Admissible Surplus Dynamics and the Government Debt Puzzle

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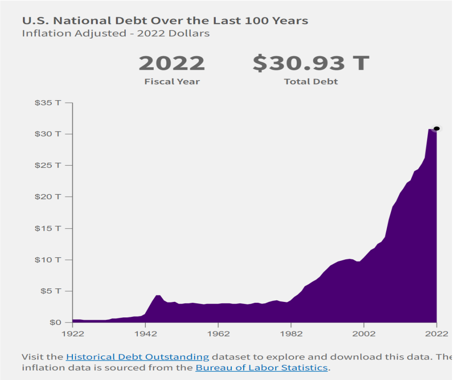
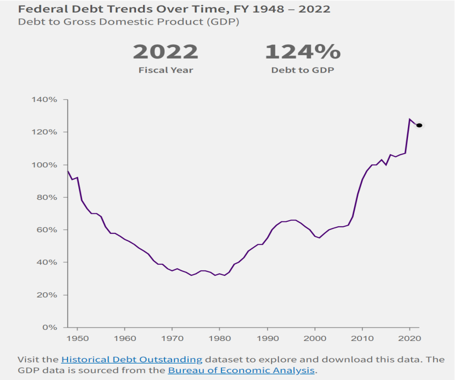
High Debt Level

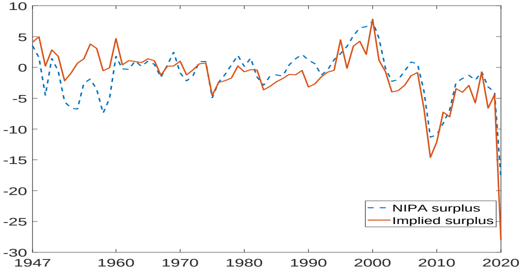
- Succession of crisis (Subprime, Covid, Ukraine) have resulted in extreme levels of Govt debt.





source: US Treasury

Persistent Negative Government Surplus



- Questions sustainability of Gvt debt and fiscal policy.
- Yet, yields on US debt remain low.

→ The **Government Debt Valuation Puzzle**
(Jiang, Lustig, van Nieuwerburgh, and Xiaolan (2022))

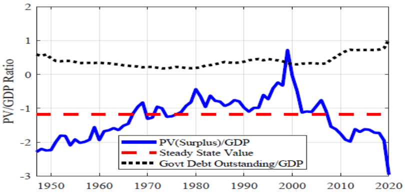
The Government Debt Puzzle

- Assuming (i) no-arbitrage and (ii) no-ponzi-schemes, JLNX obtain **debt valuation equation:**

$$\text{Total Gvt Debt} = \text{present value of future surpluses}$$

- Using affine model for surplus and realistic SDF calibrated to debt and equity they find that US debt should be worth **-129%** of GDP instead of actual **+39%**:

$\rightarrow$ Debt is overvalued by **168% valuation GAP**!



(a) Normalized by Current GDP

- Puzzling since Gvt bonds have non-negative payoffs!

Our Explanation

- We show that no-arbitrage implies Gvt debt should satisfy the debt valuation equation **at all times** and, when markets are incomplete, **for all** admissible SDFs for which it satisfies a transversality condition (**TVC**).
- This puts a high burden on the model specification: for most arbitrary surplus process, accumulated debt will **not satisfy the TVC**, especially if bond returns do not span all surplus shocks.

$\rightarrow$ The valuation gap measures **TVC** $\neq 0$.

- We show how to specify an **admissible** surplus process so that debt can satisfy the TVC.
- Fitting such a process to historical data, we can match surplus and debt dynamics without giving rise to a Debt valuation puzzle.

Related Literature

- O'Connell and Zeldes (1988), Tirole (1985)
- Bohn (1995)
- Hansen, Roberds, Sargent (1991)
- Cochrane (2022)
- Bohn (1998), Campbell, Gao, Martin (2023)
- Blanchard (2019)
- Dumas, Ehling, Yang (2022), Brunnermeier, Merkel, and Sannikov (2022a,b)
- Belo, Collin-Dufresne, Goldstein (2015)

Road map

1. The Debt valuation puzzle
2. Two Examples (with and without a GAP)
3. General Theory
4. Empirical Model
5. Empirical results

The Debt Valuation puzzle

- JLNX use no-arbitrage model with exp-affine SDF M_t :

$$
\begin{aligned}
z_{t+1} &= \Phi z_t + \Sigma^{\frac{1}{2}} \epsilon_{t+1} \\
\frac{M_{t+1}}{M_t} &= e^{-r_t - \frac{1}{2}\Lambda_t^\top \Lambda_t - \Lambda_t \epsilon_{t+1}} \\
\Lambda_t &= \lambda_0 + \lambda_1 z_t \\
r_t &= r_0 + r_1 z_t
\end{aligned}
$$

- Estimate a VAR model for the surplus

$$\mathrm{S}(z_t) = \mathrm{T}(z_t) - \mathrm{G}(z_t)$$

- Infer debt dynamics from the **accounting identity (AI)**:

$$D_{t+1} = D_t R_D(t+1) - \mathrm{S}_{t+1}$$

The Debt Valuation puzzle

- We prove that D_t **satisfies (AI) iff**:

$$D_t = E_t \left[\sum_{n=t+1}^{T} \frac{M_n}{M_t} \mathrm{S}_n \right] + E_t \left[\frac{M_T}{M_t} D_T \right]$$

- Taking the limit as $T \to \infty$ and assuming the **TVC**:

$$\lim_{T \to \infty} E_t \left[\frac{M_T}{M_t} D_T \right] = 0$$

- We obtain the **debt valuation equation**

$$D_t = E_t \left[\sum_{n=t+1}^{\infty} \frac{M_n}{M_t} \mathrm{S}_n \right]$$

Example I: generating a GAP

$$
\begin{aligned}
\mathrm{S}_{t+1} &= e^{-\kappa}\mathrm{S}_t + \sigma\epsilon_{t+1} \\
\frac{M_{t+1}}{M_t} &= e^{-r_0 - \frac{1}{2}\lambda_0^2 - \lambda_0\epsilon_{t+1}} \\
D_{t+1} &= D_t e^{r_0} - \mathrm{S}_{t+1}
\end{aligned}
$$

- We can compute explicitly:

$$
\begin{aligned}
V_t^{\mathrm{S}} &= E_t[\sum_{n=t+1}^{\infty} \frac{M_n}{M_t}\mathrm{S}_n] \\
&= \frac{\mathrm{S}_t}{(e^{r_0+\kappa}-1)} - \frac{\sigma\lambda_0}{(1-e^{-\kappa})}\frac{e^{r_0+\kappa}-e^{r_0}}{(e^{r_0}-1)(e^{r_0+\kappa}-1)}
\end{aligned}
$$

$\rightarrow$ V_t^{S} does not satisfy **(AI)**, that is

$$V_{t+1}^{\mathrm{S}} \neq V_t^{\mathrm{S}} e^{r_0} - \mathrm{S}_{t+1}$$

$\rightarrow$ A **valuation gap** $G_t = D_t - V_t^{\mathrm{S}} \neq 0$ must appear!

Example II: closing the GAP

- Define $L_t = \frac{D_t}{C_t}$ with dynamics:

$$\begin{aligned} \log \frac{C_{t+1}}{C_t} &= \mu_C - \frac{1}{2}\sigma_C^2 + \sigma_C \epsilon_{t+1} \\ L_{t+1} &= (1 - e^{-\kappa})\mu_L + e^{-\kappa} L_t + \sigma_\ell \epsilon_{t+1} \\ \frac{M_{t+1}}{M_t} &= e^{-r_0 - \frac{1}{2}\lambda_0^2 - \lambda_0 \epsilon_{t+1}} \end{aligned}$$

- Infer from AI ($\mathrm{S}_{t+1} = D_t e^{r_0} - D_{t+1}$):

$$\frac{\mathrm{S}_{t+1}}{C_{t+1}} = L_t\, e^{r_0 - \mu_C + \frac{1}{2}\sigma_C^2 - \sigma_C \epsilon_{t+1}} - L_{t+1}$$

- Since (AI) and (M) hold in this economy, we have

$$V^{\mathrm{S}}_{t,T} := \sum_{n=t+1}^{T} E_t[\tfrac{M_n}{M_t}\mathrm{S}_n] = D_t - E_t[\tfrac{M_T}{M_t} D_T]$$

$\Rightarrow$ GAP $G_t = D_t - V^{\mathrm{S}}_{t,\infty} = 0$ iff $\lim_{T\to\infty} E_t[\frac{M_T}{M_t} D_T] = 0$ (TVC)

Example II: closing the GAP

- We compute explicitly $G_{t,T} := D_t - V^{\mathrm{S}}_{t,T} = E_t[\frac{M_T}{M_t} D_T] =$

$$C_t e^{(\boldsymbol{\mu_C - r_0 - \sigma_C \lambda_0})(\mathbf{T - t})} \left\{ \mu_L + e^{-\kappa(T-t)} L_t - \sigma_L(\lambda_0 - \sigma_C) \frac{1-e^{-\kappa(T-t)}}{1-e^{-\kappa}} \right\}$$

- It follows:

$$G_t = \lim_{T\to\infty} E_t[\frac{M_T}{M_t} D_T] = 0 \quad \Longleftrightarrow \quad (\mu_C - r_0 - \sigma_C \lambda_0) < 0$$

- TVC can hold even if $r_0 < \mu_C$ (Bohn (1995)).
- This model nests the classic case where $M_t = C_t^{-\gamma}$, where
  - $r_0 = \rho + \gamma(\mu_C - \frac{1}{2}\sigma_C^2) - \frac{1}{2}\gamma^2\sigma_C^2$
  - $\lambda_0 = \gamma \sigma_C$.

Admissible Surplus Processes: a Theorem

1. *(AI)* is equivalent to *(N)* and implies that

$$\mathbb{E}_\tau\left[M^{\nu}_{\tau,t}D_t + \sum_{s=\tau+1}^{t} M^{\nu}_{\tau,s}T_s\right] = D_\tau + \mathbb{E}_\tau\left[\sum_{s=\tau+1}^{t} M^{\nu}_{\tau,s}G_s\right] \tag{1}$$

   for *all* $\nu \in \mathcal{N}$ and $0 \leq \tau \leq t$.

3. *(AI)* and *(TVC*)* jointly imply that

$$\mathbb{E}_\tau\left[\sum_{s=\tau+1}^{\infty} M^{\mu}_{\tau,s}T_s\right] = D_\tau + \mathbb{E}_\tau\left[\sum_{s=\tau+1}^{\infty} M^{\mu}_{\tau,s}G_s\right] \tag{2}$$

   for all $\tau \geq 0$ and $\mu \in \mathcal{N}$ such that *(TVC*)* holds. In particular, relative to any such process the present value of the spending claim is finite if and only if the present value of the tax claim is finite in which case

$$D_\tau = \mathbb{E}_\tau\left[\sum_{s=\tau+1}^{\infty} M^{\mu}_{\tau,s}\left(T_s - G_s\right)\right] \tag{3}$$

   for all $\tau \geq 0$.

A realistic model of surplus and debt dynamics

- Propose a (more) realistic model for debt and **admissible** surplus dynamics consistent with data.
- Fit a realistic affine pricing kernel to both Treasury bond returns and the value-weighted CRSP market return.
- Questions
  - Does an arbitrage-free model fit bond returns well?
  - Are there sources of risk that are priced in the stock market that are not spanned by bond returns; and Is the government surplus driven by some of these 'unspanned' sources of risk?
  - Does an admissible surplus process, such that (M), (AI), and (TVC) hold, fit the historical surplus series?

A pricing kernel for stocks and bonds

$$\frac{M_{t+1}}{M_t} = \exp\left\{-r_t - \frac{1}{2}\lambda_{\mathrm{p},t}^{\top}\lambda_{\mathrm{p},t} - \lambda_{\mathrm{p},t}^{\top}\epsilon_{\mathrm{p},t+1} - \frac{1}{2}\lambda_{y,t}^{2} - \lambda_{y,t}\epsilon_{y,t+1}\right\}$$

$$r_t = A^1 + B^1\mathrm{p}_t \quad \lambda_{\mathrm{p},t} = \sigma_{\mathrm{p}}^{-1}(\lambda_{\mathrm{p}0} + \lambda_{\mathrm{p}1}\mathrm{p}_t) \quad \lambda_{y,t} = \lambda_{y0} + \lambda_{y1}h_t$$

$$\begin{aligned}
\ln\frac{C_{t+1}}{C_t} &= \mu_C + \mu_{C\mathrm{p}}(\mathrm{p}_t - \bar{\mathrm{p}}) + \sigma_{C\mathrm{p}}\epsilon_{\mathrm{p},t+1} + \sigma_C\epsilon_{C,t+1} \\
\ln\frac{Y_{t+1}}{Y_t} &= \mu_Y + \mu_{Y\mathrm{p}}(\mathrm{p}_t - \bar{\mathrm{p}}) + \sigma_{Y\mathrm{p}}\epsilon_{\mathrm{p},t+1} + \sigma_Y\epsilon_{Y,t+1} \\
\mathrm{p}_{t+1} &= \bar{\mathrm{p}} + \Phi_{\mathrm{p}}(\mathrm{p}_t - \bar{\mathrm{p}}) + \sigma_{\mathrm{p}}\epsilon_{\mathrm{p},t+1} \\
h_{t+1} &= \bar{h} + \Phi_{h\mathrm{p}}(\mathrm{p}_t - \bar{\mathrm{p}}) + \Phi_h(h_t - \bar{h}) + \sigma_{h\mathrm{p}}\epsilon_{\mathrm{p},t+1} + \sigma_{hY}\epsilon_{Y,t+1} + \sigma_h\epsilon_{h,t+1}
\end{aligned}$$

- Use 5 (p) factors for yields and (h) for equity premium .
- Affine pricing solution for ZC yields $ZC_t^m = A^m + B^m\mathrm{p}_t$
- and for Price-dividend ratio $\frac{P_t^Y}{Y_t} \approx e^{\bar{z} + z_{\mathrm{p}}(\mathrm{p}_t - \bar{\mathrm{p}}) + z_h(h_t - \bar{h})}$.
- Identify three distinct priced shock components:
  1. ϵ_{p}: risk-free discount rate shocks,
  2. ϵ_Y: equity cash-flow shocks,
  3. ϵ_h: equity discount rate shocks.

Estimation methodology

1. VAR estimation of p_t to identify $(\bar{\mathrm{p}}, \Phi_{\mathrm{p}}, \Sigma_{\mathrm{p}})$ and yield-curve shocks $\epsilon_{\mathrm{p},t+1}$ with up to 5 PC
   (Duffee (2011), De los Rios (2015), Adrian, Crump, Moench (2013)).
2. Regress $g_Y(t+1)$ onto $\mathrm{p}_t, \epsilon_{\mathrm{p},t+1}$ to estimate $\mu_Y, \mu_{Y\mathrm{p}}, \sigma_{Yp}, \sigma_Y$ and dividend shocks $\epsilon_{Y,t+1}$.
3. Identify h_t and the parameters $\bar{z}, z_{\mathrm{p}}$ from a regression of $\log \frac{P_t^Y}{Y_t}$ onto p_t (with $z_h = 1$ wlog).
4. Identify the parameters $(\bar{h}, \Phi_{h\mathrm{p}}, \Phi_h, \sigma_{hY}, \sigma_h)$ and the equity discount rate shocks $\epsilon_{h,t+1}$ from a regression of h_{t+1} onto $\mathrm{p}_t, h_t, \epsilon_{\mathrm{p},t+1}, \epsilon_{Y,t+1}$.
5. Estimate risk-premia parameters $\lambda_{\mathrm{p}0}, \lambda_{\mathrm{p}1}, \lambda_{Y0}, \lambda_{Y1}$ from the cross-section of ZC yields and the log-stock-price to dividend ratio using the pricing formulas via asymptotic least squares estimator.

   (Gourieroux, Montfort, Trognon (1985), Del Rios (2015)).

Parameter Estimates

The return on government debt

- We estimate

$$\hat{R}_D(t+1) = e^{r_t} + \sum_{m=1}^{M} \hat{\omega}_m \left(e^{r_t + rx_{t+1}^{m-1}} - e^{r_t}\right) \qquad (4)$$

- Where we decompose the weight in each maturity bond into its exposure to the principal component weights:

$$\omega_m = \alpha_1 w_{1,m} + \alpha_2 w_{2,m} + \alpha_3 w_{3,m}$$

- So we estimate $\alpha_1, \alpha_2, \alpha_3$ by OLS.

The admissible surplus process

- We estimate the log debt to consumption ratio $\ell_t := \ln \frac{D_t}{C_t}$ follows an AR2 process:

$$\ell_{t+1} = \phi_1 \ell_t + \phi_2 \ell_{t-1} + (1-\phi_1-\phi_2)\bar{\ell} + \sigma_{\ell,\mathrm{p}}\epsilon_{\mathrm{p},t+1} + \sigma_{\ell,y}\epsilon_{y,t+1} + \sigma_{\ell,h}\epsilon_{h,t+1} + \sigma_{\ell}\epsilon_{\ell,t+1}$$

- Then we infer the surplus from the AI condition:

$$\frac{S_{t+1}}{C_{t+1}} = e^{\ell_t - \ln \frac{C_{t+1}}{C_t}} \hat{R}_D(t+1) - e^{\ell_{t+1}} \tag{5}$$

Data

- Yield curve data from Nasdaq website
  https://data.nasdaq.com/data/FED/SVENY-us-treasury-zerocoupon-yield-curve.

  1-year to 7-year yields from June 1961. Longer yields, up to 20 years, from October 1981.
- Market value of US debt from Dallas FED website
  https://www.dallasfed.org/research/econdata/govdebt.

  Monthly data from January 1942 to October 2022.
- Monthly returns on Gvt debt from 1790 from Hall, Payne, and Sargent.
- Stock value-weighted market portfolio (dividends, prices) from CRSP.
- Surplus from NIPA Table 3.2 from the Bureau of Economic Analysis (BEA).
- Other economic data (GDP, price level, aggregate consumption) from FRED database.

Implied versus NIPA surplus

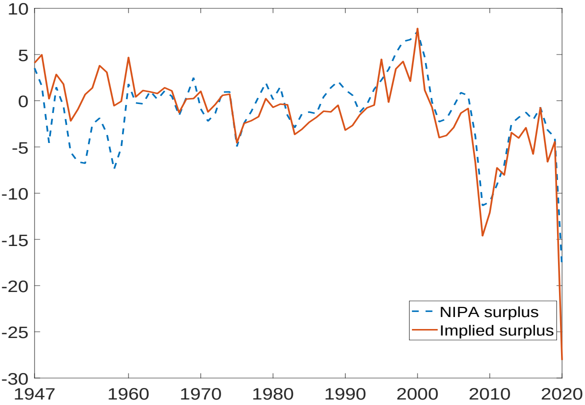


Figure: Comparison between the surplus, in percentage of aggregate consumption, implied from the AI, and the NIPA surplus.

Log Debt to GDP dynamics

- Estimate AR2 for $\ell_t = \ln \frac{D_t}{C_t}$:

$$(\ell_{t+1} - \bar{\ell}) = \phi_1(\ell_t - \bar{\ell}) + \phi_2(\ell_{t-1} - \bar{\ell}) + \epsilon_{\ell,t+1}$$

Frequency	$\bar{\ell}$	SE($\bar{\ell}$)	ϕ_1	SE(ϕ_1)	ϕ_2	SE(ϕ_2)	R^2
Quarterly	-0.158	0.403	1.272	0.056	-0.277	0.056	99.3%
Annual	-0.174	0.410	1.61	0.07	-0.66	0.08	97.1%

- Decompose residuals:

$$\hat{\epsilon}_{\ell,t+1} = \sigma_{\ell,p}\hat{\epsilon}_{p,t+1} + \sigma_{\ell,y}\hat{\epsilon}_{y,t+1} + \sigma_{\ell,h}\hat{\epsilon}_{h,t+1} + \sigma_{\ell}\epsilon_{\ell,t+1}$$

Frequency	$\sigma_{\ell,p}$	SE($\sigma_{\ell,p}$)	$\sigma_{\ell,y}$	SE($\sigma_{\ell,y}$)	$\sigma_{\ell,h}$	SE($\sigma_{\ell,h}$)	R^2
Quarterly	-0.014	0.002	-0.006	0.002	0.004	0.002	26%
Annual	-0.024	0.007	-0.001	0.006	0.008	0.006	29%

Is Debt to GDP stationary?

- The point estimates of the AR2 are consistent with stationarity (but close to the unit circle).
- Therefore we run the OLS regression:

$$\Delta \ell_{t+1} = \alpha + \gamma \ell_t + \delta_1 \Delta \ell_t + \upsilon_{t+1}$$

  and perform a one sided t-test for $\gamma < 0$ using:
  - the Augmented Dickey-Fuller test using the raw data with $\alpha \neq 0$ (ADF)
  - the more efficient Elliott-Rothenberg-Stock (1996) test based on the demeaned data with $\alpha = 0$ (ADF-GLS).

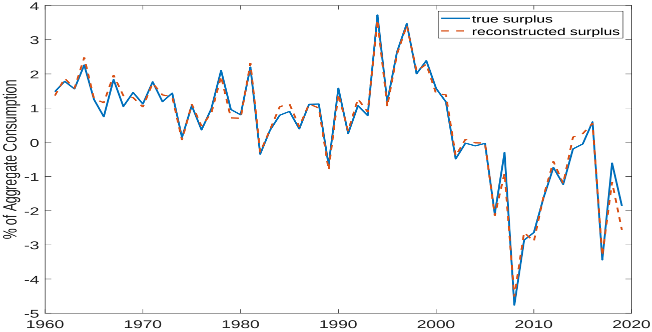
Test	γ	p-value	T-stat	Crit. Value (5%)	Reject $\gamma \geq 0$
ADF	-0.0309	0.288	-2.020	-2.898	No
ADF-GLS	-0.0307	0.042	-2.019	-1.944	Yes

Admissible surplus

$$\frac{\hat{S}_{t+1}}{C_{t+1}} = exp(l_t - g_C(t+1))\hat{R}_D(t+1) - exp(\ell_{t+1})$$

with

$$\hat{R}_D(t+1) = e^{r_t} + \sum_{m=1}^{M} \hat{\omega}_m (e^{r_t + rx^{p,m}_{t+1}} - e^{r_t})$$



Does the valuation equation hold?

- In our model economy, we can show

$$V_{t,T} := \sum_{n=t+1}^{T} E_t[\frac{M_n}{M_t} S_n] = D_t - E_t[\frac{M_T}{M_t} D_T]$$

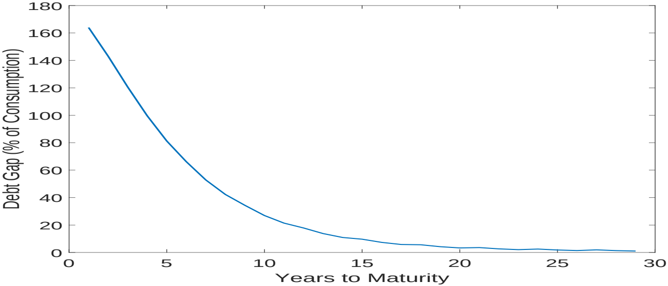
- Thus, we can define the GAP:

$$G_{t,T} := D_t - V_{t,T} = E_t[\frac{M_T}{M_t} D_T]$$

- We obtain an explicit solution:

$$G_{t,T} = C_t\, e^{Q^T_{0,t} + Q^T_{\ell,t}\ell_t + Q^T_{\ell\ell,t}\ell_{t-1} + Q^T_{\mathrm{p},t}\mathrm{p}_t + Q^T_{h,t}h_t}$$

Plot of $G_{t,T}$ as function of T



- $\lim_{T\to\infty} G_{t,T} = 0$ for our estimated parameters.

$\rightarrow$ The TVC holds.

$\rightarrow$ Debt satisifes the valuation equation $D_t = V_{t,\infty}$.

$\rightarrow$ There is no debt valuation puzzle for our estimated surplus and debt dynamics.

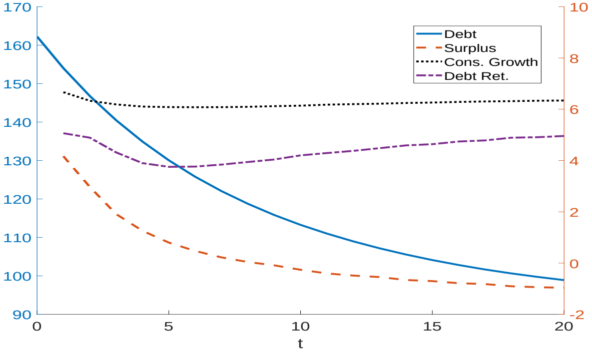
Decomposition implied by the model

- Log-linearization of AI implies (Cochrane (2022)):

$$\ell_t \approx \sum_{n=t+1}^{T} \left\{ \frac{S_n}{C_n} + g_C(n) - \ln R_d(n) \right\} + \ell_T$$

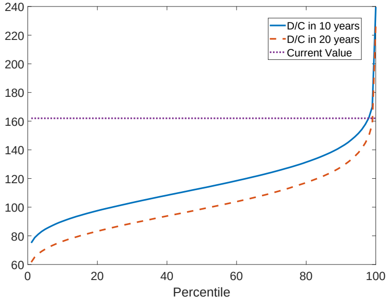
- The current level of debt to GDP can be decomposed into four components related to the future path of:
  - Surplus to consumption,
  - Consumption growth,
  - (Log) Debt returns, and
  - Future debt.
- Simulating 100,000 paths of the model starting from current state, we compute the expected path of each future component.

Decomposition: Expected Trajectory



T	ℓ_0	$\frac{D_0}{C_0}$	$\Sigma_{t=1}^{T}\hat{s}(t)$	$\Sigma_{t=1}^{T}g_C(t)$	$\Sigma_{t=1}^{T}r_D(t)$	ℓ_T	$\frac{D_T}{C_T}$
10 years	0.484	162%	0.161 (0.0007)	0.631 (0.0003)	0.417 (0.0002)	0.124 (0.0006)	115%
20 years	0.484	162%	0.096 (0.001)	1.256 (0.0005)	0.884 (0.0006)	-0.011 (0.0007)	99%

Decomposition: Quartiles



T	Quartile	ℓ_0	$\frac{D_0}{C_0}$	$\Sigma_{t=1}^{T}\hat{s}(t)$	$\Sigma_{t=1}^{T}g_C(t)$	$\Sigma_{t=1}^{T}r_D(t)$	ℓ_T	$\frac{D_T}{C_T}$
10 years	1	0.484	162%	0.392	0.697	0.462	-0.101	90%
10 years	4	0.484	162%	-0.105	0.566	0.372	0.348	109%
20 years	1	0.484	162%	0.389	1.366	1.017	-0.271	76%
20 years	4	0.484	162%	-0.242	1.148	0.753	0.245	128%

Conclusion

Supplementary Material

No-arb and AI $\Rightarrow$ Debt valuation

- (AI) and (M) imply:

$$\sum_{n=t+1}^{T} E_t[\frac{M_n}{M_t}\,\mathrm{S}_n] = \sum_{n=t+1}^{T} E_t[\frac{M_n}{M_t}(D_{n-1}R_D(n) - D_n)]$$

$$= \sum_{n=t+1}^{T} \left\{ E_t\Big[\frac{M_{n-1}}{M_t} D_{n-1}\, \underbrace{E_{n-1}[\frac{M_n}{M_{n-1}} R_D(n)]}_{=\mathbf{1}}\Big] - E_t[\frac{M_n}{M_t} D_n] \right\}$$

$$= \sum_{n=t+1}^{T} \left\{ E_t[\frac{M_{n-1}}{M_t} D_{n-1}] - E_t[\frac{M_n}{M_t} D_n] \right\}$$

$$= D_t - E_t\left[\frac{M_T}{M_t} D_T\right]$$

- This holds for any valid SDF!

Back

Derivation for example 2: how to close the gap

$$
\begin{aligned}
V^{\mathrm{S}}_{t,N} &= \sum_{n=1}^{N} E_t[\frac{M_{t+n}}{M_t}\mathrm{S}_{t+n}] && (6)\\
&= \sum_{n=1}^{N} E_t[\frac{M_{t+n}}{M_t}(D_{t+n-1}e^{r_0} - D_{t+n})] && (7)\\
&= \sum_{n=1}^{N} E_t[E_{t+n-1}[\frac{M_{t+n}}{M_t}]D_{t+n-1}e^{r_0}] - \sum_{n=1}^{N} E_t[\frac{M_{t+n}}{M_t}D_{t+n}] && (8)\\
&= \sum_{n=1}^{N} E_t[\frac{M_{t+n-1}}{M_t}D_{t+n-1}] - \sum_{n=1}^{N} E_t[\frac{M_{t+n}}{M_t}D_{t+n}] && (9)\\
&= D_t - E_t[\frac{M_{t+N}}{M_t}D_{t+N}] && (10)
\end{aligned}
$$

Back

Log-linearization of the (AI)

- Use the AI to write:

$$\ln(\mathcal{S}_{t+1} + e^{\ell_{t+1}}) = \ell_t - g_C(_{t+1}) + r_D(_{t+1})$$

where $\mathcal{S}_t = \mathrm{S}_t / C_t$, $\ell_t = \ln D_t / C_t$, $r_D(t) = \ln R_D(t)$.

- Taylor expand LHS around $\mathcal{S}_t = \check{\mathrm{S}}$ and $\ell_t = \check{\ell}$ to get:

$$\begin{aligned}
\rho_0 + \rho_\ell \ell_{t+1} + \rho_s \mathcal{S}_{t+1} &= \ell_t - g_C(_{t+1}) + r_D(_{t+1}) \\
\rho_0 &= \ln(\check{\mathrm{S}} + e^{\check{\ell}}) - \rho_\ell \check{\ell} - \rho_s \check{\mathrm{S}} \\
\rho_\ell &= \frac{e^{\check{\ell}}}{\check{\mathrm{S}} + e^{\check{\ell}}} \\
\rho_s &= \frac{1}{\check{\mathrm{S}} + e^{\check{\ell}}}
\end{aligned}$$

- Iterate forward :

$$\ell_t = \tfrac{\rho_0}{1-\rho_\ell} + \sum_{n=t+1}^{T} \rho_\ell^{n-t-1} \{\rho_s \mathcal{S}_n + g_C(n) - r_d(n)\} + \rho_\ell^{T-t} \ell_T$$

- Pick $\check{\mathrm{S}} = \check{\ell} = 0$ to get the Cochrane decomposition.

Parameter Estimates

Process	Parameter	Estimated Value	Standard Error
p	$\bar{p}$	(0.240, -0.036, 0.006, 0.001, 0.0001)	(0.003, 0.0004, 0.0001, 0.000, 0.000)
	Φ	$\begin{pmatrix} 0.854 & -0.015 & 0.001 & 0.002 & 0.001 \\ -0.120 & 0.677 & 0.005 & -0.016 & 0.007 \\ 0.872 & -0.803 & 0.113 & -0.070 & -0.033 \\ 6.960 & 1.920 & 0.239 & 0.409 & -0.047 \\ -0.158 & -0.506 & -2.456 & 0.014 & -0.005 \end{pmatrix}$	$\begin{pmatrix} 0.003 & 0.000 & 0.005 & -0.023 & -0.0112 \\ 0.000 & 0.203 & -0.162 & 0.143 & -0.534 \\ 0.005 & -0.162 & 4.331 & -0.542 & 0.807 \\ -0.023 & 0.143 & -0.542 & 16.9 & -2.00 \\ -0.012 & -0.534 & 0.807 & -2.00 & 112 \end{pmatrix}$
	σ_p	$10^{-5}\begin{pmatrix} 145.591 & 0.790 & -3.929 & -2.059 & -0.537 \\ 0.790 & 11.216 & 0.540 & -0.003 & 0.006 \\ -3.929 & 0.540 & 0.729 & 0.001 & 0.005 \\ -2.059 & -0.003 & 0.001 & 0.174 & 0.005 \\ -0.537 & 0.006 & 0.005 & 0.005 & 0.031 \end{pmatrix}$	$10^{-5}\begin{pmatrix} 32.555 & 6.391 & 1.744 & 0.860 & 0.348 \\ 6.391 & 2.508 & 0.460 & 0.221 & 0.094 \\ 1.744 & 0.460 & 0.163 & 0.056 & 0.024 \\ 0.860 & 0.221 & 0.056 & 0.039 & 0.012 \\ 0.348 & 0.094 & 0.024 & 0.012 & 0.007 \end{pmatrix}$
g_Y	μ_Y	0.087	0.015
	μ_{Yp}	(0.146, -0.584,0.302,18.61,-41.00)	(0.143,0.662,4.571,10.10,26.18)
	σ_{Yp}	(-0.009,0.013,-0.010,0.063,0.031)	(0.015,0.015,,0.015,0.016,0.016)
	σ_Y	0.107	0.002
g_C	μ_C	0.065	0.003
	μ_{Cp}	(0.089,0.524,-1.220,4.027,-0.285)	(0.025,0.117,0.804,1.784,4.605)
	σ_{Cp}	(0.007,0.005,-0.000,0.000,-0.001)	(0.003,0.003,0.003,0.003,0.003)
	σ_C	0.019	$6 * 10-5$
z	$\bar{z}$	-2.533	0.010
	z_p	(-0.087,-0.069,-3.185,-9.523,4.714)	(0.095, 0.439,3.031,6.853,17.896)
h	$\bar{h}$	0	0
	Φ_{hp}	(0.009,-0.019,0.753,1.152,12.20)	(0.075, 0.347,2.394,5.311,13.71)
	Φ_h	-0.300	0.1162
	σ_{hp}	(-0.001,-0.012, 0.002,0.003,-0.004)	(0.008,0.008,0.008,0.008,0.008)
	σ_{hY}	-0.053	0.008
	σ_h	0.055	$5.6 * 10^{-4}$
M	λ_{0p}	(0.0140,-0.001, 0, 0,0)	(0.0001,0.0004,0,0,0)
	λ_{1p}	$\begin{pmatrix} -0.055 & 1.140 & 0 & 0 & 0 \\ -0.018 & -0.137 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.0002 & 0.022 & 0 & 0 & 0 \\ 0.000 & 0.0008 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$