case $\tilde{\gamma}_0 < \tilde{\gamma}_i$ is analogous. $\hfill$ Q.E.D.

The same argument as that in the proof of Lemma B.2 implies the following result.

Lemma B.10. *Fix an agent i and assume that $\gamma_i > 1$. Then,*

$$
\begin{aligned}
\max_k \, (Z_{kT}\,D_T^{-\gamma_k})^{1-b_i} \left(\frac{\psi_k\,E[D_T\xi_T]}{E[Z_{kT}^{b_k}\,\xi_T^{1-b_k}]} \right)^{\gamma_k(1-b_i)} &\;\leq\; \xi_T^{1-b_i} \\
\leq\; C \sum_k (Z_{kT}\,D_T^{-\gamma_k})^{1-b_i} \left(\frac{\psi_k\,E[D_T\xi_T]}{E[Z_{kT}^{b_k}\,\xi_T^{1-b_k}]} \right)^{\gamma_k(1-b_i)} &
\end{aligned}
\tag{B.13}
$$

for some $C > 0$.

Following the same arguments as in the proof of Lemma B.4, we arrive at the following lemma.

Lemma B.11. *There exist $C_4 > C_3 > 0$ such that*

$$
C_4 \sum_k \left(\frac{Z_{kT}\,D_T^{-\gamma_k}}{E[Z_{kT}\,D_T^{1-\gamma_k}]} \right)^{1-b_i} \;\leq\; \frac{\xi_T^{1-b_i}}{(E[D_T\,\xi_T])^{1-b_i}} \;\leq\; C_3 \sum_i \left(\frac{Z_{kT}\,D_T^{-\gamma_k}}{E[Z_{kT}\,D_T^{1-\gamma_k}]} \right)^{1-b_i}.
$$

Now, given these bounds, the proof proceeds analogously to that of Theorem 3.1. We will need two auxiliary lemmas.

Lemma B.12. *For any $\lambda \in \Theta_J^i$, any j and any $k \neq J$ we have*

$$
\frac{E_t[W_{jT}\,D_T^{-1}\,Z_{it}^{b_i}\,(Z_{kT}\,D_T^{-\gamma_k})^{1-b_i}]\,(E[Z_{kT}\,D_T^{1-\gamma_k}])^{b_i-1}}{E_t[Z_{it}^{b_i}\,(Z_{JT}\,D_T^{-\gamma_J})^{1-b_i}]\,(E[Z_{JT}\,D_T^{1-\gamma_J}]^{-1})^{b_i-1}} \;\to\; 0
$$

almost surely.

Proof. Because $W_{jT} D_T^{-1} \le 1$, we have

$$\frac{E_t[W_{jT} D_T^{-1} Z_{it}^{b_i} (Z_{kT} D_T^{-\gamma_k})^{1-b_i}] (E[Z_{kT} D_T^{1-\gamma_k}])^{b_i-1}}{E_t[Z_{it}^{b_i} (Z_{JT} D_T^{-\gamma_J})^{1-b_i}] (E[Z_{JT} D_T^{1-\gamma_J}]^{-1})^{b_i-1}}$$

$$\le \frac{E_t[Z_{it}^{b_i} (Z_{kT} D_T^{-\gamma_k})^{1-b_i}] (E[Z_{kT} D_T^{1-\gamma_k}])^{b_i-1}}{E_t[Z_{it}^{b_i} (Z_{JT} D_T^{-\gamma_J})^{1-b_i}] (E[Z_{JT} D_T^{1-\gamma_J}]^{-1})^{b_i-1}} = e^{\sigma (\tilde{\gamma}_J - \tilde{\gamma}_k)(1-b_i) B_{\lambda T}}$$

$$\times e^{\frac{1}{2}\sigma^2 T \left((1-\lambda)\left((\delta_i b_i - \tilde{\gamma}_k (1-b_i))^2 - (\delta_i b_i - \tilde{\gamma}_J (1-b_i))^2\right) + (1-b_i)\left((1-\tilde{\gamma}_J)^2 - (1-\tilde{\gamma}_k)^2\right)\right)}.$$

$$(B.14)$$

and the claim follows by the same arguments as in the proof of Lemma B.5.
Q.E.D.

Lemma B.13. *Let $\lambda \in \Theta_J^i$ and $t = \lambda T$. Then, for all $j \ne J$ we have*

$$\frac{E_t[W_{jT} D_T^{-1} Z_{iT}^{b_i} (Z_{JT} D_T^{-\gamma_J})^{1-b_i}]}{E_t[Z_{iT}^{b_i} (Z_{JT} D_T^{-\gamma_J})^{1-b_i}]} \to 0$$

almost surely as $T \to \infty$.

Proof. We have

$$\frac{E_t[W_{jT} D_T^{-1} Z_{iT}^{b_i} (Z_{JT} D_T^{-\gamma_J})^{1-b_i}]}{E_t[Z_{iT}^{b_i} (Z_{JT} D_T^{-\gamma_J})^{1-b_i}]} = \frac{E_t[W_{jT} D_T^{-1} D_T^{\delta_i b_i - (1-b_i) \tilde{\gamma}_J}]}{E_t[D_T^{\delta_i b_i - (1-b_i) \tilde{\gamma}_J}]}.$$

Therefore, the same argument as that in (B.7) and subsequent arguments in the proof of Lemma B.6 apply with minor modifications, using the same bound (B.6). Defining

$$\eta = \frac{1}{2}\sigma^2 b_j((1-\tilde{\gamma}_J)^2 - (1-\tilde{\gamma}_j)^2), \quad \zeta = \sigma b_j (\tilde{\gamma}_J - \tilde{\gamma}_j)$$

we find that it suffices to show that

$$0 < -\zeta^{-1}\eta - (\delta_i b_i - (1-b_i) \tilde{\gamma}_J) \sigma(1-\lambda)$$

when $\zeta > 0$, and the opposite inequality should hold when $\zeta < 0$. This follows from the definition of the interval Θ_J^i. Q.E.D.

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C Proofs

Proof of Proposition 3.1. Define the function $F = F(a_1, \cdots, a_K)$ to solve $\sum_i F^{-b_i} a_i^{b_i} = 1$. Then, $U'(D_T) = F(D_T^{-\gamma_1} Z_{0T} \alpha_1^{\gamma_1}, \cdots, D_T^{-\gamma_K} Z_{KT} \alpha_K^{\gamma_K})$, where $\alpha_i = \frac{\psi_i E[D_T \xi_T]}{E[\xi_T^{1-b_i} Z_{iT}^{b_i}]}$. Therefore,

$$
\begin{aligned}
\frac{d}{dD_T} U'(D_T) &= \sum_i F_{a_i} \alpha_i^{\gamma_i} \frac{d}{dD_T} (D_T^{-\tilde{\gamma}_i} e^{(-0.5\delta_i^2 \sigma^2 - \delta_i \mu) T}) \\
&= -\sum_i F_{a_i} \alpha_i^{\gamma_i} \tilde{\gamma}_i D_T^{-\gamma_i - 1} Z_{iT} = -U'(D_T) D_T^{-1} \sum_i \tilde{\gamma}_i \lambda_{iT}.
\end{aligned}
\tag{C.1}
$$

and

$$
D_T \frac{d\tilde{\gamma}^U}{dD_T} = -\sum_i \lambda_{iT} b_i \tilde{\gamma}_i^2 + 2\tilde{\gamma}^U \sum_i \lambda_{iT} b_i \tilde{\gamma}_i - \sum_i \lambda_{iT} b_i (\tilde{\gamma}^U)^2.
$$

Applying the inequality $a^2 + b^2 \geq 2|ab|$ to

$$
a^2 = \sum_i \lambda_{iT} b_i \tilde{\gamma}_i^2, \quad b^2 = \sum_i \lambda_{iT} b_i (\tilde{\gamma}^U)^2
$$

together with the Cauchy–Schwarz inequality completes the proof of monotonicity. The fact that $\tilde{\gamma}^U$ converges to extreme effective risk aversions for extreme levels of wealth follows directly from the representation for $\tilde{\gamma}^U$. Q.E.D.

Proof of Propositions 3.3 and 4.1. Recall that price S_t and the wealth of agent i are given by $\log S_t = \log E_t[\xi_T D_T] - \log E_t[\xi_T]$ and

$$
\log W_{it} = \log \left(\frac{\psi_i E[D_T \xi_T]}{E[Z_{iT}^{b_i} \xi_T^{1-b_i}]} E_t[Z_{iT}^{b_i} \xi_T^{1-b_i}] \right) - \log E_t[\xi_T]
$$

respectively. Denoting by $\mathcal{D}_t$, the Malliavin derivative operator[58], we get

$$
\sigma_t = \mathcal{D}_t \log S_t \quad \text{and} \quad \mathcal{D}_t \log W_{it} = \sigma_t \pi_{it}.
$$

---

[58]For an expedient introduction to Malliavin derivatives, see Detemple, Garcia and Rindisbacher (2003).

Because $D_T = e^{\mu T + \sigma B_T}$, we arrive at

$$
\begin{aligned}
\mathcal{D}_t \left(\log W_{it} \right) &= \frac{E_t[\mathcal{D}_t(Z_{iT}^{b_i} \, \xi_T^{1-b_i})]}{E_t[Z_{iT}^{b_i} \, \xi_T^{1-b_i}]} - \frac{E_t[\mathcal{D}_t(\xi_T)]}{E_t[\xi_T]} \\
&= \frac{E_t[\, b_i \, \delta_i \, \sigma \, Z_{iT}^{b_i} \, \xi_T^{1-b_i} \, - \, (1 - b_i) \, Z_{iT}^{b_i} \, \xi_T^{1-b_i} \, \sigma \, \tilde{\gamma}^U(D_T)]}{E_t[Z_{iT}^{b_i} \, \xi_T^{1-b_i}]} + \sigma \, \frac{E_t[\tilde{\gamma}^U(D_T) \, \xi_T]}{E_t[\xi_T]}
\end{aligned}
\tag{C.2}
$$

and, similarly,

$$
\mathcal{D}_t \left(\log S_t \right) = \sigma \left(1 - (E_t^Q[D_T])^{-1} \operatorname{Cov}_t^Q(\tilde{\gamma}^U(D_T), D_T) \right). \tag{C.3}
$$

It remains to show the expression for the market price of risk. By the above,

$$
\frac{dE_t[\xi_T \, D_T]}{E_t[\xi_T \, D_T]} = U_t dW_t \,, \quad \frac{dE_t[\xi_T]}{E_t[\xi_T]} = V_t dW_t \,,
$$

$$
U_t = \sigma \left(1 - \frac{E_t[\tilde{\gamma}^U(D_T) \, \sigma \, \xi_T \, D_T]}{E_t[\xi_T \, D_T]} \right) \,, \quad V_t = -\sigma \, E_t^Q[\tilde{\gamma}^U(D_T)] \,.
$$

Applying Ito's formula, we get

$$
\mu_t = r + \frac{1}{2}(V_t^2 - U_t^2 + (U_t - V_t)^2) = r + V_t (V_t - U_t)
$$

and the claim follows. Q.E.D.

Proof of Theorem 3.1. Let $\lambda \in \Pi_J$ and $t = \lambda T$. By Propositions 3.4 and 3.6, it suffices to show that $E_t^Q[W_{iT} \, D_T^{-1}] \to 0$ almost surely for any $i \neq J$.

By Lemma B.4,

$$
\begin{aligned}
E_t^Q[W_{iT} \, D_T^{-1}] \leq \frac{C_1}{C_2} &\left(\frac{E_t[\, W_{iT} \, D_T^{-1} \, Z_{JT} \, D_T^{-\gamma_J}]}{E_t[Z_{JT} \, D_T^{-\gamma_J}]} \right. \\
&\left. + \frac{\sum_{j \neq J} E_t[W_{iT} \, D_T^{-1} \, Z_{jT} \, D_T^{-\gamma_j}] \, E[Z_{jT} \, D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{JT} \, D_T^{-\gamma_J}] \, E[Z_{JT} \, D_T^{1-\gamma_J}]^{-1}} \right)
\end{aligned}
\tag{C.4}
$$

and the claim follows from Lemmas B.5 and B.6. Q.E.D.

Proof of Proposition 3.6. The first claim of Proposition 3.6 follows directly from (B.6). Indeed, since $B_T/T \to 0$ almost surely, the exponent in (B.6) with $J = 0$ is given by

$$T\left(\frac{1}{2}\sigma^2 b_i((1-\tilde{\gamma}_0)^2 - (1-\tilde{\gamma}_i)^2) + \sigma\, b_i\,(\tilde{\gamma}_0 - \tilde{\gamma}_i)\, B_T/T\right) \qquad \text{(C.5)}$$

and behaves asymptotically as $T\frac{1}{2}\sigma^2 b_i((1-\tilde{\gamma}_0)^2 - (1-\tilde{\gamma}_i)^2)$.

Because $(1-\tilde{\gamma}_0)^2 < (1-\tilde{\gamma}_i)^2$ by the definition of the agent 0, the expression (C.5) converges to 0 almost surely. By (B.6), $W_{iT}\, D_T^{-1}$ also converges to zero almost surely as $T \to \infty$.

It remains to prove the claim for the dividend-forward measure. The same argument as that in (C.4) implies that

$$E_t^{Q^D}[W_{iT}\, D_T^{-1}] \leq \frac{C_1}{C_2}\left(\frac{E_t[W_{iT}\, D_T^{-1}\, Z_{JT}\, D_T^{1-\gamma_J}]}{E_t[Z_{JT}\, D_T^{1-\gamma_J}]}\right.$$
$$\left. + \frac{\sum_{j\neq J} E_t[W_{iT}\, D_T^{-1}\, Z_{jT}\, D_T^{1-\gamma_j}]\, E[Z_{jT}\, D_T^{1-\gamma_j}]^{-1}}{E_t[Z_{JT}\, D_T^{1-\gamma_J}]\, E[Z_{JT}\, D_T^{1-\gamma_J}]^{-1}}\right) \qquad \text{(C.6)}$$

and the claim follows from Lemmas B.8 and B.9. $\qquad$ Q.E.D.

Proof of Theorem 4.1. Let $t = \lambda T$ with $\lambda \in \Theta_J^i$. By Proposition 4.2, it suffices to show that only agent J survives with respect to the Q^{W_i}-measure. The same argument as that in (C.4) and (C.6), together with Lemma B.11, implies that

$$E_t^{Q^{W_i}}[W_{jT}\, D_T^{-1}] = \frac{E_t[W_{jT}\, D_T^{-1}\, Z_{iT}^{b_i}\, \xi_T^{1-b_i}]}{E_t[Z_{iT}^{b_i}\, \xi_T^{1-b_i}]}$$
$$\leq \frac{C_3}{C_4}\left(\frac{E_t[W_{jT}\, D_T^{-1}\, Z_{iT}^{b_i}\, (Z_{JT}\, D_T^{-\gamma_J})^{1-b_i}]}{E_t[Z_{iT}^{b_i}\, (Z_{JT}\, D_T^{-\gamma_J})^{1-b_i}]}\right.$$
$$\left. + \sum_{k\neq J} \frac{E_t[W_{jT}\, D_T^{-1}\, Z_{it}^{b_i}\, (Z_{kT}\, D_T^{-\gamma_k})^{1-b_i}]\, (E[Z_{kT}\, D_T^{1-\gamma_k}])^{b_i-1}}{E_t[Z_{it}^{b_i}\, (Z_{JT}\, D_T^{-\gamma_J})^{1-b_i}]\, (E[Z_{JT}\, D_T^{1-\gamma_J}]^{-1})^{b_i-1}}\right) \qquad \text{(C.7)}$$

and the claim follows from Lemmas B.12 and B.13. $\qquad$ Q.E.D.

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