Concavity of the Consumption Function with Recursive Preferences

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Abstract

Carroll and Kimball (1996) show that the consumption function for an agent with time-separable, isoelastic preferences is concave in the presence of income uncertainty. In this paper I show that concavity breaks down if we abandon time-separability. Namely, if an agent maximizing an isoelastic recursive utility has preferences for early resolution of uncertainty, there always exists a distribution of income risk such that *consumption function is not concave in wealth.* I also derive sufficient conditions guaranteeing that the consumption function is concave if the agent has preferences for late resolution of uncertainty.

1 Introduction

The theory of the consumption function (defined as the map between current wealth and current consumption) is one of the cornerstones of modern macroeconomics. Individual agents' consumption and saving decisions aggregate in the economy and influence business cycle fluctuations and growth. Therefore, it is important to understand the origins of these decisions.

One of the useful and intuitive properties of the consumption function is known as the "decreasing marginal propensity to consume," which refers to the fact that marginal propensity to consume (MPC) out of wealth declines with the level of wealth or transitory income. In mathematical terms, this means that the consumption function is concave in wealth. Important implications of this property of concavity have been discussed in the literature at least since Keynes (1935). For example, concavity generates persistence in the consumption growth, and implies that the cross-sectional distribution of wealth is an important variable for the aggregate dynamics.[1] Carroll (1997) shows that the concavity of the consumption function is crucial for the theory of "buffer-stock" target saving behavior. Kimball (1990) shows that the concavity of the consumption function implies that the effective risk aversion of an agent's value function is higher than the risk aversion of the utility function. As a result, financial risk-taking decisions become wealth-dependent. More generally, the behaviour of MPC is intimately linked to the theory of precautionary savings. See, for example, Kimball (1990), Carroll and Samwick (1997), Huggett and Ospina (2001), Huggett (2004), Parker and Preston (2005), and Carroll (2009). There is also some empirical evidence (Browning and Lusardi, 1996; Souleles, 1999; Souleles, Parker, and Johnson, 2006) suggesting that the MPC is indeed decreasing in the level of wealth.

Zeldes (1989) was the first to show (through numerical experiments) that uninsurable income shocks make the consumption function concave.[2] In an important paper, Carroll and Kimball (1996) rigorously proved that the consumption function is indeed concave, assuming (i) time-separable Hyperbolic Absolute Risk Aversion (HARA) preferences and (ii) a positive precautionary saving motive. In particular, their result covers the most important case of isoelastic (Constant Relative Risk Aversion, CRRA) utility function.

While the time-separable CRRA utility specification remains an important element of many macro-models, it has a major drawback, in that Elasticity of Intertemporal Substitution (EIS) is given by the reciprocal of risk aversion. This is quite inconvenient, especially because the sensitivity of the rate of consumption growth to changes in real rate is given by EIS, and it is not clear why it ought to be directly linked to risk preferences. The creation of the recursive utility theory by Kreps and Porteus (1978), Epstein and Zin (1989, 1991), and Duffie and Epstein (1992) made it possible to separate the effects of EIS and risk aversion. Since then, the Constant Elasticity of

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[1]See, e.g., Blundell and Preston (1998) and Carroll, Slakalek and Tokuoka (2013).

[2]See also Caballero (1991) and Aiyagari (1994).

Intertemporal Substitution - Constant Relative Risk Aversion (CEIS-CRRA) utility specification (often referred to as the isoelastic recursive utility, or the Epstein-Zin preferences) has become a major workhorse model in the macro-finance literature. See, for example, Weil (1989, 1993),[3] Epstein and Zin (1991), Bansal and Yaron (2004), Hansen, Heaton, Roussanov, and Lee (2007), Hansen, Heaton and Li (2008), and Hansen and Scheinkman (2012). Given the role the CEIS-CRRA preferences specification plays in economics, it is important to understand the optimal saving and consumption decisions they imply. This is the goal of the present paper.

My main finding is that the concavity of the consumption function depends crucially on the preferences for timing the resolution of uncertainty. Let us denote by ψ and γ the EIS and risk aversion of the CEIS-CRRA agent, respectively. First, considering a two-period model, I show that when the agent prefers late resolution of uncertainty (i.e., when $\psi^{-1} > \gamma$), concavity of the consumption function is preserved. By contrast, if the agent prefers early resolution of uncertainty, there always exists a distribution of uninsurable income shocks for which the consumption function is convex. Then, I extend the concavity result to dynamic settings and show that concavity is preserved if $\psi^{-1} > \gamma$ and $\psi^{-1} \geq 2$ and $\gamma > \frac{\psi^{-1}-2}{3}$.[4] Furthermore, if income shocks are independent consumption function is always concave when $\gamma \leq \psi^{-1} \leq 2$.

The intuition for my result is as follows. An agent prefers early resolution of uncertainty when the risk aversion effect dominates the inter-temporal substitution effect. If the agent anticipates large uncertainty about income tomorrow, he may find it optimal to reduce today's consumption more than proportionally with respect to his current wealth and to increase the buffer stock of savings against future shocks. As his wealth becomes sufficiently large, the MPC converges to one. As a result, MPC becomes an increasing function of wealth, which is below one for low wealth levels. By contrast, when the substitution effect dominates the risk aversion effect, the agent will worry more about today's consumption than about tomorrow's consumption as long is he is sufficiently constrained, implying that MPC is decreasing. The proof of the result is non-trivial and is based on subtle properties of the derivatives of the value function.

2   Setup

Time is discrete, $t = 0, \cdots, T$. Uncertainty is described by a filtration $\mathcal{F}_t$, $t = 0, \cdots, T$ satisfying standard conditions. An economic agent faces uninsurable income shocks given by a bounded stochastic process w_t, adapted to $\mathcal{F}_t$. In or-

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[3]Weil (1993) studies an interesting class of recursive preferences, the constant EIS-constant absolute risk aversion utility. He shows that, under this specification, the consumption function is linear in wealth.

[4]While inequality $\psi^{-1} > \gamma$ is necessary for the validity of the result, inequality $\gamma > \frac{\psi^{-1}-2}{3}$ is a technical condition needed for the proof. It can probably be relaxed, but I have not been able to prove this.

der to reallocate wealth across time periods, the agent can use risk-free bonds that pay a risk-free interest rate r_t at time t. The process r_t is predictable with respect to the filtration $\mathcal{F}_t$. The agent maximizes a recursive functional J_t defined on the set of positive and bounded consumption stream c_t, $t = 0, \cdots, T$, subject to the inter-temporal budget constraint

$$ c_t \;=\; x_{t-1} e^{r_t} \;+\; w_t \;-\; x_t $$

where x_t is the (adapted) process of savings, which can take arbitrary values, $x_t \in \mathbb{R}$. We assume that the utility functional J_t is the CEIS-CRRA recursive utility defined via[5]

$$ J_t \;=\; \left(c_t^{1-\lambda} \;+\; e^{-\rho} \left(E_t[J_{t+1}^{1-\gamma}] \right)^{\frac{1-\lambda}{1-\gamma}} \right)^{1/(1-\lambda)} , \quad J_{T+1} = 0 \,. $$

Here, γ is the coefficient of relative risk aversion, $\psi = \frac{1}{\lambda}$ is the elasticity of intertemporal substitution, and ρ is the time discount rate. The case $\gamma > \psi^{-1}$ (respectively, $\gamma < \psi^{-1}$) corresponds to preferences for early (respectively, late) resolution of uncertainty. See Kreps-Porteus (1978) and Epstein-Zin (1989). We can then define the value function as

$$ V_\tau(y) \;\equiv\; \max_{c_t, x_t, t \geq \tau} J_\tau((c_t)_{t \geq \tau}) $$

assuming that the income process of the agent is given by $\tilde{w}_t$, $t > \tau$, where we have defined the process $\tilde{w}_t$ via $\tilde{w}_t = w_t, t > \tau$ and $\tilde{w}_\tau = y$.

3 Main results

The next result shows that, for the case $T = 1$, preferences for late resolution of uncertainty are both necessary and sufficient for the concavity of the consumption function.

THEOREM 3.1 *Let $T = 1$. Then the consumption function c_0 is a concave function of wealth if*

$$ \lambda \geq \gamma. $$

Conversely, if $\lambda < \gamma$, there exists an income shock w_1 and an open set $\Omega \subset \mathbb{R}$ such that $c_0(w_0)$ is strictly convex for $w_0 \in \Omega$.[6]

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[5]This functional is unique, up to an ordinarily equivalent transformation. See, Kreps-Porteus (1978) and Epstein-Zin (1989). The particular form of normalization chosen in my paper is used purely for technical convenience.

[6]While the result only claims that the consumption function is convex on some set, extensive numerical experiments indicate that, in many cases, the consumption function is actually strictly convex everywhere.

I now discuss the proof of Theorem 3.1. To illustrate the new effects due to time-nonseparability, note that we can rewrite the maximization problem as

$$\max_{x_0} \left((w_0 - x_0)^{1-\lambda} + e^{-\rho} \left(E[(w_1 + e^r x_0)^{1-\gamma}] \right)^{\frac{1-\lambda}{1-\gamma}} \right)$$

where $r \equiv r_1$. Differentiating, we get the first order condition

$$g(x_0) \equiv E[(w_1 + e^r x_0)^{1-\gamma}]^{\frac{\beta}{\lambda}} E[(w_1 + e^r x_0)^{-\gamma}]^{-\frac{1}{\lambda}} = e^{(r-\rho)/\lambda} c_0, \qquad (3.1)$$

where we have defined

$$\beta = \frac{\lambda - \gamma}{1 - \gamma}.$$

As I show in the Appendix, $g(x)$ is monotone increasing; hence, (3.1) is equivalent to $x_0 = g^{-1}(e^{(r-\rho)/\lambda} c_0)$. The budget constraint

$$c_0 + g^{-1}(e^{(r-\rho)/\lambda} c_0) = w_0$$

uniquely determines the consumption function $c_0 = c_0(w_0)$. Thus, c_0 is concave if and only if g^{-1} is convex, which is in turn equivalent to g being concave. Therefore, Theorem 3.1 is a direct consequence of the following lemma.

LEMMA 3.2 *The function $g(x)$ is monotone increasing. It is concave if $\lambda \geq \gamma$. However, if $\gamma > \lambda$, there exists a random variable w_1 such that $g(x)$ is not concave for x in some open set of $\mathbb{R}$.*

It is important to note that the convexity result of Theorem 3.1 immediately extends to the case $T > 1$. Indeed, assuming that the income process is zero for $T > 1$ and using homogeneity of isoelastic preferences, we get that the time-zero consumption function c_0 coincides with that for the car $T = 1$ but with a different discount factor $e^{-\rho}$.

We now turn to the dynamic version of the problem. In this case, we can write down the dynamic programming equation as

$$V_t(y) = \max_x \left((y - x)^{1-\lambda} + e^{-\rho} \left(E_t[(V_{t+1}(e^{r_{t+1}} x + w_{t+1}))^{1-\gamma}] \right)^{\frac{1-\lambda}{1-\gamma}} \right)^{1/(1-\lambda)}.$$
$$(3.2)$$

Standard results about envelopes of concave functions, combined with the fact that the function $(x_1^{1-\lambda} + x_2^{1-\lambda})^{1/(1-\lambda)}$ is jointly concave in (x_1, x_2) imply that the following is true.

LEMMA 3.3 *The function $V_t(y)$ is increasing and concave for any $t \geq 0$.*

Using the envelope theorem, we and differentiating (3.2), we get

$$\left((1 - \lambda)^{-1} V_t^{1-\lambda}(y) \right)' = c_t(y)^{-\lambda}. \qquad (3.3)$$

Differentiating (3.3), we arrive at the following result.

LEMMA 3.4 *The consumption function $c_t(y)$ is concave if and only if*

$$(V_t(y)^{1-\lambda})''' (V_t(y)^{1-\lambda})' \geq \frac{\lambda+1}{\lambda} \left(\left(V_t(y)^{1-\lambda} \right)'' \right)^2. \qquad (3.4)$$

Lemma 3.4 is a direct analog of Lemma 3 in Carroll and Kimball (1996). Thus, proving concavity reduces to proving that the value function V_t satisfies (3.4). The proof of this result in Carroll and Kimball (1996) is based on an important inter-temporal aggregation property. Namely, they show that if the value function V_{t+1} satisfies condition (3.4) and the von Neumann-Morgenstern utility function satisfies the same condition, then so does V_t.[7] In the time-separable case, Carroll and Kimball present a short and elegant argument to prove this important result. Namely, their key observation is that if a function $V(x)$ satisfies $V'''V' \geq k(V'')^2$ then so does the function $E[V(x+w)]$ for any random variable w. However, to apply the argument of Carroll and Kimball in the time-nonseparable case, one would need to show that the function $E[V(x+w)^{\frac{1-\gamma}{1-\lambda}}]^{\frac{1-\lambda}{1-\gamma}}$ inherits the property $V'''V' \geq k(V'')^2$. I have not been able to prove this result and, if fact, I conjecture that it is not true in general. While it may still be true that the aggregation result does hold in some more general form, I have only been able to prove it under additional conditions. The following is true.

LEMMA 3.5 *Suppose that the agent has preferences for late resolution of uncertainty, i.e. $\lambda > \gamma$. If $V_{t+1}(y)$ satisfies (3.4) and we have that*

- *either $\lambda > 2$ and $\gamma \geq \frac{\lambda-2}{3}$,*

- *or $\lambda \leq 2$ and the income shock w_t is independent of $\mathcal{F}_{t-1}$ for each $t \geq 1$.*

Then, $V_t(y)$ also satisfies (3.4).

Lemmas 3.4 and 3.5 immediately yield the following result.

PROPOSITION 3.6 *Suppose that the agent has preferences for late resolution of uncertainty and that*

(1) either $\lambda > 2$ and $\gamma \geq \frac{\lambda-2}{3}$,

(2) or $\lambda \leq 2$ and the income shock w_t is independent of $\mathcal{F}_{t-1}$ for each $t \geq 1$.

Then the consumption function is concave.

In conjecture that, in general, the assumption of uncorrelated shocks in item (2) of Proposition 3.6 cannot be relaxed. The reason is that with correlated shocks the value function depends on w_t not only through the current wealth, but also through the impact of w_t on the distribution of future shocks. This introduces additional non-linear dependence into the value function and

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[7]See Lemma 2 in Carroll and Kimball (1996).

distorts the consumption/saving behaviour. Note however that item (2) allows the distribution of w_t to change over time in an arbitrary fashion. In particular, it covers the standard life-cycle income specification with mean w_t exhibiting a hump-shaped dependence on the "age" t.

It is also interesting to know whether Proposition 3.6 covers situations that are considered to be empirically relevant. In the finance literature, the most commonly used specification is $\psi \approx 1.5$ and $\gamma \geq 5$, used in Bansal and Yaron (2004). In the macroeconomics literature, there is less consensus on what the "true" value of EIS is. For example, Hall (1988) finds an estimate of ψ that is very low (less than 0.2), while Ogaki and Reinhart (1998) find an estimate of EIS that is slightly above one. Given that the risk aversion parameter γ is commonly assumed to be above one, only Hall's estimate would imply preferences for early resolution of uncertainty.

4 References

Aiyagari, S. R., 1994, "Uninsured Idiosyncratic Risk and Aggregate Saving," The Quarterly Journal of Economics 109(3), 659-684.

Bansal, R., and A. Yaron, 2004, "Risks for the Long-Run: A Potential Resolution of Asset Pricing Puzzles,?" Journal of Finance 59(4), 1481-1509.

Blundell, R., and I. Preston, 1998, "Consumption Inequality And Income Uncertainty," The Quarterly Journal of Economics 113(2), 603-640.

Browning, M., and A. Lusardi, 1996, "Household Saving: Micro Theories and Micro Facts," Journal of Economic Literature 34(4), 1797-1855.

Caballero, R. J, 1991, "Earnings Uncertainty and Aggregate Wealth Accumulation," American Economic Review 81(4), 859-871.

Carroll, C. D., and M. S. Kimball, 1996, "On the Concavity of the Consumption Function," Econometrica 64(4), 981-992.

Carroll, C. D., 1997, "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," Quaterly Journal of Economics 112(1), 1-55.

Carroll, C. D., and A. A. Samwick, 1997, "The Nature of Precautionary Wealth," Journal of Monetary Economics 40(1), 41-71.

Carroll, C. D., 2009, "Precautionary Saving and the Marginal Propensity to Consume out of Permanent Income," Journal of Monetary Economics 56(6), 780-790.

Carroll, C. D., J. Slacalek, and K. Tokuoka, 2013, "The Distribution of Wealth and the Marginal Propensity to Consume", working paper.

Duffie, D., and L. G. Epstein, 1992, "Stochastic Differential Utility," Econometrica 60(2), 353-394.

Epstein, L. G., and S. E. Zin, 1989, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework", Econometrica 57(4), 937-969.

Epstein, L. G., and S. E. Zin, 1991, "Substitution, Risk Aversion, and the Temporal Behaviour of Consumption and Asset Returns: An Empirical Investigation," Journal of Political Economy 99, 555-576.

Hall, R. E., 1988, "Intertemporal Substitution in Consumption," Journal of Polytical Economy 96, 339-357.

Hansen, L. P., J. Heaton, N. Roussanov, and J. Lee, 2007. "Intertemporal Substitution and Risk Aversion," in Handbook of Econometrics, ed. by J. Heckman, and E. Leamer, vol. 6B. North Holland, Amsterdam, The Netherlands.

Hansen, L. P., J. Heaton, and N. Li, 2008, "Consumption Strikes Back: Measuring Long-run Risk," Journal of Political Economy 116(2), 260-302.

Hansen, L. P., and J. A. Scheinkman, 2012, "Recursive Utility in a Markov Environment with Stochastic Growth," Proceedings of the National Academy of Sciences 109(30), 11967-11972.

Hardy, G. H., J. E. Littlewood, and G. Pólya, 1988, Inequalities. Cambridge Mathematical Library. Cambridge: Cambridge University Press.

Huggett, M., and S. Ospina, 2001, "Aggregate Precautionary Savings: When is the Third Derivative Irrelevant?," Journal of Monetary Economics 48(2), 373-396.

Huggett, M., 2004, "Precautionary Wealth Accumulation," Review of Economic Studies 71, 769-781.

Kimball, M. S., 1990, "Precautionary Saving in the Small and in the Large," Econometrica 58(1), 53-73.

Kreps, D. M. and E. L. Porteus, 1978, "Temporal Resolution of Uncertainty and Dynamic Choice Theory", Econometrica 46(1), 185-200.

Ogaki, M., and C. M. Reinhart, 1998, Measuring Intertemporal Substitution: The Role of Durable Goods, Journal of Political Economy 106(5), 1078-1098.

Parker, J. A., and B. Preston, 2005, "Precautionary Saving and Consumption Fluctuations," American Economic Review 95(4), 1119-1143.

Souleles, N. S., 1999, "The Response of Household Consumption to Income Tax Refunds," American Economic Review 89(4), 947-958.

Souleles, N. S., J. A. Parker, and D. S. Johnson, 2006, "Household Expenditure and the Income Tax Rebates of 2001," American Economic Review 96(5), 1589-1610.

Weil, P., 1989, "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," Journal of Monetary Economics 24(3), 401-421.

Weil, P., 1993, "Precautionary Savings and the Permanent Income Hypothesis," Review of Economic Studies 60(2), 367-383.

Zeldes, Stephen P, 1989, "Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence," The Quarterly Journal of Economics-Press 104(2), 275-298.

A Proofs

Proof of Lemma 3.2. We have

$$
\begin{aligned}
g'(x) \;=\;& \frac{\beta}{\lambda}\, E[(w_1+x)^{1-\gamma}]^{\frac{\beta}{\lambda}-1}(1-\gamma)\, E[(w_1+x)^{-\gamma}]^{1-\frac{1}{\lambda}} \\
+\;& E[(w_1+x)^{1-\gamma}]^{\frac{\beta}{\lambda}}\, \frac{\gamma}{\lambda} E[(w_1+x)^{-\gamma}]^{-1-\frac{1}{\lambda}}\, E[(w_1+x)^{-\gamma-1}] \\
=\;& E[(w_1+x)^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\, E[(w_1+x)^{-\gamma}]^{1-\frac{1}{\lambda}} \\
+\;& \frac{\gamma}{\lambda}\, E[(w_1+x)^{1-\gamma}]^{\frac{\beta}{\lambda}-1} E[(w_1+x)^{-\gamma}]^{-1-\frac{1}{\lambda}} \\
\times\;& \Big(E[(w_1+x)^{-\gamma-1}]\, E[(w_1+x)^{1-\gamma}] \;-\; \big(E[(w_1+x)^{-\gamma}] \big)^2 \Big)
\end{aligned}
\tag{A.1}
$$

By the Cauchy-Schwarz inequality,

$$
E[(w_1+x)^{-\gamma-1}]\, E[(w_1+x)^{1-\gamma}] \;\geq\; \big(E[(w_1+x)^{-\gamma}] \big)^2,
$$

and hence $g'(x) \geq 0$. Denote

$$
f(\alpha) \;\equiv\; E[(w_1+x)^{-\alpha}].
$$

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Then, differentiating (A.1), we get

$$
\begin{aligned}
g''(x) \; = \; & \frac{\gamma}{\lambda}\frac{\beta}{\lambda} f(\gamma-1)^{\frac{\beta}{\lambda}-1}(1-\gamma)f(\gamma)\,f(\gamma)^{-1-\frac{1}{\lambda}}f(\gamma+1) \\
& \frac{\gamma}{\lambda}\left(-\frac{1}{\lambda}-1\right)f(\gamma-1)^{\frac{\beta}{\lambda}}f(\gamma)^{-\frac{1}{\lambda}-2}(-\gamma)f(\gamma+1)^2 \\
& -\frac{\gamma}{\lambda}(\gamma+1)f(\gamma-1)^{\frac{\beta}{\lambda}}f(\gamma)^{-\frac{1}{\lambda}-1}(\gamma+2) \\
& +\left(1-\frac{\gamma}{\lambda}\right)\left(\frac{\beta}{\lambda}-1\right)f(\gamma-1)^{\frac{\beta}{\lambda}-2}(1-\gamma)\,f(\gamma)\,f(\gamma)^{1-\frac{1}{\lambda}} \\
& +\left(1-\frac{\gamma}{\lambda}\right)\left(1-\frac{1}{\lambda}\right)f(\gamma-1)^{\frac{\beta}{\lambda}-1}f(\gamma)^{-\frac{1}{\lambda}}(-\gamma)f(\gamma+1) \\
= \; & \frac{1}{\lambda^2}f(\gamma-1)^{\frac{\beta}{\lambda}-2}f(\gamma)^{-\frac{1}{\lambda}-2} \\
& \times\Big[\gamma(\lambda-\gamma)f(\gamma-1)f(\gamma)^2 f(\gamma+1)\;+\;\gamma^2(\lambda+1)\,f(\gamma-1)^2\,f(\gamma+1)^2 \\
& -\gamma\lambda\,(\gamma+1)\,f(\gamma-1)^2 f(\gamma)f(\gamma+2)\;+\;(\lambda-\gamma)(\beta-\lambda)(1-\gamma)f(\gamma)^4 \\
& -\gamma(\lambda-\gamma)(\lambda-1)f(\gamma-1)f(\gamma)^2 f(\gamma+1).\Big] \\
= \; & -\frac{1}{\lambda^2}f(\gamma-1)^{\frac{\beta}{\lambda}-2}f(\gamma)^{-\frac{1}{\lambda}-2} \\
& \times\Big[\gamma^2\,(\lambda+1)\,f(\gamma-1)\big(f(\gamma)\,f(\gamma+2)-f(\gamma+1)^2\big)\;+\;\gamma\,(\lambda-\gamma)\times \\
& \big[(f(\gamma-1)^2 f(\gamma)\,f(\gamma+2)+f(\gamma)^4-2f(\gamma-1)\,f(\gamma)^2 f(\gamma+1)\big) \\
& +\lambda\,f(\gamma)^2\big(f(\gamma-1)\,f(\gamma+1)-f(\gamma)^2\big)\big]\Big]
\end{aligned}
\tag{A.2}
$$

By the Cauchy–Schwarz inequality,

$$
f(\gamma)\,f(\gamma+2) \;\geq\; f(\gamma+1)^2 \;,\;\; f(\gamma-1)\,f(\gamma+1)\;\geq\; f(\gamma)^2
$$

and therefore

$$
\begin{aligned}
& f(\gamma-1)^2\,f(\gamma)\,f(\gamma+2)\;+\;f(\gamma)^4\;-\;2f(\gamma-1)\,f(\gamma)^2\,f(\gamma+1) \\
& \geq\; f(\gamma-1)^2\,f(\gamma+1)^2\;+\;f(\gamma)^4\;-\;2f(\gamma-1)\,f(\gamma)^2\,f(\gamma+1) \\
& =\;(f(\gamma-1)\,f(\gamma+1)-f(\gamma)^2)^2\;\geq\;0,
\end{aligned}
\tag{A.3}
$$

and the required concavity follows.

To prove the last statement, suppose that w_1 is such that $(w_1+x)^{-1/N}$ is uniformly distributed on $[0,2]$. Then, $f(\alpha)=0.5\int_0^2 z^{N\alpha}dz=\frac{1}{N\alpha}2^{N\alpha-1}$ When N is sufficiently large, $f(\gamma+2)/f(\gamma+1)$ and $f(\gamma+1)/f(\gamma)$ can also be made arbitrarily large. Thus, choosing a large N, we can make the terms in (A.2), containing $f(\gamma+2)$ dominate all other terms. If $\gamma>\lambda$, this will imply that $g(x)$ is locally convex. $\qquad\square$

Proof of Proposition 3.4. Let $\tilde{V} = V^{1-\lambda}$. Then,

$$\tilde{V}''_t = -\lambda\, c_t^{-\lambda-1}\, c'_t$$

and

$$\tilde{V}'''_t = \lambda\,(\lambda+1)\, c_t^{-\lambda-2}\,(c'_t)^2 - \lambda\, c_t^{-\lambda-1}\, c''_t.$$

Hence,

$$\lambda\,\tilde{V}'''\,\tilde{V}' - (\lambda+1)\,(\tilde{V}'')^2 = -\lambda^2\, c_t^{-\lambda-1}\, c''$$

and the claim follows. $\qquad\qquad\square$

Proof of Lemma 3.5. For simplicity, I omit the index $t+1$ and denote $V = V(x, w_{t+1}) \equiv V_{t+1}(x)$. I will also assume for simplicity that $\rho = r = 0$. All the calculations can be easily modified for the case of non-zero rates. I will also assume without loss of generality that $t = 0$.

The agent is facing the problem.

$$\max_x \left\{ (y-x)^{1-\lambda} + \left(E[V(x+w_1, w_1)^{1-\gamma}] \right)^{\frac{1-\lambda}{1-\gamma}} \right\}$$

and the first order condition is

$$E[V^{1-\gamma}]^{-\beta}\, E[V^{-\gamma}\, V'(x)] = c_t^{-\lambda},$$

or, equivalently,

$$g(x) = c_t$$

with

$$g(x) \equiv E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\, E[V^{-\gamma}\, V']^{-\frac{1}{\lambda}},$$

where V' denote the derivative with respect to wealth. The same argument as in the proof of Theorem 3.1 implies that we need to show that $g(x)$ is concave. We have

$$\begin{aligned}
g'(x) &= \frac{\gamma}{\lambda}\, E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\, E[V^{-\gamma}\, V']^{-1-\frac{1}{\lambda}}\, E[V^{-\gamma-1}\,(V')^2] \\
&+ \frac{\lambda-\gamma}{\lambda}\, E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\, E[V^{-\gamma}\, V']^{-\frac{1}{\lambda}+1} \qquad\qquad\text{(A.4)} \\
&- \frac{1}{\lambda}\, E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\, E[V^{-\gamma}\, V']^{-\frac{1}{\lambda}-1}\, E[V^{-\gamma}\, V''],
\end{aligned}$$

and therefore

$$
\begin{aligned}
g''(x) \;=\;& \frac{\gamma\beta\,(1-\gamma)}{\lambda^2}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}}\,E[V^{-\gamma-1}\,(V')^2] \\[4pt]
&+ \frac{\gamma}{\lambda}\left(-\frac{1}{\lambda}-1\right)E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-2}\,E[-\gamma\,V^{-\gamma-1}\,(V')^2 + V^{-\gamma}\,V'']\,E[V^{-\gamma-1}\,(V')^2] \\[4pt]
&+ \frac{\gamma}{\lambda}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-1}\,E[-(\gamma+1)V^{-\gamma-2}\,(V')^3 + 2\,V^{-\gamma-1}\,V'\,V''] \\[4pt]
&+ \frac{\lambda-\gamma}{\lambda}\,\frac{\beta-\lambda}{\lambda}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-2}\,(1-\gamma)\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}+2} \\[4pt]
&+ \frac{\lambda-\gamma}{\lambda}\left(1-\frac{1}{\lambda}\right)E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}E[V^{-\gamma}V']^{-\frac{1}{\lambda}}\,E[-\gamma\,V^{-\gamma-1}\,(V')^2 + V^{-\gamma}\,V''] \\[4pt]
&- \frac{\beta}{\lambda^2}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\,(1-\gamma)\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}}\,E[V^{-\gamma}\,V''] \\[4pt]
&+ \frac{1}{\lambda}\left(\frac{1}{\lambda}+1\right)E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-2}\,E[-\gamma\,V^{-\gamma-1}\,(V')^2 + V^{-\gamma}\,V'']\,E[V^{-\gamma}\,V''] \\[4pt]
&- \frac{1}{\lambda}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-1}\,E[-\gamma\,V^{-\gamma-1}\,V'\,V'' + V^{-\gamma}\,V'''] \\[4pt]
\;=\;& -\frac{1}{\lambda^2}E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-2}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-2} \\[4pt]
&\times \Big[\gamma^2\,(\lambda+1)\,E[V^{1-\gamma}]\left(E[V^{-\gamma}\,V']\,E[V^{-\gamma-2}(V')^3] - E[V^{-\gamma-1}(V')^2]^2\right) + \gamma\,(\lambda-\gamma)\times \\[4pt]
&\Big[\left(E[V^{1-\gamma}]^2\,E[V^{-\gamma}\,V']\,E[V^{-\gamma-2}(V')^3] + E[V^{-\gamma}\,V']^4 - 2\,E[V^{1-\gamma}]\,E[V^{-\gamma}\,V']^2\,E[V^{-\gamma-1}(V')^2]\right) \\[4pt]
&+ \lambda\,E[V^{-\gamma}\,V']^2\left(E[V^{1-\gamma}]\,E[V^{-\gamma-1}(V')^2] - E[V^{-\gamma}\,V']^2\right)\Big]\Big] \\[4pt]
&- \frac{\gamma(1+\lambda)}{\lambda^2}E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\,E[V^{-\gamma}V']^{-\frac{1}{\lambda}-2}\,E[V^{-\gamma}V'']E[V^{-\gamma-1}\,(V')^2] \\[4pt]
&+ 2\frac{\gamma}{\lambda}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}\,E[V^{-\gamma}V']^{-\frac{1}{\lambda}-1}\,E[V^{-\gamma-1}V'\,V''] \\[4pt]
&+ \frac{(\lambda-\gamma)(\lambda-1)}{\lambda^2}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\,E[V^{-\gamma}V']^{-\frac{1}{\lambda}}E[V^{-\gamma}V''] \\[4pt]
&- \frac{\beta(1-\gamma)}{\lambda^2}E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}}\,E[V^{-\gamma}\,V''] \\[4pt]
&- \frac{(1+\lambda)\,\gamma}{\lambda^2}\,E[V^{1-\gamma}]^{\frac{\beta}{\lambda}}E[V^{-\gamma}V']^{-\frac{1}{\lambda}-2}\,E[V^{-\gamma-1}\,(V')^2]\,E[V^{-\gamma}\,V'']
\end{aligned}
\tag{A.5}
$$

$$+ \frac{1+\lambda}{\lambda^2} E[V^{1-\gamma}]^{\frac{\beta}{\lambda}} E[V^{-\gamma} V']^{-\frac{1}{\lambda}-2} E[V^{-\gamma} V'']^2$$

$$+ \frac{\gamma}{\lambda} E[V^{1-\gamma}]^{\frac{\beta}{\lambda}} E[V^{-\gamma} V']^{-\frac{1}{\lambda}-1} E[V^{-\gamma-1} V' V'']$$

$$- \frac{1}{\lambda} E[V^{1-\gamma}]^{\frac{\beta}{\lambda}} E[V^{-\gamma} V']^{-\frac{1}{\lambda}-1} E[V^{-\gamma} V''']$$

$$= K - \frac{\gamma^2 (\lambda+1)}{\lambda^2} E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-2} E[V^{-\gamma} V']^{-\frac{1}{\lambda}-2} E[V^{1-\gamma}] \left(E[V^{-\gamma} V'] E[V^{-\gamma-2}(V')^3] \right.$$

$$- E[V^{-\gamma-1}(V')^2]^2 \big) + \frac{1}{\lambda^2} E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1} E[V^{-\gamma} V']^{-\frac{1}{\lambda}-2}$$

$$\times \left(- 2\gamma (1 + \lambda) E[V^{1-\gamma}] E[V^{-\gamma-1}(V')^2] E[V^{-\gamma} V''] \right.$$

$$+ 3\gamma\lambda E[V^{1-\gamma}] E[V^{-\gamma}V'] E[V^{-\gamma-1}V'V'']$$
$$+ (\lambda - \gamma)(\lambda - 2) E[V^{-\gamma}V']^2 E[V^{-\gamma}V'']$$
$$+ (1 + \lambda) E[V^{1-\gamma}] E[V^{-\gamma} V'']^2$$

$$\left. - \lambda E[V^{1-\gamma}] E[V^{-\gamma}V'] E[V^{-\gamma} V'''] \right),$$

$$(A.6)$$

where

$$K \equiv -\frac{1}{\lambda^2} E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-2} E[V^{-\gamma} V']^{-\frac{1}{\lambda}-2} \gamma (\lambda - \gamma) \times$$
$$\left[\left(E[V^{1-\gamma}]^2 E[V^{-\gamma} V'] E[V^{-\gamma-2}(V')^3] + E[V^{-\gamma} V']^4 \right. \right.$$
$$\left. - 2 E[V^{1-\gamma}] E[V^{-\gamma} V']^2 E[V^{-\gamma-1}(V')^2] \right)$$
$$\left. + \lambda E[V^{-\gamma} V']^2 \left(E[V^{1-\gamma}] E[V^{-\gamma-1}(V')^2] - E[V^{-\gamma} V']^2 \right) \right]$$

$$(A.7)$$

is negative by a few applications of the Cachy-Schwarz, just like in the proof of Theorem 3.1. By direct calculation, (3.4) is equivalent to

$$\lambda V''' \geq (\lambda + 1) \left((V'')^2 (V')^{-1} + \lambda (\lambda - 2) V' V'' V^{-1} \right) \qquad (A.8)$$

Using this inequality, we get

$$
\left(E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\, E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-2}\right)^{-1}\left(g''(x)\;-K\right)
$$

$$
\leq\;-\frac{\gamma^2\,(\lambda+1)}{\lambda^2}\left(E[V^{-\gamma}\,V']\,E[V^{-\gamma-2}(V')^3]\;-\;E[V^{-\gamma-1}(V')^2]^2\,\right)
$$

$$
+\left(-2\frac{\gamma}{\lambda^2}\,(1+\lambda)\,E[V^{1-\gamma}]\,E[V^{-\gamma-1}(V')^2]\,E[V^{-\gamma}\,V'']\right.
$$

$$
+\;3\frac{\gamma}{\lambda}\,E[V^{1-\gamma}]\,E[V^{-\gamma}V']\,E[V^{-\gamma-1}V'V'']
$$

$$
+\;\frac{(\lambda-\gamma)\,(\lambda-2)}{\lambda^2}\,E[V^{-\gamma}V']^2\,E[V^{-\gamma}V'']
$$

$$
+\;\frac{1+\lambda}{\lambda^2}\,E[V^{1-\gamma}]\,E[V^{-\gamma}\,V'']^2
$$

$$
-\;E[V^{1-\gamma}]\,E[V^{-\gamma}V']\,E[V^{-\gamma}\,(\lambda+1)\left((V'')^2\,(V')^{-1}\,+\,\lambda\,(\lambda-2)\,V'\,V''\,V^{-1}\right)]\bigg)
$$

$$
\tag{A.9}
$$

Define the quadratic polynomial

$$
P(\alpha)\;\equiv\;-(\lambda+1)\frac{\gamma^2}{\lambda^2}\,E[V^{1-\gamma}]\left(E[V^{-\gamma}\,V']\,E[V^{-\gamma-2}\,V'\,(V'-\alpha\,V\,(V')^{-1}\,V'')^2]\right.
$$

$$
-\;E[V^{-\gamma-1}\,V'\,(V'-\alpha\,V\,(V')^{-1}\,V'')\,]^2\bigg)
$$

$$
=\;-(\lambda+1)\frac{\gamma^2}{\lambda^2}\,E[V^{1-\gamma}]\left(E[V^{-\gamma}\,V']\,E[V^{-\gamma-2}\,(V')^3]\right.
$$

$$
-\;2\alpha\,E[V^{-\gamma}\,V']\,E[V^{-\gamma-1}\,V'\,V'']\,+\,\alpha^2\,E[V^{-\gamma}\,V']\,E[V^{-\gamma}\,(V')^{-1}(V'')^2]
$$

$$
-\;E[V^{-\gamma-1}(V')^2]^2\,-\,\alpha^2\,E[V^{-\gamma}\,V'']^2\,+\,2\alpha\,E[V^{-\gamma-1}(V')^2]\,E[V^{-\gamma}V'']\bigg)
$$

$$
\tag{A.10}
$$

By the Cauchy-Schwarz inequality, $P(\alpha)\leq 0$. Adding and subtracting the terms on the last two lines of (A.10) to the right-hand side of (A.9), we get that the following inequality holds true for all $\alpha\in\mathbb{R}$:

$$
\begin{aligned}
g(x)\;\leq\;&K+E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-2}\,P(\alpha)\\
&+\;E[V^{1-\gamma}]^{\frac{\beta}{\lambda}-1}\,E[V^{-\gamma}\,V']^{-\frac{1}{\lambda}-2}\,Q(\alpha),
\end{aligned}
\tag{A.11}
$$

where we have defined

$$
\begin{aligned}
Q(\alpha) \; \equiv \; & (1+\lambda)\left(2\alpha\frac{\gamma^2}{\lambda^2} - 2\frac{\gamma}{\lambda^2}\right) E[V^{1-\gamma}]\,E[V^{-\gamma-1}(V')^2]\,E[V^{-\gamma}\,V''] \\
& + \left(3\frac{\gamma}{\lambda} - 1 + \frac{2}{\lambda} - 2(\lambda+1)\frac{\gamma^2}{\lambda^2}\alpha\right) E[V^{1-\gamma}]\,E[V^{-\gamma}V']\,E[V^{-\gamma-1}V'V''] \\
& + \frac{(\lambda-\gamma)\,(\lambda-2)}{\lambda^2}\,E[V^{-\gamma}V']^2\,E[V^{-\gamma}V''] \\
& + (\lambda+1)\left(\frac{1}{\lambda^2} - \frac{\gamma^2}{\lambda^2}\alpha^2\right) E[V^{1-\gamma}]\,E[V^{-\gamma}\,V'']^2 \\
& + (\lambda+1)\left(2\alpha\frac{\gamma^2}{\lambda^2} - \frac{2\gamma}{\lambda^2}\right) E[V^{1-\gamma}]\,E[V^{-\gamma}\,(V')^2]\,E[V^{-\gamma}\,V''] \\
& + (\lambda+1)\left(\frac{\gamma^2}{\lambda^2}\alpha^2 - \frac{1}{\lambda^2}\right) E[V^{1-\gamma}]\,E[V^{-\gamma}\,V']\,E[V^{-\gamma}(V')^{-1}\,(V'')^2]
\end{aligned}
$$

$$(A.12)$$

Since K and $P(\alpha)$ are non-positive, and we have full freedom to choose the parameter α, it remains to show that the minimum of $Q(\alpha)$ over α is negative. By the Cauchy-Schwarz inequality,

$$
E[V^{-\gamma}V']\,E[V^{-\gamma}\,(V')^{-1}\,(V'')^2] \; - \; E[V^{-\gamma}V'']^2 \; \geq \; 0\,,
$$

and hence $Q(\alpha)$ is convex in α and its minimum is attained at

$$
\alpha^* \; = \; \frac{E[V^{-\gamma}V']\,E[V^{-\gamma-1}\,V'\,V''] \; - \; E[V^{-\gamma-1}(V')^2]\,E[V^{-\gamma}V'']}{E[V^{-\gamma}V']\,E[V^{-\gamma}\,(V')^{-1}\,(V'')^2] \; - \; E[V^{-\gamma}V'']^2}\,.
$$

The corresponding value is given by

$$
\begin{aligned}
Q(\alpha^*) \; = \; & \left(E[V^{-\gamma}V']\,E[V^{-\gamma}\,(V')^{-1}\,(V'')^2] \; - \; E[V^{-\gamma}V'']^2\right)^{-1} \\
& \times \Bigg(- \left(E[V^{-\gamma}V']\,E[V^{-\gamma-1}\,V'\,V''] \; - \; E[V^{-\gamma-1}(V')^2]\,E[V^{-\gamma}V'']\right)^2 \\
& \times 2\,(\lambda+1)\frac{\gamma^2}{\lambda^2}\,E[V^{1-\gamma}] \\
& + \Bigg(\left(3\frac{\gamma}{\lambda} - 1 + \frac{2}{\lambda}\right) E[V^{1-\gamma}]\,E[V^{-\gamma}V']\,E[V^{-\gamma-1}V'V''] \\
& + \frac{(\lambda-\gamma)\,(\lambda-2)}{\lambda^2}\,E[V^{-\gamma}V']^2\,E[V^{-\gamma}V'']\Bigg) \\
& \times \left(E[V^{-\gamma}V']\,E[V^{-\gamma}\,(V')^{-1}\,(V'')^2] \; - \; E[V^{-\gamma}V'']^2\right)\Bigg)
\end{aligned}
$$

$$(A.13)$$

By Lemma $V' \geq 0$ and $V'' \leq 0$, and therefore $Q(\alpha^*) < 0$ if

$$
\lambda \; \geq \; 2
$$

and
$$3\frac{\gamma}{\lambda} - 1 + \frac{2}{\lambda} \geq 0 \iff \gamma \geq \frac{\lambda - 2}{3}.$$

Suppose now that $\lambda \leq 2$ and w_t is independent of $\mathcal{F}_{t-1}$. Then, $V(x+w_1, w_1) = V(x + w_1)$, and (A.8) implies that $V''' \geq 0$. Let us show that, under this condition,

$$E[V^{1-\gamma}]\, E[V^{-\gamma}V']\, E[V^{-\gamma-1}V'(-V'')] \; \geq \; E[V^{-\gamma}V']^2\, E[V^{-\gamma}(-V'')]. \quad \text{(A.14)}$$

To this end, define a new measure M with the density $\frac{V^{1-\gamma}}{E[V^{1-\gamma}]}$. Then, denoting the expectation under this measure by $E^M[\cdot]$, we can rewrite the desired inequality (A.14) as

$$E^M[V^{-2}V'(-V'')] \; \geq \; E^M[V^{-1}V']\, E^M[V^{-1}(-V'')]. \quad \text{(A.15)}$$

Since $V'' \leq 0 \leq V'$ and $V''' \geq 0$, we get that the functions V'/V and $(-V'')/V$ are both monotone decreasing. Inequality (A.15) follows therefore from the Chebyshev sum inequality stating that

$$E[f(X)g(X)] \; \geq \; E[f(X)]E[g(X)]$$

for any random variable X and any decreasing functions f, g.[8] Thus,

$$
\begin{aligned}
&\left(3\frac{\gamma}{\lambda} - 1 + \frac{2}{\lambda}\right) E[V^{1-\gamma}]\, E[V^{-\gamma}V']\, E[V^{-\gamma-1}V'(-V'')] \\
&+ \frac{(\lambda - \gamma)(\lambda - 2)}{\lambda^2} E[V^{-\gamma}V']^2\, E[V^{-\gamma}(-V'')] \\
&\geq \left(3\frac{\gamma}{\lambda} - 1 + \frac{2}{\lambda} + \frac{(\lambda - \gamma)(\lambda - 2)}{\lambda^2}\right) E[V^{-\gamma}V']^2\, E[V^{-\gamma}(-V'')] \\
&= \frac{2\gamma(\lambda + 1)}{\lambda^2} E[V^{-\gamma}V']^2\, E[V^{-\gamma}(-V'')] \geq 0,
\end{aligned}
\quad \text{(A.16)}
$$

and therefore $Q(\alpha^*)$ is non-positive. The proof is complete. $\qquad\square$

---

[8]See Hardy, Littlewood, and Pólya (1988).