Noisy Arrow-Debreu Equilibria *

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Abstract

I develop a noisy rational expectations equilibrium model with a continuum of states and a full set of options that render the market complete. I show a major difference in equilibrium behavior between models with constant absolute risk aversion (CARA) and non-CARA preferences. First, when informed traders have non-CARA preferences, all equilibria are fully revealing, independent of the amount of noise in the supply. Second, when informed traders have CARA preferences, but uninformed traders have non-CARA preferences, the set of equilibria contains a *fully revealing* equilibrium and a *minimally revealing* equilibrium. The latter reveals the minimal possible amount of information and is highly inefficient: In this equilibrium, Arrow-Debreu state prices are not sufficient to recover the information contained in the noisy aggregate demand and supply. My results have important implications for price discovery through options.

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1 Introduction

Options play a pivotal role in the functioning of financial markets. For almost any liquid exchange-traded instrument there exists a whole menu of options written on the instrument payoff. These options allow investors to hedge and share non-linear risks,[1] as well as take leveraged bets on private information in a precise, state-contingent fashion. However, due to the intrinsically non-linear nature of option payoffs, standard noisy rational expectations models have little to say about

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[1] Options are actively traded by both retail and institutional investors. For example, in 2012, more than a billion option contracts were traded at the Chicago Board of Exchange (the largest option exchange in the world), with a total dollar volume of more than half a trillion USD. See https://www.cboe.com/data/marketstats-2012.pdf. In the standard Black and Scholes (1973) theory of option pricing, options are redundant because markets are dynamically complete. In reality, however, option prices are exposed to jumps, stochastic volatility, and other types of unhedgeable risks. See, for example, Bollerslev and Todorov (2011). In addition, continuous option hedging underlying the Black-Scholes argument is typically impossible because of transaction costs (see Leland, 1985, and Cvitanic, Shreve, Soner, 1995, who show that a buy-and-hold strategy is optimal for super-hedging (i.e., hedging from above) a call option).

how private information gets incorporated into option prices. To study this question, I develop a noisy rational expectations equilibrium model with a continuum of states and a full set of options that render the market complete. The model allows for an arbitrary probability distribution of states, arbitrary signal structures, and general trader preferences. Market completeness allows me to characterize equilibrium price discovery explicitly in terms of the *variation of Arrow-Debreu state prices across states.* Specifically, I identify two types of equilibria in my model: In a more efficient equilibrium, information is revealed through the quadratic variation of state prices across states; in a less efficient equilibrium, information is revealed through a weighted linear variation of state prices across states.

I show a major difference in equilibrium behavior between models with constant absolute risk aversion (CARA) and non-CARA preferences. First, if informed traders have non-CARA preferences, *all equilibria are fully revealing,* independent of the amount of noise in the supply. Second, when informed traders have CARA preferences, but uninformed traders have non-CARA preferences, the set of equilibria contains a *fully revealing* equilibrium and a *minimally revealing* equilibrium. The latter reveals the minimal possible amount of information and is highly inefficient: In this equilibrium, Arrow-Debreu state prices are not sufficient to recover the information contained in the noisy aggregate demand and supply. However, this happens only for specific patterns of options prices that I characterize. When such a pattern is realized, investors cannot distinguish whether prices are good or bad news and need to use non-redundant information contained in the (signed) trading volume.

Since markets are complete, option prices can be characterized directly in terms of the unique Arrow-Debreu state prices and the equilibrium is equivalent to that with a full set of Arrow-Debreu securities contingent on state realizations.[2] In equilibrium, the agents' first-order conditions imply that Arrow-Debreu state prices (henceforth, state prices) are given by the product of the marginal utility of consumption of the representative agent and the probability density reflecting the agent's equilibrium beliefs. Through market clearing, aggregate consumption in a given state equals the total supply of Arrow-Debreu securities for that state; as a result, log Arrow-Debreu state prices (as a function of state) become a noisy signal regarding the entire underlying probability density and can be decomposed into the *smooth component* (probabilities) and the *rough component* (supply shocks). One important novelty of this paper is the observation that, when the state space is continuous, tools from stochastic calculus can be used to solve the problem of inferring probabilities from state prices, with the continuous state playing the role of continuous time. These tools allow me to show that, when supply shocks are additive across states and follow a diffusion process (as in the Kyle (1985) model), and the representative agent has non-CARA preferences, the two components can be perfectly isolated using the *realized quadratic variation of state prices.* This result is driven by two effects: (1) Smooth functions have zero quadratic variation and (2) the rate of change of the representative agent's marginal utility in response to supply shocks is given by the representative agent's absolute risk aversion. Consequently, the *realized quadratic variation of Arrow-Debreu state*

[2] In such a noisy Arrow-Debreu equilibrium, markets are complete with respect to state realizations, but not with respect to the realizations of supply shocks. However, I will abuse the notation and call such markets "complete."

prices reveals this risk aversion in all states of the world. If risk aversion is non-constant (i.e., the representative agent has non-CARA preferences), the supply shock (and hence also the probability density) can be uniquely recovered from state prices.[3] This *recovery is non-parametric:* As soon as the noisy (supply-driven) part of state prices is pinned down, it can be uniquely separated from the smooth (probability density) part. In particular, uninformed agents do not need to know anything about the structure of the underlying density, except that it is smooth.[4]

An important insight from the above discussion is that the manner in which equilibrium state prices reveal information depends on the risk aversion: State prices with a CARA representative agent reveal information only through their level, whereas with a non-CARA representative agent the rate of change in equilibrium state prices with respect to supply shocks reveals additional information. Furthermore, my results suggesting that the standard approach to recovering state prices from option prices with kernel smoothing[5] may lead to losses of important information: The *non-smooth fluctuations of state prices across states contain information about supply shocks* and can be exploited to extract more precise fundamental information.

Equilibrium state prices are expressed in terms of beliefs and preferences of the representative agent. While market completeness implies that such a representative agent always exists, the agents' "aggregated" preferences and beliefs are determined in equilibrium by the distribution of individual preferences and information in the economy. Thus, the nature of the representative agent strongly depends on the presence of non-CARA agents in the economy. To explain the underlying economic mechanism, I introduce an important notion in the noisy rational expectations equilibrium (REE) theory: *the residual supply.* As noted by Kreps (1977) in the context of non-noisy REE, market clearing implies that, in equilibrium, any given uninformed trader can recover total demand of other (the continuum of) uninformed traders, which (by market clearing) is equivalent to observing the residual supply (i.e., total asset supply net of the informed traders' demand). This observation was further developed by Breon-Drish (2012, 2014), who was the first to systematically use residual supply to study non-linear noisy REE.[6] A key consequence of this observation is that residual supply contains the minimal amount of information that is revealed in any noisy REE. In a noisy Arrow-Debreu equilibrium, residual supply is a rough (non-smooth) function of the state. The Wilson (1968) efficient risk-sharing rule implies that the rate of change of the residual supply with respect to supply shocks is given by the ratio of informed risk tolerance to the aggregate risk tolerance. According to the same argument as above, this rate can be recovered from the realized quadratic variation of the residual supply. As a result, if the risk tolerance of informed agents is non-constant, then any noisy REE is fully revealing. However, when the informed agents' risk tolerance is constant, *prices can reveal more information than is contained in the residual supply.* I show that when informed traders have CARA preferences while uninformed traders have non-CARA preferences, equilibrium is always non-unique, and the set of equilibria contains two

[3] This result is straightforward if risk aversion is strictly monotone. If it is not monotone, the recovery result still holds if risk aversion is nowhere flat.

[4] However, they need to know the preferences of the representative agent.

[5] See, for example, Ait-Sahalia and Lo (1998) and Vogt (2014).

[6] See also Kyle (1989), who used residual supply to study strategic trading.

extreme elements: the fully revealing equilibrium and the minimally revealing equilibrium. When uninformed traders believe that the rate of change in equilibrium state prices with respect to supply shocks reveals these shocks, they act accordingly, and their beliefs become self-fulfilling. In contrast, if uninformed agents use a simple linear filtering rule, as in standard CARA-Normal models, these beliefs are consistent with the *minimally revealing ("bad") equilibrium* that reveals exactly the information contained in the residual supply. Surprisingly, despite the availability of a continuum of prices that can be used to learn about the signal, these prices may be insufficient to recover the information contained in the residual supply: *Too many prices reveal too little information.* There is also a key difference in the mechanisms of price discovery in these two equilibria. In the fully revealing equilibrium, information is revealed through the quadratic variation of state prices and is therefore *non-linear* (quadratic) in these changes. In contrast, in the minimally revealing equilibrium, information is revealed through a weighted *linear* combination of changes in the state prices.

The mechanism underlying the inefficiency of the "bad" equilibrium differs from the simple self-fulfilling beliefs channel for the "good" equilibrium. Through market clearing, the residual supply employed by any given uninformed trader is used to filter the signal and coincides with the total demand of the entire "crowd" of uninformed traders, which in turn depends on the beliefs of this crowd. Thus, the correct belief is computed through Bayesian updating, which is conditional on the beliefs of the crowd, and an REE corresponds to a *fixed point of this belief's correspondence.* If the beliefs' correspondence has multiple fixed points, state prices by themselves are not sufficient to determine which of these fixed points corresponds to the unique correct beliefs and additional information contained in the (signed) trading volume is necessary to pin down the correct posterior. I show that noisy Arrow-Debreu equilibria are naturally prone to this form of market inefficiency due to the intrinsically non-monotonic behavior of optimal demand in complete markets.

Even though the full revelation result hinges on the assumption of a continuum of Arrow-Debreu securities, the main implications remain valid when the state space is discrete: State prices reveal "much more" information with non-CARA preferences than with CARA preferences. The assumption of a continuum of states leads to a particularly clean characterization of the primary effects, just as the assumption of continuous trading does in the Kyle (1985) model.[7] While the model in this paper is static, one can draw interesting parallels with continuous time models of dynamically complete markets in which Arrow-Debreu equilibria are implemented by trading a few long-lived securities (see Duffie and Huang, 1985, and Hugonnier, Malamud, and Trubowitz, 2012). In such equilibria, observable asset prices are naturally linked to the product of physical probabilities and the marginal utilities of consumption, just as they are in the static model. When informed agents have non-CARA preferences, the realized volatility of asset prices across time reveals information about supply shocks, just as the realized volatility of prices across states does in my model. This effect may make equilibria fully revealing. Investigating efficiency in such

[7]In Appendix C, I show that an "almost full revelation result" holds true when the state space is "sufficiently dense." Furthermore, while in the main text I only consider a model with two trader classes (informed and uninformed), I show in Appendix C that my results also hold true for models with dispersed private information.

dynamic markets is an important topic for future research.

Black (1975) was the first to suggest that options might play an important role in price discovery because informed traders should prefer options to stocks due to their embedded leverage. Numerous papers have since empirically investigated the way private information is incorporated into option prices. See, for example, Jennings and Starks (1986), Easley, O'Hara, and Srinivas (1998), Pan and Poteshman (2003), Chakravarty, Gulen, and Mayhew (2004), Roll, Schwartz, and Subrahmanyam (2009, 2010), and Augustiin, Brenner, and Subrahmanyam (2014), as well as the references therein.

While the empirical literature on this topic is vast, only a few papers have analyzed the role of options in price discovery theoretically. Back (1993) introduces trading in a single at-the-money call option into a continuous-time version of the Kyle (1985) insider trading model with a single, privately informed strategic trader. He shows that, in general, asymmetric information makes it impossible to price options by arbitrage, and the introduction of option trading can cause the volatility of the underlying asset to become stochastic. Biais and Hillion (1994) consider a single-period model of insider trading and study how the introduction of a nonredundant option affects information revelation and risk sharing. They assume that the asset payoff takes only three values, and hence a single option is sufficient to complete the market; quite surprisingly, they show that the introduction of the option has ambiguous consequences on the informational efficiency of the market and the profitability of insider's trades.

Brennan and Cao (1996) introduce derivatives (options with a quadratic payoff) into a version of the Hellwig (1980) model. In their model, there are no noise traders, and noise arises because agents (assumed to have CARA preferences) hedge their endowment shocks. Brennan and Cao (1996) find that, in a single-period model, introducing a quadratic derivative in zero net supply immediately allows the agents to achieve a Pareto-efficient allocation despite the fact that option price does not reveal any information. They also extend their analysis to allow for multiple quadratic derivatives and multiple trading rounds. My findings imply that their results depend crucially on the CARA-Normal setup and on the type of derivatives traded: A slight deviation from the CARA-Normal setting leads option trading to participate in price discovery, and the allocation is not Pareto-efficient unless the equilibrium is fully revealing. Furthermore, even in a CARA-Normal setting, options reveal information if they have discontinuous payoffs (such as, e.g., digital options).

Easley, O'Hara, and Srinivas (1998) study a model in which the asset has a binary payoff and agents always trade a single unit of the underlying, a put, or a call option with a competitive market maker who sets bid-and-ask prices. They find that option volume plays an important informational role and confirm these findings with intraday option data. My results imply that, in a minimally efficient equilibrium, option volume (modeled by the residual supply) indeed contains non-redundant information. Interestingly, recent empirical findings (Kagkadis et al., 2014) suggest that the realized quadratic variation of trading volume across option strikes contains information about future returns, consistent with the predictions of my model.

Cao (1999), Massa (2002), and Huang (2014) study the effects of derivatives on information acquisition. Cao (1999) shows that introducing non-linear derivatives in zero net supply in a version of the Hellwig (1980) model increases incentives for information acquisition, even though derivative prices do not reveal any additional information. Massa (2002) considers a continuous time model with CARA agents and finds that derivative trading conveys additional information and has non-trivial effects on the underlying price dynamics. Huang (2014) introduces a full set of call and put options in zero net supply into the Grossman and Stiglitz (1980) model. In his model, markets are complete and therefore the equilibrium in his model can be viewed as a "noisy Arrow-Debreu equlibrium." Huang (2014) finds that option prices do not reveal any information (as in Brennan and Cao, 1996), but introducing options may have surprising non-monotonic effects on incentives to acquire information.

Vanden (2008) considers a non-linear, multi-period REE model in which investors can trade the underlying as well as a log-claim on the underlying and studies the interaction between information quality and option prices under a non-standard definition of noise trading. Back and Crotty (2014) develop a continuous time version of the Kyle (1985) model to study (both theoretically and empirically) the informational role of stock and bond trading volume. In the presence of debt, stocks and bonds become, respectively, call and put options on the underlying payoff, and hence their model is equivalent to that with a stock and a single traded option. All of these papers assume that a single-option contract is available for trading and therefore cannot study price discovery across strikes.[8] Furthermore, all of these papers assume CARA or risk neutrality for all agents and therefore cannot study wealth effects on price discovery. Finally, since in my model option prices are influenced by the exogenous noise of traders' supply, my paper is also related to the demand-based option pricing theory of Garleanu, Pedersen, and Poteshman (2009).

My paper also belongs to the literature on noisy REE models that extends beyond the standard CARA-Normal setting of Grossman (1976), Grossman and Stiglitz (1980), Hellwig (1980), Diamond and Verrecchia (1981), and Admati (1985).[9] See, for example, Gennotte and Leland (1990), Ausubel (1990a,b), Bhattacharya and Spiegel (1991), Foster and Viswanathan (1993), Rochet and Vila (1994), DeMarzo and Skiadas (1998, 1999), Spiegel and Subrahmanyam (2000), Barlevy and Veronesi (2000, 2003), Bagnoli, Viswanathan, and Holden (2001), Peress (2004), Yuan (2005), Adrian (2009), Daley and Green (2013), and Banerjee and Green (2014).[10] With the exception of Peress (2004), who studies approximate equilibria in a two-state setting with CRRA agents, all of these papers assume either CARA or risk neutrality.[11] With the single exception of Peress (2004), who derives approximate solutions assuming CRRA preferences and a small binary risk, all models that I am aware of assume either CARA preferences or risk neutrality.[12]

[8] The only exception is Huang (2014), but in his model option prices do not reveal any information.

[9] See Brunnermeier (2001), Vives (2008), and Veldkamp (2011) for excellent textbook expositions.

[10] In my model, all traders are competitive and behave as price-takers. It would be interesting to extend the analysis to settings with strategic traders, such as those in Admati and Pfleiderer (1988), Holden and Subrahmanyam (1992), and Foster and Viswanathan (1993, 1996), in which traders submit market orders, and in Kyle (1989), Bhattacharya and Spiegel (1991), Back (1993), Vayanos (1999), Vives (2011), and Rostek and Veretka (2012), in which traders submit demand schedules. Malamud and Zhang (2014) takes a first step in this direction.

[11] Bernardo and Judd (2000) compute equilibria numerically in a version of the Grossman-Stiglitz model with constant relative risk aversion (CRRA) preferences, a single asset with log-Normal payoffs, and Gaussian noise in the supply. Unfortunately, in the setting that they consider, equilibrium fails to exist: When payoffs are log-Normal, CRRA preferences imply that both agents have to be long the asset to maintain positive consumption. Hence, they cannot absorb negative aggregate supply.

[12] Mertens and Hassan (2014) consider a noisy dynamic stochastic general equilibrium (DSGE) model with Epstein-

The most closely related papers are those by Breon-Drish (2012, 2014), Albagli, Hellwig, and Tsyvinski (2012, 2013), Palvölgyi and Venter (2014), and Chabakauri, Yuan, and Zachariadis (2014). Albagli, Hellwig, and Tsyvinski (2012, 2013) assume that agents are risk neutral, with bounds on their positions, and develop a noisy REE model with a single asset that allows for fairly general payoff distributions and dispersed private information. Breon-Drish (2012, 2014) develops a non-linear extension of the Grossman and Stiglitz (1980) single-asset model that allows for arbitrary payoff and noise distributions but requires CARA preferences for all agents. In particular, under the assumption that demand functions are continuous, Breon-Drish shows that equilibrium is unique and only the residual supply is revealed in this equilibrium. I demonstrate that in my model abandoning the CARA assumption implies other (more efficient) equilibria with continuous demand functions. Breon-Drish (2012) shows that, in certain cases, residual supply contains non-redundant information relative to that contained in the price. I show that this is in fact a typical outcome for the minimally revealing equilibrium of my model when uninformed agents' preferences are non-CARA.[13]

In a contemporaneous work, Chabakauri, Yuan, and Zachariadis (2014) study noisy REE with a discrete state space. They assume that all agents have CARA preferences and only consider equilibria that reveal the residual supply.[14] At the same time, their model allows for a rich set of incomplete market structures and derivative securities. Both Breon-Drish (2014) and Chabakauri, Yuan, and Zachariadis (2014) assume that the underlying probability density is exponentially affine in the signal.

Palvölgyi and Venter (2014) show other discontinuous and nonlinear equilibria in the Grossman-Stiglitz model. Interestingly enough, some of these discontinuous equilibria are "almost fully revealing."

The result regarding non-parametric reconstruction of the probability density from equilibrium state prices (see discussion above) links my paper to the important literature on recovering physical probabilities from observed asset prices. See Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Hansen (2012), and Ross (2013). These papers assume that the representative agent's marginal utility is not known and must be estimated but information is symmetric. In particular, Ross (2013) derives sufficient conditions under which physical probabilities can be perfectly recovered from option prices (the recovery theorem). My non-parametric recovery formula can be viewed as a complement to the Ross recovery theorem for the case of asymmetric information: One can first use the Ross approach to derive an estimate of the representative agent's marginal utility, and then use my formula to derive a "correction" to the Ross formula. One of the main insights from my results is that the "rough" (noisy) part of observable state prices should not be ignored:[15]

Zinn preferences and dispersed private information. They assume that markets are complete, but that securities are traded before the private information is realized and, hence, contingent claims prices do not reveal any information. In addition, they make a very strong assumption that continent claims exist contingent on private signal realizations.

[13] Empirically, one could use volume information as a proxy for the residual supply. Existing theoretical work on the information content of trading volume (Blume, Easley, and O'Hara, 1994; Schneider, 2009) suggests that observing volume is particularly valuable when there is uncertainty about information quality. My results show that volume information can be valuable due to wealth effects, even if the quality of information is known.

[14] That is, the minimally revealing equilibria (in my terminology).

[15] The common econometric approach used in the empirical research is to smooth observed state prices using, for

The non-smooth part of state prices contains information about supply shocks and this information can be recovered by looking at the variation in state prices and/or residual supply across states. Interestingly, recent empirical evidence (Kagkadis et al., 2014) supports this prediction of my model.

The result suggesting that Arrow-Debreu equilibria are often fully revealing has an interesting connection with experimental findings: As Plott and Sunder (1988) show, the availability of Arrow securities is crucial for market efficiency. That economic agents can make precise state-contingent bets on their private information allows prices to very quickly and efficiently aggregate information. Furthermore, Plott and Sunder (1988) find evidence that uninformed agents do learn from the residual supply: In the observed experimental behavior, uninformed agents wait until informed agents submit their orders in Arrow securities and then make their trading decisions conditional on observing these orders. These experimental findings suggest that the mechanism through which Arrow-Debreu state prices reveal information in my model may be closely related to the "true" manner in which information revelation works when markets are complete.

2 Model Setup

There are two time periods, $t = 0, 1$. The time $t = 1$ state of the world is given by the realization of a random variable X that takes values in an interval $[X_*, X^*]$, $-\infty < X_* \le X^* < +\infty$.[16] As is common in the noisy REE literature, I assume that a risk-free asset with the risk-free rate $1 + r$ is available in a perfectly elastic supply.

Throughout this paper, the following assumption is always present:

Assumption 2.1 *Markets are complete with respect to the state realization: A complete set of securities (options) spans the entire range of X.*

From the fundamental theorem of asset pricing (see, e.g., Dybvig and Ross, 2003), no arbitrage implies the existence of positive state prices, that is, prices of Arrow Debreu contingent claims, $M(x)$, $x \in [X_*, X^*]$, paying one unit of consumption good in state x and nothing in any other state. Since the market is complete, these state prices are unique. The value of an asset paying $W(X)$ at $t = 1$ is then given by the following:

$$\int_{X_*}^{X^*} M(x)W(x)dx\,.$$

Given the risk-free rate, equilibrium state prices must also satisfy the no-arbitrage condition:

$$\int_{X_*}^{X^*} M(x)dx \;=\; (1+r)^{-1}\,. \tag{1}$$

example, some kernel density.

[16] The assumption of boundedness is imposed for technical reasons and can be relaxed. While it formally excludes standards densities such as, for example, a Gaussian density, we can approximate any unbounded distribution with a bounded one by selecting X_* sufficiently small and X^* sufficiently large.

Since the state space is continuous, market completeness requires the existence of a continuum of securities. While this assumption is made for analytical tractability, the immense development of derivative markets over the last few decades effectively implies an "almost infinite" number of derivative securities traded for any given risk.[17] For example, a full set of European call and put options with arbitrary strikes K and payoffs $(X-K)^+$ and $(K-X)^+$ is sufficient to make the market complete. See Ross (1976) and Breeden and Litzenberger (1978).[18]

One may ask how close modern derivative markets come to being markets with a "continuous state." As an illustration, I consider one of the most active and liquid derivatives markets: the market for call and put options on the S&P 500 index and related exchange traded funds (ETFs). To measure how "dense" the state space is for traded options, one can look at the grid of moneyness (the quotient of the option strike to the price of the underlying) for which these options are available for trading. For example, for the date and maturity pair April 7, 2015, and April 17, 2015, SPX options with moneyness varying from 0.24 to 1.22 were available for trading, with an average moneyness grid step of 0.025, and about 230 different moneyness available for trading. The steps are non-homogeneously distributed, with steps for deep out-of-the-money options being larger than those for their at-the-money counterparts. For the moneyness range of $[0.8, 1.2]$ the grid step is approximately 0.003. While the steps of 0.025 and 0.003 are already very fine, one can go even further if one exploits options on leveraged ETFs; that is, options whose payoff is given by $(LX-K)^+$ and $(K-LX)^+$ where $L>1$ is the leverage factor. By rescaling, these options can be viewed as options on the underlying with the payoffs $(X-K/L)^+$ and $(K/L-X)^+$, respectively.[19]Thus, even if K lives on the same grid as for the plain vanilla options, the effective strike K/L lives on a much finer grid. For example, for the S&P 500, options are actively traded for the products with the tickers SPXL, SPXS, UPRO, and SPXU that have a leverage factor $L=3$, while actively traded options on the ETF products with the tickers SSO and SDS have a leverage factor $L=2$. Exploiting these leveraged derivative products, we obtain a moneyness grid with an average step of 0.015, while in the range $[0.8, 1.2]$ the grid step is approximately 0.0016. Thus, while a "truly continuous" state space is naturally impossible to achieve, modern financial markets are coming close to this target, at least for some instruments.[20] The growth of the leveraged ETF market and the corresponding option market suggests that continuity of the state space will become reality for a growing set of instruments. The emergence of these new derivatives serving as "continuity refinements" suggests

[17]Derivatives are actively traded by both retail and institutional investors. For example, in 2012, more than a billion option contracts were traded at the Chicago Board of Exchange (the largest option exchange in the world), with a total dollar volume of more than half a trillion USD. See https://www.cboe.com/data/marketstats-2012.pdf. For almost any liquid exchange-traded instrument, there exists an entire menu of options written on the instrument payoff. "Derivatives trading is now the world's biggest business, with an estimated daily turnover of over US$2.5 trillion and an annual growth rate of around 14%" (from *Building the Global Market: A 4000 Year History of Derivatives* by Edward J. Swan). Of course, the fact that there are many securities available for trading does not, in general, imply market completeness because of transaction costs and other sources of incompleteness.

[18]In Appendix D, I provide details on how an Arrow-Debreu equilibrium can be implemented with simple option contracts such as call and puts or binary options.

[19]While this is true for options with very short maturity, additional volatility corrections are needed for options with longer maturity. See Leung and Sircar (2014).

[20]In Appendix C, I show that my main results still hold true when the state space is discrete but sufficiently "dense."

that investors may be interested in expressing very precise state contingent views, as in my model.

The market is populated by three types of traders, *I*nformed and *U*ninformed traders, with masses $\lambda_i,\ i = I, U$, and noise traders N. Informed traders receive a signal $s \in \mathbb{R}^m,\ m \geq 1$ at time $t = 0$. Conditional on the realization of the signal s, the state of the world X is distributed with a density denoted as $\eta(x, s)$. I use $C^k[X_*, X^*]$ to denote the set of k-times continuously differentiable functions on $[X_*, X^*]$, and $C^+[X_*, X^*]$ to denote the set of positive continuous functions on $[X_*, X^*]$.

Assumption 2.2 $\eta(x, s) \in C^1[X_*, X^*]$, *is strictly positive for every realization of* s, *and the function* $\frac{\eta'(x,s)}{\eta(x,s)} = \frac{d}{dx} \log \eta(x, s)$ *is bounded on compact subsets of* $[X_*, X^*] \times \mathbb{R}^m$. *Signals are non-redundant in that* $\eta(\cdot, s_1) \not\equiv \eta(\cdot, s_2)$ *for all* $s_1 \neq s_2$.

Allowing $\mathcal{P}$ to denote the set of all probability distributions on $\mathbb{R}^m$, I assume that uninformed traders have an initial prior distribution $\pi_0 \in \mathcal{P}$, with compact support for the signal of the informed traders and can only use information contained in asset prices to form inferences about this signal. Traders of type $i = U, I$ maximize expected utility u_i of their time $t = 1$ consumption, W_i, conditional on the information they have at time $t = 0$.

Assumption 2.3 *For each* $i = I, U$, *utility function* u_i *is strictly concave on* $(m_i, +\infty)$ *with some* $m_i \geq -\infty$, *is four times continuously differentiable and satisfies the standard Inada conditions* $u_i'(m_i) = \infty,\ u_i'(+\infty) = 0$. *Furthemore,* $m_i = -\infty$ *for at least one of the two agents' classes.*[21]

I assume that each agent of class $i = I, U$ is initially endowed with $w_i > 0$ units of consumption good at $t = 0$, while noise traders supply $Z(x)$ of Arrow-Debreu securities corresponding to state $x,\ x \in [X_*, X^*]$. I assume that $Z(x)$ satisfies the following assumption:

Assumption 2.4 *There exists a probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *supporting a one-dimensional Brownian motion* B_t. *The filtration* $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ *is the usual augmentation of the filtration generated by the Brownian motion* B_t.[22] *Furthermore, there exists a strictly positive function* $\sigma(x) \in C[X_*, X^*]$ *such that* $Z(x)$ *is given by*

$$Z(x) \;=\; Z_{X_*} + \int_{X_*}^{x} \sigma(y) dB_{y-X_*} \,.$$

The variable Z_{X_*} *is independent of the Brownian motion* B_t.

Assumption 2.4 means that $Z(x)$ is a continuous limit of a random walk, $Z(x) \approx Z_{X_*} + \sum_k \sigma(k) Y_k$ where Y_k are independent and identically distributed variables with zero mean. That is, supply shocks are additive across levels of the state X, and the magnitude of these shocks is

[21] This assumption is made to make sure that markets can always clear even for large realizations of noise traders' demand: If the supports of both utilities are bounded from below, markets may fail to clear because there might not exist prices guaranteeing that the agents' terminal wealth belongs to the domain of definition of their utilities. By making endowment level Z_{X_*} sufficiently large, we can always achieve that the probability of negative consumption realizations is arbitrarily close to zero.

[22] See Karatzas and Shreve (1991).

state-specific and given by $\sigma(k)$.[23] The function $\sigma(x)$ allows me to specify a rich set of correlation structures for supply shocks across states: By the Ito isometry formula, we have

$$\mathrm{Cov}(Z(x), Z(y)) \;=\; \mathrm{Var}(Z_{X_*}) \;+\; \int_{X_*}^{\min\{x,y\}} \sigma^2(t)dt\,,\; x,\; y > X_*\,. \tag{2}$$

Assumption 2.4 introduces the simplest possible supply shock structure, analogous to that of Brownian supply shocks in the Kyle (1985) model. However, all of my results can be extended to a large class of additive noise structures (see Appendix D). Noisy supply in Arrow securities is equivalent to noisy supply in the underlying option contracts that span the asset space. For example, an Arrow security can be viewed as a long-short portfolio of two binary options with neighboring strikes or a butterfly portfolio of three call options with neighboring strikes. Assumption 2.4 then means that *each option contract has its own supply shock* and these *supply shocks are imperfectly correlated* across option contracts (see (2)).[24]

The most important implication of the noisy structure of supply shocks is that they are *non-smooth* as a function of x : While closely related Arrow securities have closely related shocks ($Z(x)$ is continuous[25]), there is always some imperfect correlation due to residual, state-specific (or, equivalently, contract-specific) shocks. The imperfect correlation of supply shocks guarantees that $Z(x)$ is non-smooth. This behavior of supply shocks over states is analogous to that of demand shocks in the Kyle (1985) model over time: While the total demand of noise traders over time is continuous, there is always some additional noise at every time instant, and this makes total demand non-smooth as a function of time.

To explain the exact nature of this non-smoothness, I will introduce an important object from continuous time stochastic calculus: the *realized quadratic variation.* Specifically, for any given function $f(x)$ and an interval $[a, b]$, the realized quadratic variation of f on $[a, b]$ is defined via

$$[f]_{[a,b]} = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^2\,, \tag{3}$$

where $\Pi = \{a = x_1 < \cdots < x_n = b\}$ ranges over partitions of the interval $[a, b]$ and $\|\Pi\| = \max_i |x_i - x_{i-1}|$. Quadratic variation has several important properties, which are summarized in the following lemma. See Karatzas and Shreve (1991).

Lemma 2.1 *The following is true:*

(1) Any continuously differentiable function has zero quadratic variation.

(2) If f has zero quadratic variation, then $[f + g]_{[a,b]} = [g]_{[a,b]}$ for any g.

[23] Essentially no assumptions need to be made about the distribution of the supply shocks Y_k : The central limit theorem guarantees that, in the continuous limit, the process becomes Gaussian. This follows from a version of the Donsker theorem (Donsker, 1952), which implies that a continuous limit of a random walk is a Brownian motion. See Appendix C.

[24] Recent empirical results offer evidence of noise trading in option markets and for the (imperfect) correlation in this noise trading across strikes. See, for example, Lemmon and Ni (2012) and Xu et al. (2013).

[25] The assumption of continuity is not necessary here. In fact, if supply shocks contain jumps, these jumps can be immediately recovered from the implied jumps in the equilibrium state prices.

(3) For any continuously differentiable function φ, almost surely, we have

$$[\varphi(Z(x))]_{[a,b]} \;=\; \int_a^b (\varphi'(Z(t)))^2\,\sigma^2(t)\,dt\,.$$

Items (1) and (2) emphasize that quadratic variation only matters for "very non-smooth" functions and is invariant to the addition of a smooth function. Item (3) shows how this variation can be explicitly computed for a diffusion process. The properties of the quadratic variation summarized in Lemma 2.1 play a major role in my analysis: They allow me to *separate between the smooth and rough parts of Arrow-Debreu state prices.* This separation result can most easily be understood if we draw an analogy with continuous and purely discontinuous functions: If we have a function of a bounded variation, it can be uniquely decomposed into a continuous component and a pure-jump component, and the two can be easily separated because one is much "rougher" than the other.

To illustrate the primary effects, let us consider the case when the mass of uninformed traders is zero. Let $W_I(x)$ be the terminal consumption of informed traders. A classical result from the Arrow-Debreu theory implies that an economic agent always chooses consumption to equalize the ratio of marginal costs and marginal benefits across states. This means that the marginal utility of consumption, $u_I'(W_I(x))$, multiplied by the physical probabilities, $\eta(x,s)$, must be proportional to state prices:

$$y_I\,M(x) \;=\; \eta(x,s)\,u_I'(W_I(x)) \tag{4}$$

for some $y_I > 0$. Through market clearing, we arrive at $W_I(x) = Z(x)$ (demand equals supply) for all x. Substituting this into (4) and taking the logs, we arrive at the following formula for equilibrium state prices:

$$\log M(x) \;=\; -\log y_I + \underbrace{\log \eta(x,s)}_{\textit{smooth probabilities}} + \underbrace{\log u_I'(Z(x))}_{\textit{rough supply shocks}}\,. \tag{5}$$

Formula (5) is a fundamental decomposition of state prices in a noisy Arrow-Debreu equilibrium: Under asymmetric information, log state prices reveal a combination of probabilities and supply shocks. While supply shocks are noisy and hence are not smooth, the probability density is smooth. As a result, (5) is also a decomposition of the state prices into the sum of "smooth" and "rough" components. Defining $\varphi(w) \;\equiv \log u_I'(w)$, we note that the rate of change of the marginal utility with respect to supply shocks is given by the negative of the agent's absolute risk aversion, $\varphi'(w) = \frac{u_I''(w)}{u_I'(w)} \;=\; -A_I(w)$. Applying Lemma 2.1, we immediately arrive at the identities:

$$[\log M(x)]_{[a,b]} \;=\; \int_a^b (A_I(Z(x)))^2(\sigma(x))^2 dx \;\Rightarrow\; \frac{\partial}{\partial b}[\log M(x)]_{[a,b]} \;=\; (A_I(Z(b)))^2\,\sigma^2(b) \tag{6}$$

for all $a, b \in (X_*, X^*)$. Thus, *equilibrium state prices reveal the rate at which marginal utility responds to supply shocks.* If the absolute risk aversion is strictly monotone, we immediately recover the supply shocks and the physical density for all states b :

$$Z(b) \;=\; A_I^{-1}\left(\left(\frac{\frac{\partial}{\partial b}[\log M(x)]_{[a,b]}}{\sigma^2(b)}\right)^{1/2}\right)\,,\;\log\eta(b,s) \;=\; \log M(b) \;+\; \log y_I \;-\; \log u_I'(Z(b))\,, \quad (7)$$

and the constant y_I can be uniquely pinned down by the fact that $\eta(b,s)$ is a probability density. A useful feature of the full revelation result (7) is that it is non-parametric: If an econometrician knows the functional form of the utility u_I and the noise trading intensity $\sigma(x)$, no parametric assumptions regarding the density $\eta(x,s)$ are needed, except for the assumption that it is C^1.[26]

While the argument presented above hinges on the fact that risk aversion is invertible, it is possible to show that this result also holds under general shapes of risk aversion, assuming the latter is "sufficiently non-constant." To state the general full revelation result, a formal definition of a noisy REE is required.

Definition 2.1 *Let $\mathcal{B}(X_*,X^*)$ denote the space of Borel-measurable functions on (X_*,X^*). A noisy Arrow-Debreu equilibrium is*

- *a state prices map*

$$\mathbf{M} : \mathbb{R}\times\mathcal{C}[X_*,X^*] \to C[X_*,X^*]$$

from signal and noise realizations, $(s,Z(\cdot))$, to Arrow-Debreu state prices $M(\cdot)$;

- *a beliefs map*

$$\mathbf{\Pi} : \mathcal{C}[X_*,X^*] \;\to\; \mathcal{P},$$

from state prices to the posterior distribution of uninformed traders, $\pi \;=\; \mathbf{\Pi}(M(\cdot))$;

- *optimal consumption functionals $W_I(x,s,M(\cdot))$ and $W_U(x,\pi,M(\cdot))$,*

such that

(a) $\mathbf{\Pi}$ is rational given the functional form of the map $\mathbf{M}$.

(b) W_I and W_U maximize agents' expected utilities.

(c) Markets clear in every state: $\lambda_I W_I(x,s,\mathbf{M}) + \lambda_U W_U(x,\mathbf{\Pi},\mathbf{M}) = Z(x)$ for all $x\in(X_*,X^*)$ and any realization of (s,Z).

To pursue an argument analogous to that presented above, we need to derive a version of formula (5) for the case when both types I and U are present in the market. To this end, we must solve for the equilibrium risk-sharing rule conditional on the agents' beliefs.

I start with a characterization of optimal consumption functionals. To state the next lemma, I must introduce some notation. Let $J_i(w), i=I,U$ denote the inverse of the marginal utility u_i'

[26] In Appendix D, I show that this result still holds if $Z(x)$ is differentiable but is less smooth than the physical density.

defined via $u_i'(J_i(w)) = w,\ i = I, U$. Also let

$$\eta(x,\pi(\cdot)) \equiv \int_{\mathbb{R}^m} \eta(x,s)d\pi(s) \tag{8}$$

be the posterior distribution of X conditional on the uninformed beliefs $\pi(\cdot)$ about the signal s. With this notation, we can rewrite $\eta(x,s)$ as $\eta(x,s) = \eta(x,\delta_s(\cdot))$, where δ_s is the point mass at s. Since markets are complete, a characterization of optimal consumption follows by a straightforward application of the first-order conditions (4): Since the marginal utility of consumption is proportional to the ratio of state prices to physical probabilities, applying the inverse marginal utility provides the optimal consumption, and the proportionality factor is pinned down by the budget constraint.

Lemma 2.2 *The optimal consumption of an agent of class i with beliefs $\pi(\cdot)$ is given by*

$$W_i(x,\pi(\cdot),M(\cdot)) = J_i\left(\frac{y_i M(x)}{\eta(x,\pi(\cdot))}\right),$$

where the Lagrange multiplier $y_i(\pi(\cdot),M(\cdot))$ is the unique solution to

$$\int_{X_*}^{X^*} J_i\left(\frac{y_i M(x)}{\eta(x,\pi(\cdot))}\right) M(x)dx = w_i.$$

Substituting optimal consumption into the market-clearing condition, we arrive at the equation defining equilibrium state prices:

$$\lambda_I\, J_I\left(\frac{y_I M(x)}{\eta(x,s)}\right) + \lambda_U\, J_U\left(\frac{y_U M(x)}{\eta(x,\pi(\cdot))}\right) = Z(x)\,. \tag{9}$$

The risk-sharing rule (9) defines the consensus (representative) consumer in the sense of Jouini and Napp (2007): $M(x)$ is the probability-weighted marginal utility of the representative consumer. Since markets are complete, allocation (9) of the aggregate risk Z across the two classes of agents is *interim efficient* given private information and the information revealed by prices.[27] This allocation solves the social planner's problem:

$$\max_{W_I,W_U,\ \lambda_I W_I+\lambda_U W_U=Z}\left(y_I^{-1}\lambda_I\int \eta(x,s)u_I(W_I(x))dx + y_U^{-1}\lambda_U\int \eta(x,\pi(\cdot))u_U(W_U(x))dx\right).$$

Indeed, an allocation is interim efficient if the probability-weighted marginal utilities are equalized across agents: $y_I^{-1}\eta(x,s)u_I'(W_I(x)) = y_U^{-1}\eta(x,\pi(\cdot))u_U'(W_U(x))$. This is precisely the case for the allocation (9).

Suppose now that the equilibrium is fully revealing. In this case, $\eta(x,\pi(\cdot)) = \eta(x,s)$, and the allocation is also ex-post efficient. In particular, beliefs aggregation is not an issue, and we can reformulate equilibrium risk sharing (9) in terms of the representative consumer's marginal utility.

[27] See Holmström and Myerson (1983) for a general theory and DeMarzo and Skiadas (1993) for the first application of interim efficiency to the theory of (non-noisy) REE.

Lemma 2.3 *Let $u(w)$ be a concave function such that*

$$\lambda_I\, J_I(y_I u'(w)) \;+\; \lambda_U\, J_U(y_U u'(w)) \;=\; w\,. \tag{10}$$

If the equilibrium is fully revealing, then equilibrium state prices are given by

$$M^*(s,x) \;\equiv\; \eta(x,s)\, u'(Z(x))\,. \tag{11}$$

Conversely, if state prices (11) reveal the signal s, then (11) is a fully revealing equilibrium.

Formula (10) defines the representative consumer: an imaginary agent whose preferences are defined in such a way that, absent heterogeneity in beliefs, equilibrium prices in our heterogeneous agents' economy coincide with those populated by the single representative agent. Lemma 2.3 is a direct consequence of (9) and shows that a fully revealing equilibrium is a self-fulfilling phenomenon. Indeed, suppose that M^* are fully revealing, and let $\mathbf{\Pi}$ be the equilibrium beliefs functional recovering the signal from state prices. That is, $\eta(x, \mathbf{\Pi}(M^*(s,\cdot))) \;=\; \eta(x,s)$. Then, we have

$$\begin{aligned} Z(x) &= \lambda_I\, J_I\left(\frac{y_I M^*(s,x)}{\eta(x,s)}\right) \;+\; \lambda_U\, J_U\left(\frac{y_U M^*(s,x)}{\eta(x,s)}\right) \\ &=\; \lambda_I\, J_I\left(\frac{y_I M^*(s,x)}{\eta(x,s)}\right) \;+\; \lambda_U\, J_U\left(\frac{y_U M^*(s,x)}{\eta(x,\mathbf{\Pi}(M^*(s,\cdot)))}\right)\,, \end{aligned} \tag{12}$$

which means that equilibrium is fully revealing.

Formula (7) suggests that equilibrium is fully revealing when the aggregate risk tolerance is sufficiently non-constant. The classic Wilson (1968) characterization of Pareto-efficient risk sharing implies that the aggregate risk tolerance $T(w) \;=\; -\frac{u'(w)}{u''(w)}$ is given by the sum of individual risk tolerances,

$$T(w) \;=\; T_I(W_I) \;+\; T_U(W_U)\,,$$

where

$$T_i(w) \;=\; -\lambda_i \frac{u_i'(w)}{u_i''(w)},\;\; i = I, U$$

is the total risk tolerance of class i and $W_i = J_i(y_i\, u'(w)),\;\; i = I, U$. Differentiating and using the identity $u_i'(J_i(w)) = w,\;\; i = 1, 2$, we determine that the sensitivity of the aggregate risk tolerance to wealth changes is a weighted average of individual risk tolerances:[28]

$$T'(w) \;=\; \frac{1}{T(w)}(T_I'(W_I)\lambda_I^{-1} T_I(W_I) + T_U'(W_U)\lambda_U^{-1} T_U(W_U))\,. \tag{13}$$

Thus, while it may happen that $T'(w) \equiv 0$ for some combination of $\lambda_I,\;\lambda_U$, this cannot be the case for generic weights λ_I, λ_U if one of the two risk tolerances is non-constant. I formalize this

[28] By direct calculation,

$$T'(w) = \frac{u''(w)}{u'(w)}(y_I\, T_I'(J_I(u'(w)))J_I'(y_I u'(w)) y_I u'(w) + \; T_U'(J_U(y_U\, u'(w)))J_U'(y_U u'(w)) y_U u'(w)),$$

and the claim follows from $J_i'(w) = 1/u_i''(J_i(w))$.

observation in the following lemma:

Lemma 2.4 *Suppose that at least one of the risk tolerances* $T_i(w) = -\frac{u_i'(w)}{u_i''(w)},\ i = I, U$ *is such that the set* $\{w : T_i'(w) = 0\}$ *has a Lebesgue measure of zero. Then, the set* $\{w : T'(w) = 0\}$ *always has a Lebesgue measure of zero, except possibly for a countable set of values of the quotient* λ_I/λ_U.

According to the discussion preceding (7), state prices reveal the risk tolerance $T(Z(x))$. As I show in the Appendix, Lemma 2.4 implies that $T(Z(x))$ and $M(x)$ can be used to uniquely determine $\eta(x, s)$. The following is true:

Theorem 2.1 *Suppose that at least one of the risk tolerances* $T_i(w) = -\frac{u_i'(w)}{u_i''(w)},\ i = I, U$ *is such that the set* $\{w : T_i'(w) = 0\}$ *has a Lebesgue measure of zero. Then, there always exists a fully revealing equilibrium given by* (11)*, except possibly for a countable set of values of the quotient* λ_I/λ_U*. In contrast, if both* $T_i,\ i = I, U$ *are constant, then there does not exist a fully revealing equilibrium.*

How surprising is the full revelation result of Theorem 2.1? At first glance, one is tempted to say: "Of course, if you give me a continuum (infinite number) of prices, this boils down to a continuum of noisy signals about the underlying signal. Not surprisingly, equilibria are fully revealing." However, this is actually not the reason why equilibria are fully revealing (and not even true, as shown in the last statement of Theorem 2.1; see also Theorem 2.2 below).[29] The reason is that price changes allow traders to figure out the noisy supply and thus back out with certainty the density $\eta(x, s)$. The intuition behind this result is as follows: An informed trader who is contemplating the purchases of Arrow securities corresponding to states x and $x + dx$ has very similar probabilities assigned to these states because the rate of change of $\eta(x, s)$ over a small interval dx is very small due to the assumption that $\eta(x, s)$ is smooth. Hence, the difference in the prices of the two Arrow securities arising from the change in η is small. In contrast, the change in the state prices due to the supply shock is much larger due to its noisy structure (as in the Kyle (1985) model for the noisy supply across time). When preferences exhibit non-constant absolute risk aversion, the change in the state prices over a small interval of states is much more informative about $Z(x)$ than it is about the signal s. In the continuous state limit, this allows uninformed traders to figure out $Z(x)$ exactly and hence pins down the realization of s.

From Theorem 2.1, a fully revealing equilibrium exists if and only if at least one of the risk tolerances $T_i(w)$ is "sufficiently non-constant." Since utilities $u_I,\ u_U$ enter (10) symmetrically, we determine that the existence of fully revealing equilibria depends *symmetrically* on u_I and u_U: *It does not matter whose preferences exhibit wealth effects.* What matters is the behavior of the aggregate risk tolerance. Naturally, if both agents have CARA preferences, $T(w)$ is always constant and hence the rate of change of the representative marginal utility does not reveal any additional information: With CARA preferences, information is only revealed through the level of prices, and equilibrium cannot be fully revealing. Another interesting and unique case occurs when both

[29] In the continuous-time Kyle (1985) model, prices also reveal a continuum of signals about the underlying state, but full revelation does not occur until the insider starts trading infinitely aggressively at the terminal horizon.

classes of agents have decreasing absolute risk aversion (DARA) preferences: In this case, (13) implies that $T(w)$ is increasing (see also Hara et al., 2007) and hence the explicit recovery formula (7) holds with A_I replaced by the aggregate risk aversion $A = 1/T$. I formalize this observation in the following proposition:

Proposition 2.1 *Suppose that both T_I and T_U are monotone increasing. Then, there exists a fully revealing equilibrium in which*

$$Z(b) \;=\; A^{-1}\left(\left(\frac{\frac{\partial}{\partial b}[\log M(x)]_{[a,b]}}{\sigma^2(b)}\right)^{1/2}\right)\,,\; \log\eta(b,s) \;=\; \log M(b) \;+\; \log y_I \;-\; \log u'(Z(b))\,, \quad (14)$$

where $A(w) = -\frac{u''(w)}{u'(w)}$ and u is defined in (10).

It is important to understand that the non-parametric recovery formula (14) is not a fully self-contained result: Applying formula (14) to real data requires estimating both u and $\sigma(x)$. The volatility of noise trading, $\sigma(x)$, could, for example, be proxied by the option trading volume, say, for call and put options at a given strike price.[30] As for the risk aversion A, one can either use a parametric specification (such as, e.g., the popular CRRA) or estimate preferences using the recovery theorem of Ross (2013) applied to a "smoothed version" of the estimated state prices M. This way formula (14) can be viewed as a complement to the Ross recovery theorem for the case of asymmetric information. Specifically, formula (14) exploits the fact that *non-smooth fluctuations of state prices across states contain information about supply shocks.* The quadratic variation in formula (14) is designed to extract information about these supply shocks from equilibrium state prices.

Theorem 2.1 is an existence result. It does not provide any insight into the possibility of the existence of other, non-fully revealing equilibria or their potential structure. This naturally leads us to the following question: *What is the minimal amount of information that can be revealed in an equilibrium?* My analysis of this question is based on the important concept of *residual supply*, introduced by Kreps (1977) in the context of non-noisy REE and further developed by Breon-Drish (2012, 2014), who was the first to systematically use residual supply to study non-linear noisy REE. Specifically, in any equilibrium, the market-clearing condition $\lambda_I W_I \;+\; \lambda_U W_U = Z$ implies that uninformed agents can observe the *residual supply*

$$D = Z - \lambda_I W_I\,. \quad (15)$$

Indeed, since each uninformed trader can compute W_U, the trader can also compute $D = W_U$. Therefore, residual supply represents a lower bound for the amount of information that can be revealed in any equilibrium. This observation implies that an REE will fail to exist in the standard sense (see Definition 2.1) if the information contained in $D(x)$ cannot be recovered from asset

[30] Noise trading is often associated with retail traders. One can use available data on traders' positions to back out trading volume by retail investors.

prices. As Kreps (1977) notes in this case, to restore equilibrium existence, we must allow agents to condition their demand directly on the residual supply.[31] I will call an equilibrium in which prices reveal information contained in the residual supply *weakly efficient.*[32] Motivated by this concept, I introduce the following definition:

Definition 2.2 *A noisy Arrow-Debreu equilibrium in the sense of Definition 2.1 is called weakly efficient. An equilibrium in which agents can condition their demand directly on the residual supply, and in which state prices do not fully reveal information contained in this residual supply, is called weakly inefficient.*

My next goal is to understand the nature of information contained in the residual supply. To this end, I will compute the sensitivity of the residual supply to noisy supply shocks, $Z(x)$. Differentiating equation (9) with respect to Z, we obtain the Wilson (1968) risk-sharing rule: Each agent marginally takes on the amount of risk that is proportional to the ratio of the agent's risk tolerance to the aggregate risk tolerance,

$$\lambda_I \frac{\partial}{\partial Z} W_I \;=\; \frac{T_I(W_I)}{T_I(W_I) + T_U(W_U)}\,.$$

According to this formula, the sensitivity of the residual supply to supply shocks is given by

$$\frac{\partial D}{\partial Z} \;=\; \frac{T_U(W_U)}{T_I(W_I) + T_U(W_U)}\,.$$

As in (6), this sensitivity can be computed directly from the realized quadratic variation of D. Since in equilibrium uninformed traders observe $T_U(W_U) \;=\; T_U(D)$, they can infer the realization of $T_I(W_I) = T_I(J_I(y_I M(x)/\eta(x,s)))$ from the residual supply. As above, if T_I is monotonic, $\eta(x,s)$ can be non-parametrically recovered from prices and the residual supply using

$$\frac{\partial}{\partial b}[D]_{[a,b]} \;=\; \left(\frac{\partial D}{\partial Z}\right)^2 \sigma^2(b)\,,$$

and therefore

$$\frac{1}{\eta(b,s)} \;=\; \frac{1}{y_I M(b)} u_I'\left(T_I^{-1}\left(\left(\left(\frac{\frac{\partial}{\partial b}[D]_{[a,b]}}{\sigma^2(b)}\right)^{-1/2} - 1\right) T_U(D(b))\right)\right)\,. \tag{16}$$

In the Appendix, I show that monotonicity is in fact not necessary and this recovery result holds

[31] This is the approach used in the recent literature on non-linear REE. See, for example, Breon-Drish (2012) and Banerjee and Green (2014). Modern financial markets contain a lot of direct information about demand and supply. For example, futures and options exchanges publish data on trading volume and open interest; data on institutional stock holdings are also often available (with delays); the Commodity Futures Trading Commission (CFTC) also regularly publishes data on positions of different classes of traders. The ideal example is a limit order book market, in which the order book reveals aggregate demand and supply at every time instant. Experimental evidence (Plott and Sunder, 1988) also suggests that agents actively use order flow data to infer private information.

[32] This should not be confused with the concept of weak-form efficiency that commonly refers to a situation in which the sequence of past prices reveals all the available information.

whenever A_I is sufficiently non-constant. Specifically, the following is true:

Theorem 2.2 *Suppose that the set $\{w : A_I'(w) = 0\}$ has a Lebesgue measure of zero. Then any noisy Arrow-Debreu equilibrium is fully revealing. In contrast, if A_I is constant, there always exists a non-fully revealing equilibrium.*

Theorem 2.2 complements the result of Theorem 2.1 and emphasizes a natural asymmetry in the effects of informed and uninformed agents' preferences on equilibrium behavior. Specifically, if informed traders have non-CARA preferences, then equilibrium is always fully revealing because the sensitivity of the residual supply to noisy supply shocks reveals the level of these shocks. In contrast, if only uninformed agents have non-CARA preferences, then equilibrium is always non-unique, and a fully revealing (and hence efficient) equilibrium co-exists with a non-fully revealing (and hence less efficient) equilibrium. The latter equilibrium can be chosen to only reveal information contained in the residual supply. Properties of such non-fully revealing equilibria are examined in the next section.

It is also instructive to compare the full revelation result of Theorem 2.2 with full revelation, which is often attained in the continuous time limit in models of strategic trading (see, e.g., Holden and Subrahmanyam, 1992; Vayanos and Chau, 2008; Caldentey and Stacchetti, 2010). In these models, a risk-neutral strategic trader takes infinitely large bets on his or her private information as the trading frequency increases. The same happens in the Kyle (1985) model as the time approaches final horizon. The mechanism in my model is different: All agents are price-takers and risk averse, and full revelation is obtained due to wealth effects of informed trading.

Interestingly, Breon-Drish (2014) shows that, when all agents have CARA preferences and there is a single risky asset, there exists a unique continuous equilibrium, and this equilibrium only reveals information contained in the residual supply. Theorem 2.2 shows that this uniqueness result generally breaks down in my model: With non-CARA preferences, there exists a continuous equilibrium that reveals more information than is contained in the residual supply. The mechanism behind this result is that of self-fulfilling expectations: If uninformed agents believe that the rate at which equilibrium state prices respond to supply shocks reveals these supply shocks, then this is indeed the case in equilibrium if uninformed agents have non-CARA preferences. In contrast, if they believe that this rate does not reveal any additional information, then this is also true in equilibrium. Note that the fully revealing equilibrium has another surprising property: In this equilibrium, *prices reveal strictly more information in the presence of uninformed traders.* Specifically, if $\lambda_U = 0$ and informed traders have CARA preferences, prices can never reveal all of the available information. However, the introduction of an arbitrarily small mass $\lambda_U > 0$ of uninformed traders leads to the emergence of a fully revealing equilibrium.

In the real world, option prices can be available for a rather dense set of strikes, but these strikes always live on a bounded interval. One may therefore ask whether some information is lost if only state prices for a sub-interval of x values are observable. The next result shows that no information is lost if the conditional density $\eta(x, s)$ is a real-analytic function of x for any realization of s.[33] Real analyticity is helpful for recovery because of the following result: A real-analytic function is either identically zero or is non-zero almost everywhere (see Krantz and Parks, 2002). In particular, the entire conditional density is uniquely determined by the values it takes on an arbitrarily small interval. Pick an arbitrary interval of states $[a,b] \subset [X_*, X^*]$. Since, under the hypothesis of Theorem 2.2 the values of state prices $M(x),\ x \in [a,b]$ can be used to uniquely recover $\eta(x,s),\ x \in [a,b]$, we arrive at the following result:

Proposition 2.2 *Suppose that $\eta(x,s)$ is a real-analytic function of x for any realization of s. Suppose also that the set $\{w:\ A_I'(w)=0\}$ has a Lebesgue measure of zero. Then, for any interval of states $[a,b] \subset [X_*, X^*]$, the values $M(x),\ x \in [a,b]$ uniquely determine the posterior density $\eta(x,s)$.*

In other words, under the hypothesis of Proposition 2.2, knowing the state prices for an arbitrarily small interval of states is sufficient to recover the entire probability density $\eta(x,s)$: *Too few prices reveal too much information.* While this result may seem surprising at first sight, one should remember that we still need infinitely many prices for the full revelation to hold, as we are replacing a large infinite set with a small (but still infinite) set.

3 Minimally Revealing Equilibria and Information in the Aggregate Demand and Supply

The goal of this section is to study non-fully revealing equilibria. According to Theorem 2.2, they exist only if informed traders have CARA preferences at least for some interval of consumption. Throughout this section, I will impose the following assumption:

Assumption 3.1 $u_I(x) = -A_I^{-1} e^{-A_I x}$ *for some* $A_I > 0$.

By Theorems 2.1 and 2.2, Assumption 3.1 always implies that equilibrium is non-unique if A_U is non-constant. The set of equilibria is quite large, but it contains two extreme points: a fully revealing equilibrium and a minimally revealing equilibrium. The latter reveals the minimal possible amount of information. It is also the equilibrium that has been studied in a majority of the existing noisy REE literature: In a CARA-Normal setting, a linear equilibrium is always minimally revealing.[34] The minimally efficient equilibrium is important because it establishes a natural lower bound on the amount of information that can be revealed in any equilibrium. The goal of this section is to study this equilibrium. From the above discussion, this is the equilibrium that reveals exactly the information contained in the residual supply.

[33] Recall that a function $f(x)$ is real analytic if it can be represented by its convergent Taylor series, in a small neighborhood of any point x. See Krantz and Parks (2002). All standard densities (e.g., Normal, log-Normal, chi-squared, uniform, exponential) are real analytic.

[34] The only exception I am aware of is Palvölgyi and Venter (2014), who show that, in addition to the minimally revealing equilibrium, there are other much more efficient equlibria in the Grossman and Stiglitz (1980) model.

Under Assumption 3.1, inverse marginal utility is logarithmic, $J_I(x) = -A_I^{-1}\log x$, and the informed demand is given by

$$W_I(x) \;=\; -A_I^{-1}\Big(\log(y_I\, M(x)) \;-\; \log\eta(x,s)\Big)\,. \tag{17}$$

This is a unique property of CARA preferences: Since the sensitivity of optimal consumption to beliefs is given by the absolute risk tolerance, for CARA preferences this sensitivity is independent of the price level, and hence the optimal consumption is an additively separable function of prices and beliefs. The functional $y_I = y_I(s, M(\cdot))$ is defined by the informed traders' budget constraint and only shifts the level of the informed traders' demand by a constant.

According to (17), we can rewrite the residual supply $D(x) = Z(x) - \lambda_I W_I(x)$ as

$$D(x) \;=\; T_I \log M(x) \;+\; Y(x)$$

with

$$Y(x) \;\equiv\; Z(x) - T_I \log\eta(x,s) + T_I \log y_I\,. \tag{18}$$

A key observation is that a constant shift in the level of the residual supply does not reveal any additional information due to market clearing and budget constraints. Indeed, in equilibrium, we have

$$\int_{X_*}^{X^*} D(x)M(x)dx \;=\; \int_{X_*}^{X^*} \lambda_U W_U(x)M(x)dx \;=\; w_U\,. \tag{19}$$

Thus, observing the changes $dD(x)$ in the residual supply, uninformed traders can immediately recover its value at X_* by substituting $D(x) = D(X_*) + \int_{X_*}^{x} dD(y)$ into (19). Therefore, learning from the residual supply is equivalent to learning from

$$dY(x) \;=\; dZ(x) - T_I\frac{\eta'(x,s)}{\eta(x,s)}dx \tag{20}$$

because $Y(X_*)$ does not reveal any additional information. The problem of learning from $dY(x)$ can be solved using the Zakai equation (see Zakai, 1969):[35] The posterior distribution $\pi(s)$, conditional on observing the changes in $Y(x)$ for all $x \in [X_*, X^*]$, is given by

$$d\pi(s) \;=\; \nu\, e^{\mathcal{G}(s)}\, d\pi_0(s)\,, \tag{21}$$

where ν is a normalization factor and the posterior log density $\mathcal{G}(s)$ satisfies

$$\mathcal{G}(s) \;\equiv\; \int_{X_*}^{X^*} \frac{\delta(x,s)}{\sigma^2(x)}dY(x) \;-\; \frac{1}{2}\int_{X_*}^{X^*} \frac{\delta^2(x,s)}{\sigma^2(x)}dx \tag{22}$$

with

$$\delta(x,s) \;\equiv\; -T_I\,\frac{d}{dx}\log\eta(x,s) \;=\; -T_I\,\frac{\eta'(x,s)}{\eta(x,s)}\,. \tag{23}$$

[35] In fact, (21) can be derived by a simple application of the Bayes rule if we discretize the state space and then take the continuous limit, replacing sums by integrals.

The quantity

$$\mathcal{S}(x,s) \;\equiv\; \frac{\delta(x,s)}{\sigma^2(x)}$$

appearing in (22) is the *signal-to-noise ratio,* defining how much weight an uninformed agent places on the information contained in the change $dY(x)$. Note that the posterior density is non-parametric and can depend in a non-trivial way on observed option prices (as summarized by $Y(\cdot)$) even if the densities $\eta(x,s)$ have a relatively simple form. According to (22), *the amount of information revealed in a minimally revealing equilibrium is independent of the nature of equilibrium prices and risk sharing:* It depends only on the noise and signal realization.[36] However, the equilibrium itself may potentially be non-unique: While there is only one posterior density that uninformed agents can have for a given realization of $(s, Z(\cdot))$, there might be multiple ways for agents to share their endowment risks in equilibrium. These equilibria may differ in their degree of allocative efficiency, but they will always have the same degree of informational efficiency.

As discussed above, conditional on the posterior beliefs (21), computing an equilibrium is reduced to solving an (interim-efficient) risk-sharing problem for agents with heterogeneous beliefs given by $\eta(x,s)$ and $\eta(s,\pi(\cdot))$, respectively. Since all involved quantities are bounded, standard arguments can be used to establish equilibrium existence.[37] The following is true:

Proposition 3.1 *A minimally revealing noisy Arrow-Debreu equilibrium always exists.*

Following Definitions 2.1 and 2.2, we are naturally led to the question of whether residual supply contains non-redundant information in a minimally revealing equilibrium (i.e., the question of whether this equilibrium is weakly inefficient). This question is particularly important in the context of option markets (see, e.g., Easley, O'Hara, and Srinivas (1998), who find that option volume plays an important informational role, and Kagkadis et al. (2014), who find that the realized quadratic variation of trading volume across option strikes contains information about future returns). If in equilibrium the residual supply contains non-redundant information, I will often abuse the notation and say that option volume and open interest contain non-redundant information.[38] With this notation, the rest of this section is devoted to the question: *Under what conditions do option volume and open interest contain non-redundant information?*

To answer this question, we will exploit the market-clearing condition that pins down the residual supply in terms of beliefs and prices. Substituting

$$Y(x) = D(x) - T_I \log M(x) \;=\; \lambda_U J_U\left(\frac{y_U M(x)}{\eta(x,\pi(\cdot))}\right) - T_I \log M(x) \tag{24}$$

[36] Breon-Drish (2014) makes the same observation in the one asset case.

[37] I assume that agents can condition their demand directly on the residual supply. This is the approach used in the literature on non-linear REE, initiated by Kreps (1977). See, for example, Breon-Drish (2012) and Banerjee and Green (2014).

[38] The exact link between the residual supply, volume, and open interest depends on the way in which trades are executed. For example, if informed traders submit their orders simultaneously with noise traders, and uninformed traders only trade afterward (at time $t = 0+$), then option volume and open interest immediately reveal the absolute value of the residual supply.

into (21), we obtain the following integral equation[39] for the posterior log density $\mathcal{G}(s)$:

$$\mathcal{G}(s) = \int_{X_*}^{X^*} \mathcal{S}(x,s) d\left(\lambda_U J_U\left(\frac{y_U(\mathcal{G})M(x)}{\eta(x,\mathcal{G}(\cdot))}\right) - T_I \log M(x)\right) - \frac{1}{2}\int_{X_*}^{X^*} \frac{\delta^2(x,s)}{\sigma^2(x)} dx\,, \tag{25}$$

where

$$\eta(x,\mathcal{G}(\cdot)) \;\equiv\; \eta(x, \nu\, e^{\mathcal{G}(s)}\, d\pi_0(s))$$

and where $y_U(\mathcal{G})$ is the Lagrange multiplier y_U for the uninformed traders' budget constraint, which depends on $\mathcal{G}$ through the posterior density $\eta(x,\mathcal{G}(\cdot))$. An uninformed agent who cannot condition his or her demand directly on the residual supply (option volume and open interest) must solve this integral equation given the observed state prices $M(\cdot)$. While we know that this equation always has a solution given by (21), there may be other solutions. In fact, any solution to this integral equation is a candidate for the REE beliefs map, and without additional information, an uninformed agent cannot distinguish which of these solutions is the right one. I summarize these findings in the following proposition:

Proposition 3.2 *Given a realization of state prices $M(x)$, $x \in [X_*, X^*]$, option volume and open interest contain non-redundant information if and only if (25) has multiple solutions $\mathcal{G}(\cdot)$.*

Breon-Drish (2012) was the first to show that with one risky asset prices may fail to reveal information contained in the residual supply if we depart from the standard CARA-Normal setting. He constructed several examples where such an inefficiency arises and derived sufficient conditions for the residual supply to be revealed (see Breon-Drish, 2014). I refer the reader to Breon-Drish (2012) for a detailed discussion of this important phenomenon. To understand the underlying economic mechanism, I will consider the following reduced-form example. Suppose that there is only one tradable security, and that informed traders submit a linear demand schedule[40] $T_I(s-p)$, where p is the price. The total noisy supply Z has a log-concave density with full support. The quantity $Y = -T_I s + Z$ plays the role of the residual supply. Uninformed traders submit a schedule $W_U(\pi, p)$, where $\pi = \Pi(Y)$ summarizes their posterior beliefs and is a function of the residual supply. The market-clearing condition then takes the form

$$\lambda_U W_U(\pi(Y), p) \;-\; T_I p \;=\; Y\,. \tag{26}$$

Under natural conditions, demand schedule $W_U(\pi, p)$ is monotone decreasing in p and hence (26) defines p as a function of $Y : p = p(Y)$. Since the distribution of Z (and hence that of Y) has full support, equilibrium is weakly efficient if and only if p is monotone in Y. By log-concavity, π is always monotone decreasing in Y. Suppose that high s is always "good news."[41] Then, W_U is monotone

[39]It is important to note that the stochastic integral on the right-hand side of (25) is formally defined only for almost every sample path of the semi-martingale process $M(x)$. However, integrating by parts, we can rewrite it as an expression that is well defined for any continuous $M(x)$ and coincides with the stochastic integral almost surely. See the Appendix for details.

[40]This is the informed demand schedule in the Grossman and Stiglitz (1980) model with s being the expected mean payoff.

[41]For example, $\eta(x,s)$ increases in s in the sense of first-order stochastic dominance.

increasing in π and therefore by the implicit function theorem p is monotone decreasing in Y, and the equilibrium is weakly efficient.[42] The main question is: Do such monotonicity properties hold in some form to a noisy Arrow-Debreu world? As I will argue, the answer is generally "no" because of the specific manner in which the optimal demand depends on the private information in an Arrow-Debreu setting. Indeed, in an Arrow-Debreu equilibrium, we have $W_U(\pi, M) = J_U\left(\frac{y_U(\pi)M(x)}{\eta(x,\pi)}\right)$. By the budget constraint, $\int_{X_*}^{X^*} J_U\left(\frac{y_U(\pi)M(x)}{\eta(x,\pi)}\right) M(x)dx$ is independent of π, and, hence, optimal demand cannot be monotone for all x : An increase in the demand in some states must always be compensated by a decrease in the demand in other states. This effect is amplified by the non-monotonicity of $\eta(x,\pi)$ with respect to π : Since it is a probability density, a change in π always increases the density in some states and at the same time decreases it in other states, leading to a corresponding adjustment of optimal consumption across states. For example, if high s is "good news," the agents will change consumption allocation by increasing it in states with high realizations of X at the cost of decreasing it in states with low realizations of X. This intrinsic non-monotonicity of demand with respect to beliefs in Arrow-Debreu equilibria suggests that these equilibria may often fail to be weakly efficient. Below I show that this is indeed the case.

To identify the mechanisms that may lead to weakly inefficient equilibria, let us first consider the benchmark case when uninformed traders have CARA preferences. A key intuition that we can derive from (26) is that equilibrium efficiency is closely linked to sensitivity of demand with respect to prices. In the case of CARA preferences and complete markets, uninformed demand is separable in prices and beliefs state by state, except for the budget constraint channel that operates through the Lagrange multiplier $y_U(\mathcal{G})$. This wealth effect only influences the overall level of uninformed demand, but not the way that this demand changes across states. Therefore, (25) takes the form

$$\begin{aligned} &\mathcal{G}(s) - T_U \int_{X_*}^{X^*} \mathcal{S}(x,s)\frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})}dx \\ &= -\int_{X_*}^{X^*} \mathcal{S}(x,s)d((T_I+T_U)\log M(x)) - \frac{1}{2}\int_{X_*}^{X^*} \frac{\delta^2(x,s)}{\sigma^2(x)}dx \end{aligned} \tag{27}$$

As we can see from (27), the manner in which the agents learn from changes in the residual supply is determined by the map on the left-hand side of (27). By Proposition 3.2, equilibrium is weakly efficient if and only if the map of the left-hand side of (27) is bijective for almost every realization of $M(\cdot)$: Indeed, if the map is not bijective, there will be a set of $M(\cdot)$ of positive measure such that (27) has multiple solutions, and each of these solutions will correspond to a candidate posterior density; without additional information, the agent cannot distinguish which candidate solution is the correct posterior. Intuitively, this bijectivity depends on the richness of the span of the functions $\mathcal{S}(x,\cdot) = \frac{\delta(x,\cdot)}{\sigma^2(x)} \in C(\mathbb{R}^m),\ x \in [X_*, X^*]$. Indeed, (27) implies that the adjusted log posterior density $\hat{\mathcal{G}}(s) = \mathcal{G}(s) + \frac{1}{2}\int_{X_*}^{X^*} \frac{\delta^2(x,s)}{\sigma^2(x)}dx$ is always in the span of these functions. The higher the dimension of this span, the more different dimensions there are for

[42] Breon-Drish (2014) shows that, with CARA preferences, demand monotonicity indeed holds for a large set of conditional densities. Chabakauri, Yuan, and Zachariadis (2014) extend the results of Breon-Drish to the case in which state-contingent claims are available for trading.

uninformed traders to interpret information contained in observed state prices. Therefore, it is natural to expect that the potential for inefficiencies is the lowest when $\delta(x,s) = -T_I \frac{d}{dx}\log\eta(x,s)$ has a *one-dimensional structure* in that there exists a single function $c(s)$ such that $\delta(x,s) = -T_I(a(x)+b(x)c(s))$. This one-dimensional structure of $\delta(x,\cdot)$ corresponds to the exponentially affine conditional density family $\eta(x,s) = e^{d(s)+A(x)+B(x)c(s)}$ of Chabakauri, Yuan, and Zachariadis (2014), where $d(s)$ is a normalization constant and $A(x)$ and $B(x)$ are the anti-derivatives of $a(x)$ and $b(x)$, respectively. The Gaussian density family $\eta(x,s) = e^{-(x-s)^2/(2\sigma^2)}$ used in most noisy REE models and the exponential family $\eta(x,s) = e^{d(s)+A(x)+xc(s)}$ of Breon-Drish (2014) belong to this exponentially affine class. The one-dimensional structure of such conditional densities severely restricts the manner in which uninformed traders can react to new information: Whatever information they infer from prices, their adjusted log posterior density will always be proportional to $c(s)$ (up to a normalization constant). Naturally, this structure leaves much less freedom for a multiplicity of solutions to (27): In fact, as the next result shows, the assumption of a one-dimensional span on $\delta(x,\cdot)$ always implies that minimally revealing equilibria are weakly efficient. However, absent the one-dimensional structure, equilibria may fail to be weakly efficient. To state the next result, I will need the following definition:

Definition 3.1 *I say that the density has a non-degenerate n-dimensional signal structure if there exist functions $B_j(x)$, $c_j(s)$ $j = 0,\cdots,n$, and $A(x)$ such that (i) $\eta(x,s) = e^{d(s)+A(x)+\sum_{j=1}^n B_j(x)c_j(s)}$, (ii) the functions $c_j(s)$ are linearly independent modulo a constant,*[43] *and (iii) the functions $B_j'(x)$ are linearly independent.*

$B_n(x)$ is the leading coefficient if $\lim_{|x|\to\infty}|B_j'(x)/B_n'(x)| = 0$ for all $j < n$.

The n-dimensional signal structure is quite general: In fact, it is possible to show that any density $\eta(x,s)$ can be approximated by a density with a n-dimensional signal structure with a sufficiently large n. The functions $c_j(s)$ can be interpreted as "learning risk factors": Specifically, (27) implies that the adjusted log posterior density $\hat{\mathcal{G}}(s) = \mathcal{G}(s) + \frac{1}{2}\int_{X_*}^{X^*}\frac{\delta^2(x,s)}{\sigma^2(x)}dx$ is always a linear combination of $c_j(s)$. The corresponding coefficients are risky in that they depend on the realization of supply shocks. This way, supply shocks influence learning though their impact on the coefficients of the learning risk factors. The following example illustrates other parts of Definition 3.1:

Example 1 *Suppose that $\eta(x,s)$ is Gaussian, $\eta(x,s) = d(s,X_*,X^*)e^{-(x-\mu(s))^2/(2\Sigma(s))}$, where the normalizing constant $d(s,X_*,X^*)$ ensures that η is a probability density. Then, if the signal s contains information only about the mean ($\Sigma(s) = const$) or only about the variance ($\mu(s) = \text{const}$) or if the mean is linear in variance ($\mu(s) = \mu_0 + \mu_1\Sigma(s)$), then we have a 1-dimensional signal structure. In contrast, if $\mu(s)$ and $\Sigma(s)$ are both non-constant and are linearly independent modulo a constant, then we have a non-degenerate 2-dimensional signal structure. That is, non-degeneracy is equivalent to private information about both volatility and the mean that are not perfectly aligned.*

More generally, if $\delta(x,s)$ is a polynomial so that $\eta(x,s) = e^{d(s)+\sum_{j=1}^n x^j c_j(s)}$ with an even n, then x^n is the leading coefficient and the signal structure is non-degenerate if $c_n(s)$ is non-constant.

[43] That is, no linear combination of these functions is constant.

The following is true:

Proposition 3.3 *Suppose that the signal structure is n-dimensional.*

(1) If $n = 1$, then prices reveal all information contained in the residual supply.

(2) If $n > 1$, the signal structure has a leading coefficient and $\eta(x, s)$ is real-analytic, then prices fail to reveal all information contained in the residual supply for an open set of initial priors π_0 and noise trading volatilities $\sigma(\cdot)$ whenever $-X_*$ and X^* are sufficiently large.

The first item of Proposition 3.3 confirms the intuitive argument outlined above: When the span of $\delta(x, \cdot)$ is one dimensional, learning from the residual supply is perfectly aligned across different states, and therefore equilibria are always weakly efficient. In contrast, the second item shows that even a small misalignment of learning can lead to non-monotonicities in the residual supply and, as a result, equilibria may fail to be weakly efficient and option volume and open interest may contain non-redundant information.

As an illustration of Proposition 3.3, consider the truncated Gaussian density $\eta(x, s)$ of Example 1. If (as in the Grossman and Stiglitz (1980) model) the signal s contains information only about the mean of the underlying, then the span of $\delta(x, s)$ is one dimensional and, hence, by the first item of Proposition 3.3, equilibrium is weakly efficient. However, if the signal contains an arbitrarily small amount of information about the variance of X, the second item of Proposition 3.3 implies that equilibria will fail to be weakly efficient for an open set of initial priors and supply shock volatility functions $\sigma(\cdot)$. I summarize these findings in the following corollary:

Corollary 3.1 *Suppose that $\eta(x, s)$ is the truncated Gaussian of Example 1, $\mu(s), \Sigma(s)$ are linearly independent modulo a constant and are real analytic, and $-X_*, X^*$ are sufficiently large. Then, for an open set of initial priors π_0 and noise trading volatilities $\sigma(\cdot)$, option volume and open interest contain non-redundant information.*

Corollary 3.1 has important implications for equity option markets: Since volatility is negatively correlated with returns (the so-called leverage effect), a signal informative about the mean is typically also informative about the variance. Corollary 3.1 implies that this leverage effect may be responsible for the informativeness of option volume and open interest found in the literature (see, e.g., Easley, O'Hara, and Srinivas, 1998). This result also emphasizes the main point of this section: In a noisy Arrow-Debreu equilibrium, even small deviations from the CARA-Normal setting generally lead to failure of equilibria to be weakly efficient.

Without the CARA assumption, prices influence the way in which the uninformed demand changes with the state level x : The sensitivity of demand to beliefs is given by the risk tolerance T_U, which in turn depends on the realized state prices $M(x)$. When the amount of noise in the supply shocks for some states is sufficiently low, uninformed traders place a lot of weight on the residual supply for these states when updating their beliefs. This may lead to strong non-monotonicities in uninformed demand and equilibrium may fail to be weakly efficient. The potential inefficiency of equilibria is also linked to the initial prior π_0. If most of the mass of π_0 is concentrated on some given value of s, the agents will learn very little from prices and their beliefs will essentially be insensitive to changes in state prices; as a result, equilibrium will be weakly efficient. However, if the initial prior distribution is sufficiently dispersed, learning effects become strong enough and may lead to failure of the equilibrium to be weakly efficient.

Recall that $\mathcal{P}$ is the space of all probability measures on $\mathbb{R}^m$, equipped with the total variation metric. The following theorem is the main result of this section:

Theorem 3.1 *If the set $\{w : T_U'(w) = 0\}$ has a Lebesgue measure of zero, then prices fail to reveal all information contained in the residual supply for an open set of initial priors π_0 and noise trading volatilities $\sigma(\cdot)$.*

One important implication of the failure is the way uninformed agents interpret option prices as good or bad news. If equilibrium is weakly inefficient, uninformed agents cannot distinguish based only on prices whether a given pattern of state prices is "good" or "bad" news: An increase in $M(x)$ on some interval of x means good news (high expected X) for some residual supply realizations and bad news for other residual supply realizations, both being consistent with observed option prices. This mechanism might be responsible for sudden crashes in seemingly identical environments: Even if prices "look the same as before," additional information contained in option volume and open interest may be changing and eventually lead to a large jump in equilibrium prices, even though the prices themselves do not reveal "anything bad."[44]

Theorem 3.1 also implies that arbitrarily small deviations from the CARA assumption may lead to weakly inefficient equilibria. As an illustration, suppose that conditional densities belong to the exponential family, $\eta(x, s) = e^{d(s)+A(x)+B(x)c(s)}$. Then, according to Proposition 3.3, equilibria are always weakly efficient when uninformed traders have CARA preferences. However, if uninformed risk tolerance is non-constant, there always exists an open set of supply shock volatilities $\sigma(x)$ and an open set of priors π_0 for which (25) has multiple solutions. Interestingly enough, combining Proposition 3.3 with Theorem 3.1, we determine that the combination of CARA preferences with a one-dimensional structure of $\delta(x, s)$ is essentially the only way to guarantee that equilibria are weakly efficient without imposing additional restrictions on the initial prior and the structure of supply shock volatilities.

I complete this section with a comparison of the two extreme equilibria constructed in formulae (14) (fully revealing) and (25) (minimally revealing). Both formulae emphasize the same idea: In complete markets, *price discovery occurs through the variation of Arrow-Debreu state prices across states.* However, the form of this variation can be quite different depending on the nature of the underlying equilibrium. In a fully revealing (more efficient) equilibrium, it is the quadratic variation (i.e., the sum of squares of changes in state prices) that contains important information about the underlying signal. In a minimally revealing equilibrium, it is a weighted linear variation (i.e., weighted sum of changes in state prices) that contains information about the underlying signal. In

[44] Of course, fully developing this argument would require a dynamic model, which goes beyond the scope of my paper.

an “intermediate” equilibrium, both mechanisms may come into play and uninformed agents may exploit both ways of extracting information from option prices.

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A Proofs: Fully Revealing Equilibria

Proof of Lemma 2.4. Let $f_i(w) \equiv -\lambda_i^{-1} T_i'(w)$. Then, by direct calculation,

$$\lambda_U^{-1} T'(w) = -T(w)(y_I\,(\lambda_I/\lambda_U) f_I(J_I(u'(w)))T_I(J_I(y_I u'(w))) + f_U(J_U(y_U\, u'(w)))T_U(J_U(y_U u'(w))))\,. \tag{28}$$

Without loss of generality, we may assume that f_I is non-zero Lebesgue-almost everywhere. Suppose that the claim of the lemma is not true. Then, there exists a set $\mathcal{B}$ of cardinality continuum such that for each $\lambda_I/\lambda_U \in \mathcal{B}$ the set $\chi(\lambda_I/\lambda_U) = \{w : T'(w) = 0\}$ has a positive Lebesgue measure. According to Lemma A.2, from the online Supplementary Appendix to Hugonnier, Malamud, and Trubowitz (2012),[45] there exists at least one pair of values of λ_I/λ_U for which the intersection of the corresponding sets $\chi(\lambda_I/\lambda_U)$ also has a positive Lebesgue measure. However, then, on this intersection, $f_I(J_I(u'(w))) = 0$, which contradicts the assumption that f_I is non-zero Lebesgue-almost everywhere. ■

[45] https://www.econometricsociety.org/uploads/Supmat/8783_proofs.pdf

Proof of Theorem 2.1. Since $\eta(x,s)$ is continuous, it suffices to show that the values of $\eta(x,s)$ can be uniquely recovered on an everywhere dense subset of $[X_*, X^*]$: Then, all other values of $\eta(x,s)$ can be recovered by continuity.

Suppose now toward a contradiction that the full revelation result does not hold. Then, there exists a set of sample paths of $Z(x)$ of positive measure such that for each $Z(x)$ in this set there exists a pair of C^1-densities $\eta_1(x),\ \eta_2(x)$ and another sample path $Z_1(x)$ for which

$$\frac{u'(Z(x))}{u_1'(Z_1(x))} \;=\; \frac{\eta_2(x)}{\eta_1(x)}$$

where u_1 is the utility of the representative agent conditional on the density $\eta_1(x)$. Utility u_1 may be different from u because it depends on the value of y_I, which is not known to the uninformed agents. Pick such a tuple $Z(x), Z_1(x), \eta_1(x), \eta_2(x)$. Since the quotient of two probability densities cannot be constant, there exists an interval $[a,b]$ such that $\frac{d}{dx}\frac{\eta_2(x)}{\eta_1(x)} \neq 0$. Uninformed traders observe $M(x)$ and hence through the quadratic variation of $\log M(x)$ they also observe $T(Z(x))$ and $T_1(Z_1(x))$. As a result, we ought to have $T_1(Z_1(x)) = T(Z(x))$. By Lemma 2.1, $Z(x)$ is nowhere locally constant, and hence $Z_* \equiv \min_{[a,b]} Z(x) < \max_{[a,b]} Z(x) \equiv Z^*$. Since, by Lemma 2.4, $T'(w)$ is almost surely non-zero we may assume without loss of generality (by shrinking the interval if necessary) that $T'(w)$ is non-zero on that interval and hence $T_* \equiv \min_{[Z_*,Z^*]} T(Z) < \max_{[Z_*,Z^*]} T(Z) \equiv T^*$. By the Sard theorem (Sternberg, 1964, Theorem II.3.1), almost every point $t \in [T_*, T^*]$ is regular in that the set $T^{-1}(t)$ is finite[46] and $T'(w) \neq 0$ for all $w \in T^{-1}(t)$. Pick such a regular point t and let $\bar{x}$ be such that $T(Z(\bar{x})) = t$ and let $\{w_1, \cdots, w_K\} = T^{-1}(t)$. Let $\tau(x) \equiv T(Z(x))$. Then, by the implicit function theorem, there exists an $\varepsilon > 0$ and a collection of C^1-functions $\varphi_1, \cdots, \varphi_K$ with each φ_k defined on a small neighborhood of $[t-\varepsilon, t+\varepsilon]$ and such that $T(Z) = \tau$ is equivalent to $Z \in \{\varphi_1(\tau), \cdots, \varphi_K(\tau)\}$ for all Z is a small neighborhood of $T^{-1}(t)$ and all $\tau \in (t-\varepsilon, t+\varepsilon)$. Furthermore, by the implicit function theorem, $\varphi_k'(\tau) \neq 0$ for all $\tau \in (t-\varepsilon, t+\varepsilon)$ and, by choosing ε sufficiently small, we may assume that the sets $\varphi_k((t-\varepsilon, t+\varepsilon))$ are disjointed. Since both $Z(x)$ and $Z_1(x)$ are continuous, we determine that there exists a pair k_1, k_2 such that $Z(x) = \varphi_{k_1}(\tau(x)),\ Z_1(x) = \varphi_{k_2}(\tau(x))$ for all x is a small open neighborhood of $\bar{x}$. Since φ_k are strictly monotone, we determine that there exists a strictly monotone C^1-function $\psi(z) = \varphi_{k_1}(\varphi_{k_2}^{-1}(z))$ such that $Z(x) = \psi(Z_1(x))$. By the implicit function theorem, the identity

$$\log\frac{u'(\psi(Z_1(x)))}{u_1'(Z_1(x))} \;=\; \log\frac{\eta_2(x)}{\eta_1(x)}\,,$$

implies that $Z_1(x)$ can be non-differentiable if and only if $\log\frac{u'(\psi(z))}{u_1'(z)}$ has a derivative that is identically zero on a small neighborhood of $Z_1(\bar{x})$. However, this would in turn imply that the derivative of $\log\frac{\eta_2(x)}{\eta_1(x)}$ is zero on a small neighborhood of $\bar{x}$, which contradicts the made assumption. The proof is complete. ■

Proof of Theorem 2.2. The argument in the text implies that through the quadratic variation of the residual supply $Z(x) - \lambda_I J_I(y_I M(x)/\eta(x))$ uninformed agents can observe $T_I(W_I) =$

[46] Finiteness follows from compactness of the interval $[Z_*, Z^*]$.

$T_I(J_I(y_I M(x)/\eta(x)))$. Suppose toward a contradiction that the full revelation result does not hold. Then, as in the proof of Theorem 2.1, there exists a tuple $Z(x), Z_1(x), \eta_1(x), \eta_2(x)$ such that

$$Z_1(x) - \lambda_I J_I(\hat{y}_I M(x)/\eta_2(x)) = Z(x) - \lambda_I J_I(y_I M(x)/\eta_1(x))$$

and

$$T_I(J_I(y_I M(x)/\eta_1(x))) \;=\; T_I(J_I(\hat{y}_I M(x)/\eta_2(x))).$$

By the market clearing condition (9), the fact that $Z(x)$ is not differentiable on any interval implies that $M(x)$ is also not differentiable on any interval. Pick a point x such that $J_I(y_I M(x)/\eta_1(x))$ is a regular point of T_I (this is always possible by the Sard theorem). Then, by the same argument as in the proof of Theorem 2.1, there exists a smooth and monotonic function ψ and a small interval $[x_1, x_2]$ of x values such that

$$J_I(y_I M(x)/\eta_1(x)) = \psi(J_I(\hat{y}_I M(x)/\eta_2(x)))\;,\; x \in [x_1, x_2].$$

To complete the proof, it suffices to show that the function $F(a,b) \equiv J_I(y_I ab) - \psi(J_I(\hat{y}_I a))$ has a non-zero derivative with respect to a for an open interval of values of $b = \eta_2(x)/\eta_1(x)$. Indeed, in this case, the implicit function theorem implies that $M(x)$ is continuously differentiable on an interval, leading to a contradiction.

Pick a point $\bar{x}$ as in the proof of Theorem 2.1. Then, by direct calculation,

$$\frac{\partial}{\partial b}F(a,b) \;=\; J_I'(y_I ab) y_I a \;=\; -T_I(J_I(y_I ab))/b\,.$$

Therefore, if $J_I(y_I ab)$ is a regular point of T_I, then $\frac{\partial^2}{\partial a \partial b}F(a,b)$ is non-zero and, hence, $\frac{\partial}{\partial a}F(a,b)$ cannot be identically zero. The proof of the first part of the statement is complete. The existence of a non-fully revealing equilibrium follows from Proposition 3.1. ■

Proof of Proposition 2.2. The proof follows directly from the proof of Theorem 2.2 and the fact that a real analytic function is uniquely determined by its values on any non-empty interval. ■

B Proofs: Minimally Revealing Equilibria

Proof of Proposition 3.1. Fix a signal and noise realization $(s, Z(\cdot))$. Since the posterior density $\eta(x, \pi(\cdot))$ depends only on the noise realization and signal realization, it does not interact with equilibrium prices $M(\cdot)$, and I will use $\eta_U(x)$ to denote this density and treat it as exogenously specified beliefs. What is important is that, by Assumption 2.2 and the assumption of compact support for π_0, we determine that $\mathcal{G}(s)$ is bounded on compact subsets of $\mathbb{R}^m$, and therefore $\eta_U(x)$ is continuous and strictly positive.

Define the function $f(z)$ to be the unique solution to

$$-\,T_I \log f + \lambda_U J_U(f) = z. \tag{29}$$

Let $\theta \equiv y_I/y_U$. Then, by direct calculation, we have $M = M(x,\theta)/y_U$ with

$$M(x,\theta) = \eta_U(x) f\left(Z(x) + T_I \log \frac{\theta \eta_U(x)}{\eta(x,s)}\right). \tag{30}$$

The Lagrange multiplier y_U is pinned down by the no-arbitrage condition

$$\int_{X_*}^{X^*} M(x,\theta)dx/y_U = (1+r)^{-1} \;\Rightarrow\; y_U \;=\; (1+r)\int_{X_*}^{X^*} M(x,\theta)dx\,,$$

and it remains to find θ from the individual budget constraints. Since $M(x,\theta)/y_U$ always clears the markets, it suffices to satisfy the budget constraint of uninformed traders: The budget constraint of informed traders is then satisfied by market clearing. Therefore, for any given realization of $(s, Z(\cdot))$, computing the equilibrium reduces to solving the fixed point equation for the quotient θ of Lagrange multipliers:

$$\int_{X_*}^{X^*} J_U\left(f\left(Z(x) + T_I \log \frac{\theta\eta_U(x)}{\eta(x,s)}\right)\right) \eta_U(x) f\left(Z(x) + T_I \log \frac{\theta\eta_U(x)}{\eta(x,s)}\right) dx - w_U \;=\; 0. \tag{31}$$

Equation (29) implies that $f(z)$ is monotone decreasing and $\lim_{z\to+\infty} f(z) = 0$ and $\lim_{z\to-\infty} f(z) = +\infty$. Indeed, if f remained bounded away from zero as $z \to +\infty$, then the left-hand side of (29) would also stay bounded and could not be equal to the right-hand side. The proof for $\lim_{z\to-\infty} f(z)$ is analogous. Inada conditions imply that $\lim_{x\to+\infty} J_U(x) = m_U$ and $\lim_{x\to 0} J_U(x) = +\infty$. Since sample paths of $Z(x)$ are bounded and $\log \frac{\eta(x,s)}{\eta_U(x)}$ is bounded, we conclude that the quantity $J_U\left(f\left(Z(x) + T_I \log \frac{\theta\eta_U(x)}{\eta(x,s)}\right)\right)$ uniformly converges to $+\infty$ and m_U then θ converges to $+\infty$ and 0, respectively. By Assumption 2.4, we determine that the left-hand side of (31) is negative for θ close to zero and positive for large θ. Hence, a solution to (31) always exists by the intermediate value theorem. Each of these solutions corresponds to a minimally revealing equilibrium. The proof of equilibrium existence is complete. ∎

Proof of Proposition 3.2. First, we notice that, integrating by parts, we can rewrite (25) as

$$\begin{aligned}\mathcal{G}(s) = &-\int_{X_*}^{X^*} \left(\lambda_U J_U\left(\frac{y_U(\mathcal{G})M(x)}{\eta(x,\mathcal{G}(\cdot))}\right) - T_I \log M(x) - \lambda_U Z_U(x)\right)\mathcal{S}'(x,s)dx \\ &- \frac{1}{2}\int_{X_*}^{X^*} \frac{\delta^2(x,s)}{\sigma^2(x)}dx + \left(\mathcal{S}(x,s)\left(\lambda_U J_U\left(\frac{y_U(\mathcal{G})M(x)}{\eta(x,\mathcal{G}(\cdot))}\right) - T_I \log M(x) - \lambda_U Z_U(x)\right)\right)\Big|_{X_*}^{X^*}\end{aligned} \tag{32}$$

and this expression is well defined for any function continuous $M(x)$. In equilibrium, state prices are given by (30). Since f is continuous monotone decreasing, an open set of $M(x)$ corresponds to an open set of $Z(x)$. The claim now follows because any open set in the space of continuous functions is non-null with respect to the probability measure defining Brownian motion (this follows from the Stroock-Varadhan support theorem. See Stroock and Varadhan, 1972). ∎

Proof of Proposition 3.3. In the exponentially affine case, $\mathcal{G}$ is always proportional to $c(s)$ (up to adding a constant). I will abuse the notation and use $\mathcal{G}$ to denote the corresponding constant of proportionality. Then, the bijectivity of the map is equivalent to monotonicity of the function

$$\mathcal{G} + T_U \int_{X_*}^{X^*} b(x) \frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})} dx$$

As I will now show, this function is always monotone increasing.

We have

$$\eta(x,\mathcal{G}) \;=\; e^{A(x)} \int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} d\pi_0(s)$$

and hence

$$\frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})} \;=\; a(x) + b(x)\sigma^2(x) \frac{\int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} c(s) d\pi_0(s)}{\int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} d\pi_0(s)}$$

and hence

$$\frac{\partial}{\partial \mathcal{G}} \frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})} \;=\; b(x)\sigma^2(x) \times$$

$$\frac{\int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} c^2(s) d\pi_0(s) \int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} d\pi_0(s) - \left(\int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} c(s) d\pi_0(s)\right)^2}{\left(\int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} d\pi_0(s)\right)^2}. \tag{33}$$

Introducing a new measure

$$d\hat{\pi}_0(s) \;\equiv\; \frac{e^{d(s)+(B(x)+\mathcal{G})c(s)} d\pi_0(s)}{\left(\int_{\mathbb{R}^m} e^{d(s)+(B(x)+\mathcal{G})c(s)} d\pi_0(s)\right)}$$

we determine that

$$\frac{\partial}{\partial \mathcal{G}} \frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})} \;=\; b(x)\sigma^2(x)(E[c(s)^2] - (E[c(s)])^2)$$

where E is the expectation under $d\hat{\pi}_0$. Hence, $\frac{\partial}{\partial \mathcal{G}} \frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})}$ has the same sign as $b(x)$. Therefore,

$$\frac{d}{d\mathcal{G}} \int_{X_*}^{X^*} b(x) \frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})} dx \;=\; \int_{X_*}^{X^*} b(x) \frac{\partial}{\partial \mathcal{G}} \frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})} dx \;\geq 0\,,$$

and the claim follows.

To establish the second part of the proposition, I proceed as follows. Let us pick the three states s_1, s_2, s_3 such that $c(s_1), c(s_2), c(s_3)$ are all different and $c(s)$ is not locally constant as s_3. Let

$$d\pi_0^*(s) \;=\; \frac{1}{3}(\delta_{s_1} + \delta_{s_2} + \delta_{s_3}). \tag{34}$$

Then, solving for the fixed point $\mathcal{G}(s)$ reduces to solving for the three values $\mathcal{G}_i \equiv \mathcal{G}(s_i), i = 1, 2.3$. Denote $\eta_i(x) \equiv \eta(x, s_i)$. Let $f_i(x) \equiv \eta_i(x)/\eta_3(x)$, and $\alpha_i \equiv \mathcal{G}_i - \mathcal{G}_3,\ i = 1, 2, 3$. Let also $g_i(x) \equiv$

$(\log f_i(x))' = (\frac{\eta_i'(x)}{\eta_i(x)} - \frac{\eta_3'(x)}{\eta_3(x)})$. Then,

$$\begin{aligned} \frac{\eta'(x,\mathcal{G})}{\eta(x,\mathcal{G})} &= \frac{\sum_{i=1}^3 e^{\mathcal{G}_i}\eta_i'(x)}{\sum_{i=1}^3 e^{\mathcal{G}_i}\eta_i(x)} = \frac{\sum_{i=1}^3 e^{\mathcal{G}_i-\mathcal{G}_3}\eta_i'(x)}{\sum_{i=1}^3 e^{\mathcal{G}_i-\mathcal{G}_3}\eta_i(x)} \\ &= \frac{\sum_{i=1}^3 e^{\alpha_i} f_i(x)\frac{\eta_i'(x)}{\eta_i(x)}}{\sum_{i=1}^3 e^{\alpha_i} f_i(x)} = \frac{\eta_3'(x)}{\eta_3(x)} + \frac{1}{T_I}\frac{\sum_{i=1}^2 e^{\alpha_i} f_i(x) g_i(x)}{\sum_{i=1}^2 e^{\alpha_i} f_i(x)+1} \end{aligned} \tag{35}$$

Subtracting from the fixed point equation (27) evaluated at $s_i, i = 1,2$ the same equation evaluated at s_3, we arrive at the system

$$H^j(\alpha_1,\alpha_2,T_U) \equiv \alpha_j + T_U T_I \int_{X_*}^{X^*} \sigma^{-2}(x) g_j(x) \left(\frac{\eta_3'(x)}{\eta_3(x)} + \frac{\sum_{i=1}^2 e^{\alpha_i} f_i(x) g_i(x)}{\sum_{i=1}^2 e^{\alpha_i} f_i(x)+1}\right) dx = a_j,\ j = 1,2 \tag{36}$$

for constants α_1, α_2, where

$$a_j = (T_I + T_U) T_I \int_{X_*}^{X^*} \sigma^{-2}(x) g_j(x) d\log M(x) - \frac{1}{2}\int_{X_*}^{X^*} \frac{\delta^2(x,s_j)-\delta^2(x,s_3)}{\sigma^2(x)} dx\,.$$

We have

$$\frac{\partial H^i}{\partial \alpha_j} = \delta_{ij} + T_U T_I A_{i,j}(\alpha_1,\alpha_2)$$

where

$$\begin{aligned} A_{j,1}(\alpha_1,\alpha_2) &\equiv \int_{X_*}^{X^*} e^{\alpha_1}\sigma^{-2}(x) g_j(x) f_1(x) \frac{(g_1(x)-g_2(x))e^{\alpha_2} f_2(x) + g_1(x)}{\left(\sum_{i=1}^2 e^{\alpha_i} f_i(x)+1\right)^2} dx \\ A_{j,2}(\alpha_1,\alpha_2) &\equiv \int_{X_*}^{X^*} e^{\alpha_2}\sigma^{-2}(x) g_j(x) f_2(x) \frac{(g_2(x)-g_1(x))e^{\alpha_1} f_1(x) + g_2(x)}{\left(\sum_{i=1}^2 e^{\alpha_i} f_i(x)+1\right)^2} dx \end{aligned} \tag{37}$$

for $j = 1,2$. Define

$$B_{i,j,k} \equiv (A_{i,j})_{\alpha_k}\ ,\ i,j,k = 1,2$$

and let

$$\begin{aligned} \Theta(\alpha_1,\alpha_2,T) &= \left(B_{2,1,1} - \frac{TA_{2,1}}{1+TA_{1,1}} B_{1,1,1}\right)\left(\frac{TA_{1,2}}{1+TA_{1,1}}\right)^2 \\ &- 2\left(B_{2,1,2} - \frac{TA_{2,1}}{1+TA_{1,1}} B_{1,1,2}\right)\frac{TA_{1,2}}{1+TA_{1,1}} + \left(B_{2,2,2} - \frac{TA_{2,1}}{1+TA_{1,1}} B_{1,2,2}\right). \end{aligned} \tag{38}$$

Let also

$$\Delta(\alpha_1,\alpha_2,T) \equiv (1+TA_{1,1})(1+TA_{2,2}) - T^2 A_{1,2} A_{2,1}$$

be the Jacobian of the map (H^1, H^2).

The proof of the required assertion is based on the following lemma.

Lemma B.1 *Suppose that there exists (α_1^*, α_2^*) such that $H^1_{\alpha_1}(\alpha_1^*, \alpha_2^*, T) \neq 0$, $\Delta(\alpha_1^*, \alpha_2^*, T) = 0$ and $\Theta(\alpha_1^*, \alpha_2^*, T) \neq 0$. Then, there exists an open set $\hat{P}$ of priors such that equilibrium is not weakly* *efficient except possibly for a countable set of values of T.*

Proof. Suppose without loss of generality that $\Theta(\alpha_1^*, \alpha_2^*, T) > 0$. By the implicit function theorem, there exists a smooth function φ defined on a small neighborhood of α_2^*, such that $H^1(\varphi(\alpha_2), \alpha_2, T) = H^1_{\alpha_1}(\alpha_1^*, \alpha_2^*, T)$. By direct calculation,

$$\frac{d^2}{d\alpha_2^2} H^2(\varphi(\alpha_2), \alpha_2, T) \;=\; \Theta(\alpha_1, \alpha_2, T) \;>\; 0\,,$$

whereas $\Delta(\alpha_1^*, \alpha_2^*, T) = 0$ implies $\frac{d}{d\alpha_2} H^2(\varphi(\alpha_2^*), \alpha_2^*, T) = 0$. Thus, the function $H^2(\varphi(\alpha_2), \alpha_2, T)$ is locally quadratic and convex, with a vertex at α_2^*, and therefore there exists an $\varepsilon > 0$ such that the equation $H^2(\varphi(\alpha_2), \alpha_2, T) = a_2$ has exactly two solutions $\alpha_{2,1}^*, \alpha_{2,2}^*$ in a small neighborhood of α_2^* for all $a_2 \in (H^2(\varphi(\alpha_2^*), \alpha_2^*, T), H^2(\varphi(\alpha_2^*), \alpha_2^*, T) + \varepsilon)$. Furthermore, $\frac{d}{d\alpha_2} H^2(\varphi(\alpha_{2,i}^*), \alpha_{2,i}^*, T) \approx \frac{d^2}{d\alpha_2^2} H^2(\varphi(\alpha_{2,i}^*), \alpha_{2,i}^*, T)(\alpha_{2,i}^* - \alpha_2^*) \neq 0$, which is equivalent to $\Delta(\varphi(\alpha_{2,i}^*), \alpha_{2,i}^*, T) \neq 0$.

Let $\Psi(\mathcal{G}; d\pi_0) :\; C^b(\mathbb{R}^m) \to C^b(\mathbb{R}^m)$ be the map defined as the difference between the left-hand side and the right-hand side of (27), mapping the space $C^b(\mathbb{R}^m)$ of bounded and continuous functions $\mathcal{G}$ on $\mathbb{R}^m$ into itself. Then, we determine that there exists an open set of $M(\cdot)$ in the space of continuous functions on $[X_*, X^*]$, such that the equation $\Psi(\mathcal{G}; d\pi_0^*) = 0$ has two solutions $\mathcal{G}, \hat{\mathcal{G}}, \mathcal{G} \neq \hat{\mathcal{G}}$ with π_0^* defined in (34). Furthermore, $\Delta(\varphi(\alpha_{2,i}), \alpha_{2,i}, T) \neq 0$ immediately implies that the Jacobian $D\Psi : C^b(\mathbb{R}^m) \to C^b(\mathbb{R}^m)$ evaluated at both $(\mathcal{G}, d\pi_0^*)$ and $(\hat{\mathcal{G}}, d\pi_0^*)$ has a bounded inverse on $C^b(\mathbb{R}^m)$. Therefore, by the implicit function theorem, there exists a small open neighborhood of π_0^* such that $\Psi(\mathcal{G}; d\pi) = 0$ has at least two different solutions for all initial priors π in this neighborhood. ∎

Thus, to complete the proof, it suffices to show that there exists an open set of $\sigma(x)$ such that for each such $\sigma(x)$ there exists a pair (α_1^*, α_2^*) satisfying $H^1_{\alpha_1}(\alpha_1^*, \alpha_2^*, T) \neq 0,\; \Delta(\alpha_1^*, \alpha_2^*, T) = 0$, and $\Theta(\alpha_1^*, \alpha_2^*, T) \neq 0$. We will need the following auxiliary result:

Lemma B.2 *Suppose that there exists an $x \in [X_*, X^*]$ such that $\left(\frac{g_2(x)}{g_1(x)}\right)' \left(\frac{g_2(x) f_2(x)}{g_1(x) f_1(x)}\right)' < 0$. Then, there exists a $\sigma(x) \in C^+[X_*, X^*]$ and α_1^*, α_2^* such that $\Delta(\alpha_1^*, \alpha_2^*, T) = 0$.*

Proof. The key observation is that, for the map (H^1, H^2), multiplying $\sigma^2(x)$ by a constant is equivalent to dividing T_U by the same constant. Thus, it suffices to show that there exists a $\sigma(x)$ and T_U such that the Jacobian is zero. Since Δ is quadratic in T_U, it suffices to show that the senior coefficient, $A_{1,1}A_{2,2} - A_{1,2}A_{2,1}$ is negative for some $\sigma(x)$. When $\alpha_1, \alpha_2 \to -\infty$, the latter inequality takes the form

$$\begin{aligned} &\int_{X_*}^{X^*} \sigma^{-2}(x) g_1^2(x) f_1(x) dx \int_{X_*}^{X^*} \sigma^{-2}(x) g_2^2(x) f_2(x) dx \\ &< \int_{X_*}^{X^*} \sigma^{-2}(x) g_1(x) g_2(x) f_1(x) dx \int_{X_*}^{X^*} \sigma^{-2}(x) g_1(x) g_2(x)(x) f_2(x) dx\,. \end{aligned} \tag{39}$$

Define a new measure on $[X_*, X^*]$ with the density

$$\frac{\sigma^{-2}(x)g_1^2(x)f_1(x)}{\int_{X_*}^{X^*}\sigma^{-2}(x)g_1^2(x)f_1(x)dx}.$$

Then, under this measure, we can rewrite the required inequality (39), which takes the form

$$E\left[\frac{g_2^2(X)f_2(X)}{g_1^2(X)f_1(X)}\right] \;<\; E\left[\frac{g_2(X)}{g_1(X)}\right]\, E\left[\frac{g_2(X)f_2(X)}{g_1(X)f_1(X)}\right] \tag{40}$$

The proof will now follow from the following lemma:

Lemma B.3 *Let $a(x), b(x)$ be two continuously differentiable functions. Suppose that there exists a point $\bar{x}$ such that $a'(\bar{x})b'(\bar{x}) < 0$. Then there exists an equivalent change of probability measure such that $E[a(X)b(X)] \;<\; E[a(X)]E[b(X)]$.*

Proof. Select an $\varepsilon > 0$ and an equivalent measure that puts a sufficiently small mass on the complement of $[\bar{x}-\varepsilon, \bar{x}+\varepsilon]$. On $[\bar{x}-\varepsilon, \bar{x}+\varepsilon]$ the functions $a(x), b(x)$ are anti-comonotone, and the claim follows from the known result that $E[a(X)b(X)] \;<\; E[a(X)]E[b(X)]$ whenever a and b are anti-comonotone.[47] ■

■

Finally, I will need the following auxiliary result, which follows by direct calculation:

Lemma B.4 *The sign of $\left(\frac{g_2(x)}{g_1(x)}\right)'\left(\frac{g_2(x)f_2(x)}{g_1(x)f_1(x)}\right)'$ coincides with the sign of*

$$\left(\frac{\delta(x,s_2)-\delta(x,s_3)}{\delta(x,s_1)-\delta(x,s_3)}\right)'\left(\left(\frac{\delta(x,s_2)-\delta(x,s_3)}{\delta(x,s_1)-\delta(x,s_3)}\right)' - \frac{1}{T_I}\frac{\delta(x,s_2)-\delta(x,s_3)}{\delta(x,s_1)-\delta(x,s_3)}(\delta(x,s_2)-\delta(x,s_1))\right). \tag{41}$$

Under the hypothesis of Proposition 3.3, $\min\left\{\left(\frac{g_2(x)}{g_1(x)}\right)'\left(\frac{g_2(x)f_2(x)}{g_1(x)f_1(x)}\right)'\right\} < 0.$

Proof. By direct calculation, the sign of $\left(\frac{g_2(x)f_2(x)}{g_1(x)f_1(x)}\right)'$ coincides with the sign of

$$\begin{aligned}\frac{1}{f_1f_2g_1^2}((g_2'f_2+g_2f_2')g_1f_1-(g_1'f_1+g_1f_1')g_2f_2) \;&=\; \frac{1}{g_1^2}((g_2'+g_2^2)g_1-(g_1'+g_1^2)g_2)\\ =\; \left(\frac{g_2(x)}{g_1(x)}\right)' + \frac{g_2}{g_1}(g_2-g_1)&\end{aligned} \tag{42}$$

and the claim follows because $g_i(x) = -T_I^{-1}(\delta(x,s_i)-\delta(x,s_3))$. Let $c(s) = c_n(s),\ b(x) = B_n'(x)$. Under the made assumptions, we determine that

$$\left(\frac{\delta(x,s_2)-\delta(x,s_3)}{\delta(x,s_1)-\delta(x,s_3)}\right)' \to 0 \tag{43}$$

[47]See, for example, Embrechts, Frey, and McNeil, 2005, Theorem 5.25(2).

as x approaches one of the boundaries X_*, X^*, whereas $\frac{\delta(x,s_2)-\delta(x,s_3)}{\delta(x,s_1)-\delta(x,s_3)} \to \frac{c(s_2)-c(s_3)}{c(s_1)-c(s_3)}$. Since $(\delta(x,s_2)-\delta(x,s_1)) \sim (c(s_2)-c(s_1))b(x)$. Since $b(x)b(-x)<0$ for large x, we determine

$$\left(\frac{\delta(x,s_2)-\delta(x,s_3)}{\delta(x,s_1)-\delta(x,s_3)}\right)' - \frac{1}{T_I}\frac{\delta(x,s_2)-\delta(x,s_3)}{\delta(x,s_1)-\delta(x,s_3)}(\delta(x,s_2)-\delta(x,s_1))$$

changes the sign for some x. By moving s_3 slightly and using the fact that $\nabla_{s_3} c$ is non-zero, we can find a value of s_3 such that the entire expression (41) changes sign. The proof is complete. ■

By the above,

$$T_U = T(\alpha_1,\alpha_2) \;\equiv\; \frac{A_{1,1}+A_{2,2}+\sqrt{(A_{1,1}+A_{2,2})^2+4(A_{1,2}A_{2,1}-A_{1,1}A_{2,2})}}{2(A_{1,2}A_{2,1}-A_{1,1}A_{2,2})} \;>\; 0 \qquad (44)$$

defines the unique positive value of risk tolerance for which the Jacobian Δ is zero. As I explain above, changing T_U is equivalent to changing the level of $\sigma(x)$. The identity $H^1_{\alpha_1} = 1 + T(\alpha_1,\alpha_2)A_{1,1} = 0$ can only occur when $A_{1,2}A_{2,1}=0$. Since all functions $A_{i,j}$ are real analytic and none of them is identically zero, this happens for a (possibly empty) set of (α_1,α_2) of a Lebesgue measure of zero. Thus, to complete the proof, it suffices to show that $\Theta(\alpha_1,\alpha_2,T(\alpha_1,\alpha_2))$ is almost surely non-zero. Since Θ is real analytic, it suffices to show that this holds in at least one point. Let $\alpha_1 = \alpha_2 - c = \alpha \to -\infty$ with some $c \in \mathbb{R}$. Then, a direct calculation implies that

$$e^{-\alpha}A_{i,j} \;=\; e^{c\mathbf{1}_{j=2}}\int_{X_*}^{X^*} g_i(x)f_j(x)g_j(x)dx \;+\; O(e^{\alpha})$$

whereas

$$e^{-\alpha}B_{i,j,j} \;=\; e^{c\mathbf{1}_{j=2}}\int_{X_*}^{X^*} g_i(x)f_j(x)g_j(x)dx \;+\; O(e^{\alpha})$$

and

$$e^{-\alpha}B_{i,j_1,j_2} \;=\; O(e^{\alpha})$$

when $j_1 \neq j_2$. It follows that

$$e^{\alpha}T(\alpha_1,\alpha_2) \;=\; t(c) \;+\; O(e^{\alpha})$$

where

$$t(c) = \frac{d(c)+\sqrt{d^2(c)+4e^cD}}{2e^cD}$$

with

$$D \;=\; \int_{X_*}^{X^*} g_1(x)g_2(x)f_1(x)dx \int_{X_*}^{X^*} g_1(x)g_2(x)(x)f_2(x)dx - \int_{X_*}^{X^*} g_1^2(x)f_1(x)dx \int_{X_*}^{X^*} g_2^2(x)f_2(x)dx$$

and

$$d(c) \;=\; \int_{X_*}^{X^*} (g_1^2(x)f_1(x)+e^cg_2^2(x)f_2(x))dx\,.$$

Therefore,

$$e^{-\alpha}\Theta(\alpha_1,\alpha_2,T(\alpha_1,\alpha_2)) \;=\; \theta(c)+O(e^{\alpha}),$$

where

$$
\begin{aligned}
\theta(c) = & \left(\frac{\int g_2(x)f_1(x)g_1(x)dx}{1+t(c)\int g_1^2(x)f_1(x)dx}\right)\left(\frac{t(c)e^c\int g_1(x)f_2(x)g_2(x)dx}{1+t(c)\int g_1^2(x)f_1(x)dx}\right)^2 \\
& + e^c\left(\int g_2^2(x)f_2(x)dx - \frac{t(c)\int g_2(x)f_1(x)g_1(x)dx}{1+t(c)\int g_1^2(x)f_1(x)dx}\int g_1(x)g_2(x)f_2(x)dx\right)
\end{aligned}
\tag{45}
$$

and, to complete the proof, it suffices to show that $\theta(c)$ is not identically zero. To this end, we note that as $c \to +\infty$, $t(c) \to t^* = \frac{\int g_2^2(x)f_2(x)dx}{D} > 0$ and, hence, as $c \to +\infty$,

$$
\theta(c) \;=\; e^{2c}\left(\frac{\int g_2(x)f_1(x)g_1(x)dx}{1+t^*\int g_1^2(x)f_1(x)dx}\right)\left(\frac{t^*\int g_1(x)f_2(x)g_2(x)dx}{1+t^*\int g_1^2(x)f_1(x)dx}\right)^2 \;+\; O(e^c),
$$

and the claim follows. ■

Proof of Theorem 3.1. Suppose first that the prior $d\pi_0$ is supported on two points, s_1 and s_2 :

$$
d\pi_0(a) \;=\; \frac{e^a}{1+e^a}d\delta_{s_1} \;+\; \frac{1}{1+e^a}d\delta_{s_2}\,. \tag{46}
$$

Then, the posterior measure is given by

$$
\nu e^{\mathcal{G}(s)}d\pi_0 \;=\; \frac{e^{a+\mathcal{G}(s_1)-\mathcal{G}(s_2)}}{1+e^{a+\mathcal{G}(s_1)-\mathcal{G}(s_2)}}d\delta_{s_1} \;+\; \frac{1}{1+e^{a+\mathcal{G}(s_1)-\mathcal{G}(s_2)}}d\delta_{s_2}\,.
$$

Let $\ell \;\equiv\; a+\mathcal{G}(s_1)-\mathcal{G}(s_2), q \;\equiv\; \frac{e^\ell}{1+e^\ell}$ and

$$
\eta(x,q) \;\equiv\; q\eta(x,s_1) \;+\; (1-q)\eta(x,s_2)\,.
$$

Then, (25) implies that q satisfies the fixed-point equation

$$
\Phi(q) \;=\; \hat{a}\,,
$$

where

$$
\Phi(q) \;\equiv\; \log\frac{q}{1-q} \;-\int_{X_*}^{X^*}\frac{\delta(x,s_1)-\delta(x,s_2)}{\sigma^2(x)}d\left(\lambda_U J_U\left(\frac{y_U(q)M(x)}{\eta(x,q)}\right)-T_I\log M(x)\right) \tag{47}
$$

and where I have defined

$$
\hat{a} \;\equiv\; a \;-\; \frac{1}{2}\int_{X_*}^{X^*}\frac{\delta^2(x,s_1)-\delta^2(x,s_2)}{\sigma^2(x)}dx \tag{48}
$$

The proof of Theorem 3.1 is based on the following lemma:

Lemma B.5 *Fix an $M(\cdot)$. Suppose that there exists a q^* such that $\Phi'(q^*)=0$ and $\Phi''(q^*)\neq 0$. Then, there exists an open set $\hat{P}$ of priors such that equilibrium is not weakly efficient.*

Proof. The proof is completely analogous to that of Lemma B.1. ■

Thus, to complete the proof, it remains to show that there exists an open set of $M(\cdot)$ such that for all each $M(\cdot)$ in this set there exists a q^* such that $\Phi'(q^*)=0$ and $\Phi''(q^*)\neq 0$. Differentiating the identity $u'(J(x))=x$, we get $u''(J(x))J'(x)x=u'(J(x))$, that is $J'(x)x=-\lambda_U^{-1}T_U(J(x))$. Differentiating the budget constraint, we determine

$$\frac{y_U'(q)}{y_U(q)} \;=\; \frac{\int_{X_*}^{X^*}\frac{\eta(x,s_1)-\eta(x,s_2)}{\eta(x,q)}T_U\left(J_U\left(\frac{y_U(q)M(x)}{\eta(x,q)}\right)\right)M(x)dx}{\int_{X_*}^{X^*}T_U\left(J_U\left(\frac{y_U(q)M(x)}{\eta(x,q)}\right)\right)M(x)dx}\,. \tag{49}$$

Furthermore, integrating by parts, we get

$$\begin{aligned}\Phi(q) \;\equiv\; & \log\frac{q}{(1-q)} + \int_{X_*}^{X^*}\left(\frac{\delta(x,s_1)-\delta(x,s_2)}{\sigma^2(x)}\right)'\left(\lambda_U J_U\left(\frac{y_U(q)M(x)}{\eta(x,q)}\right)-T_I\log M(x)\right)\\ - & \frac{\delta(X^*,s_1)-\delta(X^*,s_2)}{\sigma^2(X^*)}\left(\lambda_U J_U\left(\frac{y_U(q)M(X^*)}{\eta(X^*,q)}\right)-T_I\log M(X^*)\right)\\ + & \frac{\delta(X_*,s_1)-\delta(X_*,s_2)}{\sigma^2(X_*)}\left(\lambda_U J_U\left(\frac{y_U(q)M(X_*)}{\eta(X_*,q)}\right)-T_I\log M(X_*)\right)\,.\end{aligned} \tag{50}$$

Denote

$$A(x) \;\equiv\; \frac{\delta(x,s_1)-\delta(x,s_2)}{\sigma^2(x)}\,.$$

Then,

$$\begin{aligned}\Phi'(q) \;\equiv\; & \frac{1}{q(1-q)} - \int_{X_*}^{X^*}A'(x)T_U\left(J_U\left(\frac{y_U(q)M(x)}{\eta(x,q)}\right)\right)\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(x,s_1)-\eta(x,s_2)}{\eta(x,q)}\right)dx\\ & + A(X^*)T_U\left(J_U\left(\frac{y_U(q)M(X^*)}{\eta(X^*,q)}\right)\right)\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(X^*,s_1)-\eta(X^*,s_2)}{\eta(X^*,q)}\right)\\ & - A(X_*)T_U\left(J_U\left(\frac{y_U(q)M(X_*)}{\eta(X_*,q)}\right)\right)\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(X_*,s_1)-\eta(X_*,s_2)}{\eta(X_*,q)}\right)\,.\end{aligned} \tag{51}$$

The first key observation is that all of the above expressions are continuous in $M(\cdot)$ in the sense of almost certain convergence as long as the sequence remains bounded away from zero and from infinity. The same is true for $\Phi''(q)$. Thus, it suffices to show that there exists a nonnegative function $M(x)$ with a finite number of discontinuities such that $\Phi'(q)=0$ and $\Phi''(q)\neq 0$. Then, by the implicit function theorem, there exists an open set of continuous functions $M(x)$ such that the same holds for all $M(\cdot)$ in this open set and, hence, Lemma B.5 implies the required assertion.

Assuming that $M(x)$ is a function of bounded variation, we can rewrite

$$\Phi'(q) \;\equiv\; \frac{1}{q(1-q)} + \int_{X_*}^{X^*}A(x)d\left(T_U\left(J_U\left(\frac{y_U(q)M(x)}{\eta(x,q)}\right)\right)\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(x,s_1)-\eta(x,s_2)}{\eta(x,q)}\right)\right) \tag{52}$$

Let us pick $M(x)=a\eta(x,s)\mathbf{1}_{x<\bar{x}}+b\eta(x,s)\mathbf{1}_{x>\bar{x}}$ for some $\bar{x}\in(X_*,X^*)$ such that $\eta(\bar{x},s_1)-\eta(\bar{x},s_2)\neq$

0 and where a and b are chosen so that

$$\int_{X_*}^{X^*} M(x)dx = (1+r)^{-1}.$$

Then,

$$\begin{aligned}&\int_{X_*}^{X^*} A(x)d\left(T_U\left(J_U\left(\frac{y_U(q)M(x)}{\eta(x,q)}\right)\right)\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(x,s_1)-\eta(x,s_2)}{\eta(x,q)}\right)\right)\\ &= A(\bar{x})\bar{T}\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(\bar{x},s_1)-\eta(\bar{x},s_2)}{\eta(x,q)}\right) - \int_{X_*}^{X^*} A(x)\left(\frac{\eta(\bar{x},s_1)-\eta(\bar{x},s_2)}{\eta(x,q)}\right)' dx\end{aligned} \tag{53}$$

where $\bar{T} \equiv T(J_U(y_U b)) - T(J(y_U a))$. It suffices to show that there exists a choice of a, b and $\bar{x}$ such that

$$A(\bar{x})\bar{T}\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(\bar{x},s_1)-\eta(\bar{x},s_2)}{\eta(\bar{x},q)}\right) < 0.$$

Indeed, in this case, we can always choose $\sigma(x)$ sufficiently large for all $x \neq \bar{x}$ so that the term $\int_{X_*}^{X^*} A(x)\left(\frac{\eta(\bar{x},s_1)-\eta(\bar{x},s_2)}{\eta(x,q)}\right)' dx$ is negligible and, hence, by changing the value of $\sigma(\bar{x})$ we can achieve that $\Phi'(q) = 0$. If $a = b = (1+r)^{-1}$, then (49) implies that y_U' is zero. Hence, picking both a, b sufficiently close to $(1+r)^{-1}$, we can see that the sign of $\left(\frac{y_U'(q)}{y_U(q)} - \frac{\eta(\bar{x},s_1)-\eta(\bar{x},s_2)}{\eta(x,q)}\right)$ coincides with that of $-\frac{\eta(\bar{x},s_1)-\eta(\bar{x},s_2)}{\eta(\bar{x},q)}$. Since T_U' is almost surely non-zero, we can choose a, b arbitrarily close to $(1+r)^{-1}$ so that $\bar{T}$ has any desired sign. The proof is complete.

■

C Discrete State Space

In this section, I investigate the case in which the state space is discrete and private information is dispersed. Without loss of generality, I normalize the interval bounds to $X_* = 0,\ X^* = 1$ and assume that the random variable $X^{(K)}$ describing the state of the world takes a finite number of values $x_k = k/K,\ k = 0, \cdots, K$. In complete analogy with Assumption 2.4 in the main text, I assume that the noisy supply Z_k of Arrow securities corresponding to $X = x_k$ follows a random walk: $Z_k = \frac{1}{\sqrt{k}}\sum_{t=0}^{k} \sigma(t)\varepsilon_t$ where ε_t are independent and identically distributed with a positive C^1-density $n(x)$ with full support, mean zero, and a unit variance.[48]

There are N classes of agents. The agents of class i are initially endowed with a private signal $s_i \in \mathbb{R}^m$. Conditional on the realization of the signals vector $\mathbf{s} = (s_1, \cdots, s_N) \in \mathbb{R}^{N\times m}$, the state of the world $X^{(K)}$ is distributed according to

$$\text{Prob}(X^{(K)} = x_k|\mathbf{s}) = \nu^{(K)}(\mathbf{s})\eta(x_k, \mathbf{s}),$$

where the density $\eta(x, \mathbf{s})$ is assumed to be jointly C^2 in $(x, \mathbf{s})$. Here, $\nu^{(K)}(\mathbf{s})$ is a normalizing constant. The goal of this section is to study the asymptotic equilibrium behavior of this economy

[48]The full support assumption is important to prevent full revelation occurring with positive probability.

as $K \to \infty$.

I assume that agents of class i maximize the utility u_i of terminal consumption and use J_i to denote the inverse of the marginal utility $u_i', i = 1, \cdots, N$. Agents of class i have a prior distribution $\pi_{i0}(\mathbf{s}_{-i}; s_i)$ of the signals $\mathbf{s}_{-i} \equiv (s_j)_{j \neq i}$ of other classes. Having computed their posterior $\pi_i(\mathbf{s}_{-i}; s_i)$, the agents calculate

$$\eta(k, \pi_i) \;=\; \int_{\mathbb{R}^{(N-1)m}} \eta(x_k, (s_i, \mathbf{s}_{-i})) d\pi_i(\mathbf{s}_{-i}; s_i)\,,$$

and then use this density to compute their optimal consumption allocation across the states x_k. As I explain in the main text, equilibria are typically non-unique in a noisy Arrow-Debreu setting. However, as I will now demonstrate, in the limit as $K \to \infty$ all equilibria become fully revealing independent of the way in which the agents learn from prices. To state the next result, I will need the following definition:

Definition C.1 *We say that risk tolerances $T_k(w), k = 1, \cdots, N$ are in a generic position if for any i and any positive real numbers $a_j, b_j > 0$ such that $a_j \neq b_j$ for at least some value of j we have that the function*

$$F(M) \;\equiv\; \sum_{j \neq i} \lambda_j (T_j\left(J_j\left(M a_j\right)\right) - T_j\left(J_j\left(M b_j\right)\right)) \tag{54}$$

has a derivative that is non-zero Lebesgue-almost everywhere.

If $N = 2$ then it follows from the proof of Theorem 2.2 that risk tolerances are in a generic position whenever each of them has a derivative that is non-zero Lebesgue almost-everywhere. If $N > 2$, the joint behavior of risk tolerances may become more subtle, and they may happen not to be in a generic position. For example, if all agents have preferences $u_i(w) = \frac{(w-m_i)^{1-\gamma}}{1-\gamma}$ then $T_j(J_j(x)) = \frac{1}{\gamma} x^{-1/\gamma}$ is independent of j and hence

$$\sum_{j \neq i} \lambda_j (T_j\left(J_j\left(M a_j\right)\right) - T_j\left(J_j\left(M b_j\right)\right)) \;=\; \frac{1}{\gamma} M^{-1/\gamma} \sum_{j \neq i} \lambda_j (a_j^{-1/\gamma} - b_j^{-1/\gamma})$$

can be identically zero. However, introducing an arbitrarily small preferences heterogeneity into this example immediately resolves the problem: Since the functions x^{-1/γ_i} are linearly independent for heterogeneous γ_i, we immediately see that they are in a generic position. To state sufficient conditions guaranteeing that a given collection of risk tolerances is in a generic position, I will again use the concept of real analyticity, employing the same property of real analytic functions that was used in the proof of Proposition 2.2. Specifically, if $u_i(w)$ is real analytic for any i, then so is the function (54). Hence, it suffices to verify that the derivative $F'(M)$ is non-zero in at least one point to conclude that it is non-zero almost everywhere.

Lemma C.1 *Suppose that $u_i, i = 1, \cdots, N$ are real-analytic and that for each i there exists a $\gamma_i > 0$ such that $T_i(u_i(w)) \;\sim\; c_i w^{-1/\gamma_i}$ asymptotically as $w \to \infty$, and $\gamma_i \neq \gamma_j$ for $i \neq j$. Then, risk tolerances are in a generic position.*

Proof. The proof follows because $F'(M)$ is non-zero for sufficiently large M and is therefore non-zero almost everywhere due to real analyticity. ■

We are now ready to state the main result of this section.

Proposition C.1 *Suppose that the support of π_{i0} is compact for each i and that the risk tolerances are in a generic position. Then, in the limit as $K \to \infty$, all equilibria become fully revealing.*

Proof. By the Donsker theorem (Donsker, 1952), the process $Z(k)$ converges to a Gaussian process of Assumption 2.4 in the limit as $K \to \infty$. Since the support of π_0 is compact, the set of probability measures with the same support is also compact in the weak-* topology. See Billingsley (1999). Therefore, in the limit as $K \to \infty$, we can always select a sub-sequence such that the posterior beliefs converge to some measures $d\pi_i$. Since the derivatives of $\eta(x, \mathbf{s})$ are continuous in $\mathbf{s}$ and are therefore uniformly bounded, we determine that $\eta(x, \pi_i(\cdot))$ belong to $C^1[X_*, X^*]$. If the limit is not fully revealing, then by the same argument as in the proof of Theorem 2.2 we determine that $F'(M)$ has to be zero for a set of positive measure of sample path realizations of $Z(\cdot)$. However, this is impossible by Lemma C.1. ■

D Implementing Arrow-Debreu Equilibria with Options when the State Space is Continuous

As I explain in the main text, when the state space is continuous, it is natural to specify supply shocks $Z(x)$ as a stochastic process (i.e., a "sequence" of random variables, indexed by the continuous parameter x). In this section, I show how the results from the main text can be extended to a large class of supply shock processes satisfying the following heuristic "axioms:"

(a) There exists a continuum of independent and identically distributed (i.i.d.) shocks such that each $Z(x)$ is a linear combination of these shocks.

(b) The exposure of $Z(x)$ to these i.i.d. shocks only depends on x and not on the realization of shocks.

(c) $Z(x)$ is continuous in x.

Parts (a) and (b) simply assume that there are basic building blocks from which the shocks are constructed, and the exposure to these "basic shocks" is driven by some fundamental supply/demand mechanism and, hence, does not depend on the shock realization. The assumption of continuity is natural: One would expect that the supply of Arrow securities for almost identical states is also almost identical.[49] This continuity assumption together with items (a) and (b) immediately implies that $Z(x)$ should be a part of a Gaussian diffusion process: Linear combinations of i.i.d. shocks always have a Gaussian distribution in the continuous limit by the central limit theorem.[50] I formalize these observations in the following assumption:

[49] Introducing jumps only makes the inference problem easier.

[50] This result is an analog of the fact that a Brownian motion is a continuous limit of a random walk (Donsker, 1952).

Assumption D.1 *There exists a probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *supporting a Brownian motions* $B_t \in \mathbb{R}^d$ *for some* $d \in \mathbb{N}$*. The filtration* $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ *is the usual augmentation of the filtration generated by the Brownian motion* B_t*.*[51] *Furthermore,*

(1) There exists a Gaussian diffusion process $D_t = (D_{t,1}, \cdots, D_{t,d}) \in \mathbb{R}^d, t \geq 0$ *adapted to* $\mathbb{F}$*, such that* $Z(x) = D^1_{x-X_*,1}$[52] *The coefficients of the diffusion* D_t *are infinitely differentiable.*[53]

(2) $\mathrm{Var}[Z(x)|\mathcal{F}_0] \neq 0$ *for all* $x > X_*$*.*

(3) D_0 *is normally distributed.*

Discussion of Assumption D.1. Assumption D.1 allows for essentially arbitrary dependence structures in the noisy supply. In particular, $Z(x)$ need not necessarily be Markovian: Supply shocks to different levels of x may have impacts on each other: These dependencies are captured by the latent (unobservable) components of the diffusion process D_t (i.e., by $D_{t,j},\ j > 1$). Item (2) means that there are non-redundant supply shocks for each state x. The following lemma is a direct consequence of the basic properties of Ornstein-Uhlenbeck processes:

Lemma D.1 *For any* $x \in (X_*, X^*)$*, there exists a neighborhood* $(x - \varepsilon, x + \varepsilon)$ *and a* $k < d$ *such that* $Z(y) \in C^k(x - \varepsilon, x + \varepsilon)$ *and* $Z^{(k)}(y)$ *has a non-random quadratic variation* $\sigma_Z(x) \neq 0$ *and is therefore almost surely nowhere differentiable. In particular, there exists a* $k^* < d$ *such that* $Z(x)$ *is not* k^* *times differentiable on any interval.*

Now, repeating the same arguments found in the proof of Theorems 2.2 and 2.1 but applied to derivatives $\frac{d^k}{dx^k} \log M(x)$ and $\frac{d^k}{dx^k} D(x)$, respectively, we arrive at the following result:

Theorem D.1 *Suppose that* $\eta(x, s) \in C^{k^*}[X_*, X^*]$*. Then the following is true:*

- *Suppose that at least one of the risk tolerances* $T_i(w) = -\frac{u_i'(w)}{u_i''(w)},\ i = I, U$ *is such that the set* $\{w : T_i'(w) = 0\}$ *has a Lebesgue measure of zero. Then, there always exists a fully revealing equilibrium given by (11), except possibly for a countable set of values of the quotient* λ_I/λ_U*. In contrast, if both* $T_i,\ i = I, U$ *are constant, then no fully revealing equilibrium exists.*

- *Suppose that the set* $\{w :\ A_I'(w) = 0\}$ *has a Lebesgue measure of zero. Then, any REE is fully revealing.*

Lemma D.1 implies that under a general Gaussian structure of Assumption D.1 supply shocks are always non-smooth. Smoothness of $Z(x)$ is not just a matter of mathematical curiosity. It plays an important role in the implementation of Arrow-Debreu equilibria with derivative securities. This is made clear by the following example:

[51] See Karatzas and Shreve (1991).

[52] Recall that a diffusion process D_t satisfies the stochastic differential equation $dD_t = \mu(t, D_t)dt + \sigma(t, D_t)dB_t$. A diffusion process is Gaussian if and only if the diffusion matrix σ does not depend on D_t and μ is linear in $D_t:\ \mu(t, D_t) = \mu_0(t) - \mu_1(t)D_t$ for some μ_0, μ_1. See Karatzas and Shreve (1991).

[53] In fact, only a finite number of derivatives is needed, but this number depends on d_i.

Example 2 *Recall that a European call (respectively, put) option with strike K on the underlying X is a contract with payoff $(X-K)^+$ (respectively, $(K-X)^+$). Similarly, a binary call (respectively, put) option with strike K is a contract with payoff $\mathbf{1}_{X>K}$ (respectively, $\mathbf{1}_{X<K}$).*

- *Suppose that a full menu of European call and put options with arbitrary strikes and maturity $t = 1$ is available for trading. Then, markets are complete because any state contingent claim $Z(x),\ x \in (X_*, X^*)$ can be replicated with a portfolio of call and put options using the formula*[54]

$$Z(x) \;=\; Z(X_0) + Z'(X_0)(x - X_0) + \int_{X_*}^{X_0} (K - x)^+ dZ'(K) \;+\; \int_{X_0}^{X^*} (x - K)^+ dZ'(K)\,, \quad (55)$$

where X_0 is an arbitrary reference point. For all the objects in (55) to be well defined, we need to require that $Z(x)$ has at least one continuous derivative, $Z'(x)$. As a result, $Z(x)$ cannot be a Markov diffusion process. The simplest specification is to assume that $Z(x)$ is an integral of a diffusion: If we define the diffusion D_t to be

$$\begin{aligned} dD_t^1 &= D_t^2 dt \\ dD_t^2 &= \sigma(t) dB_t\,, \end{aligned} \quad (56)$$

then (for $x > X_0$) we have $Z(x) = D_x^1 = Z(X_*) + \int_{X_*}^{x} D_{t-X_*}^2 dt$ where $D_t^2 = D_0^2 + \int_0^t \sigma(s) dB_s$. In this case, the noisy supply of $Z(x)$ Arrow securities is equivalent to the noisy supply of $Z'(X_0)$ units of the underlying claim X and $dZ'(x) = \sigma(x) dB_x$ units of options. In particular, the supply of options of strike K is infinitesimally small and uncorrelated over strikes. This is analogous to the continuous-time Kyle (1985) model in which the noisy supply per instant of time is dB_t. If one wants a non-infenitesimal supply that correlates across strikes, one would need at least two continuous derivatives for $Z(x)$. For example, if we define the diffusion D_t via

$$\begin{aligned} dD_t^1 &= D_t^2 dt \\ dD_t^2 &= D_t^3 dt \\ dD_t^3 &= \sigma(t) dB_t\,, \end{aligned} \quad (57)$$

then $Z''(x) = D_{x-X_*}^3 \;=\; D_0^2 + \int_0^t \sigma(s) dB_s$ is well defined, and the noisy supply of $Z(x)$ Arrow securities is equivalent to the supply of $Z''(x)$ of call and put options with the corresponding strikes. None of these problems arises with a discrete state space: In this case, we can define the supply of Arrow securities as in the main text, $Z(x) = \sum \sigma(k) Y_k$ and then introduce the supply of call options options of strike k given by the "discrete second derivative" of Z, given by $\sigma(k)Y_k - \sigma(k-1)Y_{k-1}$.

[54] This formula follows by a direct calculation using integration by parts. See Carr and Madan (1998) and Demeterfi et al. (1999). If $W'(x)$ is a diffusion then the integral is understood in the Ito sense.

- *Suppose that a full menu of binary options is available for trading. By direct calculation,*

$$Z(x) \;=\; Z(X_0) - \int_{X_*}^{X_0} \mathbf{1}_{x<K} dZ(K) \;+\; \int_{X_0}^{X^*} \mathbf{1}_{x>K} dZ(K)\,. \tag{58}$$

  and hence the supply of $Z(x)$ of Arrow securities is equivalent to the supply of $Z'(x)$ of binary options. In particular, the model in the main text corresponds to the supply of $\sigma(x)dB_x$ of digital options of strike x.