Admissible Surplus Dynamics and the Government Debt Puzzle

Pierre Collin-Dufresne (SFI & EPFL)
Julien Hugonnier (EPFL)
Elena Perazzi (EPFL)

May 2023
High Debt Level

- Succession of crisis (Subprime, Covid, Ukraine) have resulted in extreme levels of Govt debt.

source: US Treasury
Questions sustainability of Gvt debt and fiscal policy.

Yet, yields on US debt remain low.

The **Government Debt Valuation Puzzle**
(Jiang, Lustig, van Nieuwerburgh, and Xiaolan (2022))
The Government Debt Puzzle

- Assuming (i) no-arbitrage and (ii) no-ponzi-schemes, JLNX obtain **debt valuation equation**:
  
  \[
  \text{Total Gvt Debt} = \text{present value of future surpluses}
  \]

- Using affine model for surplus and realistic SDF calibrated to debt and equity they find that US debt should be worth **-129%** of GDP instead of actual **+39%**: Debt is overvalued by **168% valuation GAP**!

- Puzzling since Gvt bonds have non-negative payoffs!
Our Explanation

- We show that no-arbitrage implies Gvt debt should satisfy the debt valuation equation at all times and, when markets are incomplete, for all admissible SDFs for which it satisfies a transversality condition (\textit{TVC}).

- This puts a high burden on the model specification: for most arbitrary surplus process, accumulated debt will not satisfy the TVC, especially if bond returns do not span all surplus shocks.

→ The valuation gap measures $\text{TVC} \neq 0$.

- We show how to specify an admissible surplus process so that debt can satisfy the TVC.

- Fitting such a process to historical data, we can match surplus and debt dynamics without giving rise to a Debt valuation puzzle.
Related Literature

- Bohn (1995)
- Hansen, Roberds, Sargent (1991)
- Cochrane (2022)
- Bohn (1998), Campbell, Gao, Martin (2023)
- Blanchard (2019)
- Dumas, Ehling, Yang (2022), Brunnermeier, Merkel, and Sannikov (2022a,b)
- Belo, Collin-Dufresne, Goldstein (2015)
Road map

1. The Debt valuation puzzle
2. Two Examples (with and without a GAP)
3. General Theory
4. Empirical Model
5. Empirical results
The Debt Valuation puzzle

- JLNX use no-arbitrage model with exp-affine SDF $M_t$:

$$
\begin{align*}
z_{t+1} &= \Phi z_t + \Sigma^{\frac{1}{2}} \epsilon_{t+1} \\
\frac{M_{t+1}}{M_t} &= e^{-r_t - \frac{1}{2} \Lambda_t^\top \Lambda_t - \Lambda_t \epsilon_{t+1}} \\
\Lambda_t &= \lambda_0 + \lambda_1 z_t \\
r_t &= r_0 + r_1 z_t
\end{align*}
$$

- Estimate a VAR model for the surplus

$$
S(z_t) = T(z_t) - G(z_t)
$$

- Infer debt dynamics from the accounting identity (AI):

$$
D_{t+1} = D_t R_D(t + 1) - S_{t+1}
$$
We prove that $D_t$ satisfies (AI) iff:

$$D_t = E_t \left[ \sum_{n=t+1}^{T} \frac{M_n}{M_t} S_n \right] + E_t \left[ \frac{M_T}{M_t} D_T \right]$$

Taking the limit as $T \to \infty$ and assuming the TVC:

$$\lim_{T \to \infty} E_t \left[ \frac{M_T}{M_t} D_T \right] = 0$$

We obtain the debt valuation equation

$$D_t = E_t \left[ \sum_{n=t+1}^{\infty} \frac{M_n}{M_t} S_n \right]$$
Example I: generating a GAP

\[
\begin{align*}
S_{t+1} &= e^{-\kappa}S_t + \sigma \epsilon_{t+1} \\
\frac{M_{t+1}}{M_t} &= e^{-r_0 - \frac{1}{2} \lambda_0^2 - \lambda_0 \epsilon_{t+1}} \\
D_{t+1} &= D_t e^{r_0} - S_{t+1}
\end{align*}
\]

We can compute explicitly:

\[
V_t^S = E_t \left[ \sum_{n=t+1}^{\infty} \frac{M_n}{M_t} S_n \right] = \frac{S_t}{(e^{r_0 + \kappa} - 1)} - \frac{\sigma \lambda_0}{(1 - e^{-\kappa})} \frac{e^{r_0 + \kappa} - e^{r_0}}{(e^{r_0} - 1)(e^{r_0 + \kappa} - 1)}
\]

\[\rightarrow V_t^S \text{ does not satisfy (AI), that is}
\]

\[V_{t+1}^S \neq V_t^S e^{r_0} - S_{t+1}\]

\[\rightarrow \text{A valuation gap } G_t = D_t - V_t^S \neq 0 \text{ must appear!}\]
Example II: closing the GAP

Define $L_t = \frac{D_t}{C_t}$ with dynamics:

\[
\begin{align*}
\log \frac{C_{t+1}}{C_t} &= \mu_C - \frac{1}{2}\sigma^2_C + \sigma_C \epsilon_{t+1} \\
L_{t+1} &= (1 - e^{-\kappa})\mu_L + e^{-\kappa}L_t + \sigma_L \epsilon_{t+1} \\
M_{t+1} &= e^{-r_0 - \frac{1}{2}\lambda^2_0 - \lambda_0 \epsilon_{t+1}}
\end{align*}
\]

Infer from AI ($S_{t+1} = D_t e^{r_0} - D_{t+1}$):

\[
\begin{align*}
\frac{S_{t+1}}{C_{t+1}} &= L_t e^{r_0 - \mu_C + \frac{1}{2}\sigma^2_C - \sigma_C \epsilon_{t+1}} - L_{t+1}
\end{align*}
\]

Since (AI) and (M) hold in this economy, we have

\[
V^{S}_{t,T} := \sum_{n=t+1}^{T} E_t \left[ \frac{M_n}{M_t} S_n \right] = D_t - E_t \left[ \frac{M_T}{M_t} D_T \right]
\]

$\Rightarrow$ GAP $G_t = D_t - V^{S}_{t,\infty} = 0$ iff $\lim_{T \to \infty} E_t \left[ \frac{M_T}{M_t} D_T \right] = 0$ (TVC)
Example II: closing the GAP

We compute explicitly $G_{t, T} := D_t - V_{t, T} = E_t \left[ \frac{M_T}{M_t} D_T \right] = C_t e^{(\mu_C - r_0 - \sigma_C \lambda_0)(T-t)} \left\{ \mu_L + e^{-\kappa(T-t)} L_t - \sigma_L (\lambda_0 - \sigma_C) \frac{1-e^{-\kappa(T-t)}}{1-e^{-\kappa}} \right\}$

It follows:

$G_t = \lim_{T \to \infty} E_t \left[ \frac{M_T}{M_t} D_T \right] = 0 \iff (\mu_C - r_0 - \sigma_C \lambda_0) < 0$

TVC can hold even if $r_0 < \mu_C$ (Bohn (1995)).

This model nests the classic case where $M_t = C_t^{-\gamma}$, where

$r_0 = \rho + \gamma (\mu_C - \frac{1}{2} \sigma_C^2) - \frac{1}{2} \gamma^2 \sigma_C^2$

$\lambda_0 = \gamma \sigma_C$. 
Admissible Surplus Processes: a Theorem

1. (AI) is equivalent to (N) and implies that

\[
E_T \left[ M_{\tau,t}^{\nu} D_t + \sum_{s=\tau+1}^{t} M_{\tau,s}^{\nu} T_s \right] = D_\tau + E_T \left[ \sum_{s=\tau+1}^{t} M_{\tau,s}^{\nu} G_s \right]
\]

(1)

for all \( \nu \in \mathcal{N} \) and \( 0 \leq \tau \leq t \).

3. (AI) and \((TVC^*)\) jointly imply that

\[
E_T \left[ \sum_{s=\tau+1}^{\infty} M_{\tau,s}^{\mu} T_s \right] = D_\tau + E_T \left[ \sum_{s=\tau+1}^{\infty} M_{\tau,s}^{\mu} G_s \right]
\]

(2)

for all \( \tau \geq 0 \) and \( \mu \in \mathcal{N} \) such that \((TVC^*)\) holds. In particular, relative to any such process the present value of the spending claim is finite if and only if the present value of the tax claim is finite in which case

\[
D_\tau = E_T \left[ \sum_{s=\tau+1}^{\infty} M_{\tau,s}^{\mu} (T_s - G_s) \right]
\]

(3)

for all \( \tau \geq 0 \).
A realistic model of surplus and debt dynamics

- Propose a (more) realistic model for debt and admissible surplus dynamics consistent with data.
- Fit a realistic affine pricing kernel to both Treasury bond returns and the value-weighted CRSP market return.

Questions
- Does an arbitrage-free model fit bond returns well?
- Are there sources of risk that are priced in the stock market that are not spanned by bond returns; and is the government surplus driven by some of these ‘unspanned’ sources of risk?
- Does an admissible surplus process, such that (M), (AI), and (TVC) hold, fit the historical surplus series?
A pricing kernel for stocks and bonds

\[
\frac{M_{t+1}}{M_t} = \exp \left\{ -r_t - \frac{1}{2} \lambda_{p,t}^\top \lambda_{p,t} - \lambda_{p,t}^\top \epsilon_{p,t+1} - \frac{1}{2} \lambda_{y,t}^2 - \lambda_{y,t} \epsilon_{y,t+1} \right\}
\]

\[r_t = A^1 + B^1 p_t \quad \lambda_{p,t} = \sigma_p^{-1} (\lambda_{p0} + \lambda_{p1} p_t) \quad \lambda_{y,t} = \lambda_{y0} + \lambda_{y1} h_t\]

\[
\ln \frac{C_{t+1}}{C_t} = \mu_C + \mu_{CP} (p_t - \bar{p}) + \sigma_{CP} \epsilon_{p,t+1} + \sigma_C \epsilon_C,t+1
\]

\[
\ln \frac{Y_{t+1}}{Y_t} = \mu_Y + \mu_{YP} (p_t - \bar{p}) + \sigma_{YP} \epsilon_{p,t+1} + \sigma_Y \epsilon_Y,t+1
\]

\[
p_{t+1} = \bar{p} + \Phi_p (p_t - \bar{p}) + \sigma_p \epsilon_{p,t+1}
\]

\[
h_{t+1} = \bar{h} + \Phi_{hp} (p_t - \bar{p}) + \Phi_h (h_t - \bar{h}) + \sigma_{hp} \epsilon_{p,t+1} + \sigma_{hY} \epsilon_{Y,t+1} + \sigma_{h\epsilon_h,t+1}
\]

- Use 5 \((p)\) factors for yields and \((h)\) for equity premium.
- Affine pricing solution for ZC yields \(ZC_t^m = A^m + B^m p_t\)
- and for Price-dividend ratio \(\frac{P^Y_{t}}{Y_t} \approx e^{\bar{Z} + z_p (p_t - \bar{p}) + z_h (h_t - \bar{h})}\).
- Identify three distinct priced shock components:
  1. \(\epsilon_p\): risk-free discount rate shocks,
  2. \(\epsilon_Y\): equity cash-flow shocks,
  3. \(\epsilon_h\): equity discount rate shocks.
Estimation methodology

1. VAR estimation of $p_t$ to identify $(\bar{p}, \Phi_p, \Sigma_p)$ and yield-curve shocks $\epsilon_{p,t+1}$ with up to 5 PC (Duffee (2011), De los Rios (2015), Adrian, Crump, Moench (2013)).

2. Regress $g_Y(t + 1)$ onto $p_t, \epsilon_{p,t+1}$ to estimate $\mu_Y, \mu_{Y_p}, \sigma_{Y_p}, \sigma_Y$ and dividend shocks $\epsilon_{Y,t+1}$.

3. Identify $h_t$ and the parameters $\bar{z}, z_p$ from a regression of $\log \frac{P_t^Y}{Y_t}$ onto $p_t$ (with $z_h = 1$ wlog).

4. Identify the parameters $(\bar{h}, \Phi_{hp}, \Phi_h, \sigma_{hY}, \sigma_h)$ and the equity discount rate shocks $\epsilon_{h,t+1}$ from a regression of $h_{t+1}$ onto $p_t, h_t, \epsilon_{p,t+1}, \epsilon_{Y,t+1}$.

5. Estimate risk-premia parameters $\lambda_{p0}, \lambda_{p1}, \lambda_{Y0}, \lambda_{Y1}$ from the cross-section of ZC yields and the log-stock-price to dividend ratio using the pricing formulas via asymptotic least squares estimator. (Gourieroux, Montfort, Trognon (1985), Del Rios (2015)).
The return on government debt

We estimate

\[
\hat{R}_D(t+1) = e^{rt} + \sum_{m=1}^{M} \hat{\omega}_m (e^{rt+\hat{r}_x_{t+1}^m} - e^{rt})
\]  \hspace{1cm} (4)

Where we decompose the weight in each maturity bond into its exposure to the principal component weights:

\[
\omega_m = \alpha_1 w_{1,m} + \alpha_2 w_{2,m} + \alpha_3 w_{3,m}
\]

So we estimate \( \alpha_1, \alpha_2, \alpha_3 \) by OLS.
The admissible surplus process

We estimate the log debt to consumption ratio $\ell_t := \ln \frac{D_t}{C_t}$ follows an AR2 process:

$$\ell_{t+1} = \phi_1 \ell_t + \phi_2 \ell_{t-1} + (1 - \phi_1 - \phi_2) \bar{\ell}$$

$$+ \sigma_{\ell,p} \epsilon_{p,t+1} + \sigma_{\ell,y} \epsilon_{y,t+1} + \sigma_{\ell,h} \epsilon_{h,t+1} + \sigma_{\ell} \epsilon_{\ell,t+1}$$

Then we infer the surplus from the AI condition:

$$\frac{S_{t+1}}{C_{t+1}} = e^{\ell_t - \ln \frac{c_{t+1}}{c_t}} \hat{R}_D(t + 1) - e^{\ell_{t+1}} \quad (5)$$
Data

- Yield curve data from Nasdaq website
  1-year to 7-year yields from June 1961. Longer yields, up to 20 years, from October 1981.

- Market value of US debt from Dallas FED website
  Monthly data from January 1942 to October 2022.

- Monthly returns on Gvt debt from 1790 from Hall, Payne, and Sargent.

- Stock value-weighted market portfolio (dividends, prices) from CRSP.

- Surplus from NIPA Table 3.2 from the Bureau of Economic Analysis (BEA).

- Other economic data (GDP, price level, aggregate consumption) from FRED database.
Implied versus NIPA surplus

Figure: Comparison between the surplus, in percentage of aggregate consumption, implied from the AI, and the NIPA surplus.
Log Debt to GDP dynamics

Estimate AR2 for $\ell_t = \ln \frac{D_t}{C_t}$:

$$(\ell_{t+1} - \bar{\ell}) = \phi_1(\ell_t - \bar{\ell}) + \phi_2(\ell_{t-1} - \bar{\ell}) + \epsilon_{\ell,t+1}$$

<table>
<thead>
<tr>
<th>Frequency</th>
<th>$\bar{\ell}$</th>
<th>SE($\bar{\ell}$)</th>
<th>$\phi_1$</th>
<th>SE($\phi_1$)</th>
<th>$\phi_2$</th>
<th>SE($\phi_2$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarterly</td>
<td>-0.158</td>
<td>0.403</td>
<td>1.272</td>
<td>0.056</td>
<td>-0.277</td>
<td>0.056</td>
<td>99.3%</td>
</tr>
<tr>
<td>Annual</td>
<td>-0.174</td>
<td>0.410</td>
<td>1.61</td>
<td>0.07</td>
<td>-0.66</td>
<td>0.08</td>
<td>97.1%</td>
</tr>
</tbody>
</table>

Decompose residuals:

$$\hat{\epsilon}_{\ell,t+1} = \sigma_{\ell,p}\hat{\epsilon}_{p,t+1} + \sigma_{\ell,y}\hat{\epsilon}_{y,t+1} + \sigma_{\ell,h}\hat{\epsilon}_{h,t+1} + \sigma_{\ell}\epsilon_{\ell,t+1}$$

<table>
<thead>
<tr>
<th>Frequency</th>
<th>$\sigma_{\ell,p}$</th>
<th>SE($\sigma_{\ell,p}$)</th>
<th>$\sigma_{\ell,y}$</th>
<th>SE($\sigma_{\ell,y}$)</th>
<th>$\sigma_{\ell,h}$</th>
<th>SE($\sigma_{\ell,h}$)</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarterly</td>
<td>-0.014</td>
<td>0.002</td>
<td>-0.006</td>
<td>0.002</td>
<td>0.004</td>
<td>0.002</td>
<td>26%</td>
</tr>
<tr>
<td>Annual</td>
<td>-0.024</td>
<td>0.007</td>
<td>-0.001</td>
<td>0.006</td>
<td>0.008</td>
<td>0.006</td>
<td>29%</td>
</tr>
</tbody>
</table>
Is Debt to GDP stationary?

▶ The point estimates of the AR2 are consistent with stationarity (but close to the unit circle).

▶ Therefore we run the OLS regression:

\[ \Delta l_{t+1} = \alpha + \gamma l_t + \delta_1 \Delta l_t + \nu_{t+1} \]

and perform a one sided t-test for \( \gamma < 0 \) using:

▶ the Augmented Dickey-Fuller test using the raw data with \( \alpha \neq 0 \) (ADF)

▶ the more efficient Elliott-Rothenberg-Stock (1996) test based on the demeaned data with \( \alpha = 0 \) (ADF-GLS).

<table>
<thead>
<tr>
<th>Test</th>
<th>( \gamma )</th>
<th>p-value</th>
<th>T-stat</th>
<th>Crit. Value (5%)</th>
<th>Reject ( \gamma \geq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADF</td>
<td>-0.0309</td>
<td>0.288</td>
<td>-2.020</td>
<td>-2.898</td>
<td>No</td>
</tr>
<tr>
<td>ADF-GLS</td>
<td>-0.0307</td>
<td>0.042</td>
<td>-2.019</td>
<td>-1.944</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Admissible surplus

\[
\frac{\hat{S}_{t+1}}{C_{t+1}} = \exp(l_t - gC(t + 1))\hat{R}_D(t + 1) - \exp(\ell_{t+1})
\]

with

\[
\hat{R}_D(t + 1) = e^{rt} + \sum_{m=1}^{M} \hat{\omega}_m (e^{r_{t+1}x_{t+1}^{p,m}} - e^{rt})
\]
Does the valuation equation hold?

In our model economy, we can show

\[ V_{t,T} := \sum_{n=t+1}^{T} E_t \left[ \frac{M_n}{M_t} S_n \right] \]

\[ = D_t - E_t \left[ \frac{M_T}{M_t} D_T \right] \]

Thus, we can define the GAP:

\[ G_{t,T} := D_t - V_{t,T} = E_t \left[ \frac{M_T}{M_t} D_T \right] \]

We obtain an explicit solution:

\[ G_{t,T} = C_t e^{Q_{0,t}^T + Q_{l,t}^T \ell_t + Q_{ll,t}^T \ell_{t-1} + Q_{p,t}^T p_t + Q_{h,t}^T h_t} \]
Plot of $G_{t,T}$ as function of $T$

$\lim_{T \to \infty} G_{t,T} = 0$ for our estimated parameters.

The TVC holds.

Debt satisfies the valuation equation $D_t = V_{t,\infty}$.

There is no debt valuation puzzle for our estimated surplus and debt dynamics.
Decomposition implied by the model

Log-linearization of AI implies (Cochrane (2022)):

\[
\ell_t \approx \sum_{n=t+1}^{T} \left\{ \frac{S_n}{C_n} + g_C(n) - \ln R_d(n) \right\} + \ell_T
\]

The current level of debt to GDP can be decomposed into four components related to the future path of:

- Surplus to consumption,
- Consumption growth,
- (Log) Debt returns, and
- Future debt.

Simulating 100,000 paths of the model starting from current state, we compute the expected path of each future component.
Decomposition: Expected Trajectory

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\ell_0$</th>
<th>$\frac{D_0}{C_0}$</th>
<th>$\sum_{t=1}^{T} \hat{s}(t)$</th>
<th>$\sum_{t=1}^{T} g_C(t)$</th>
<th>$\sum_{t=1}^{T} r_D(t)$</th>
<th>$\ell_T$</th>
<th>$\frac{D_T}{C_T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 years</td>
<td>0.484</td>
<td>162%</td>
<td>0.161 (0.0007)</td>
<td>0.631 (0.0003)</td>
<td>0.417 (0.0002)</td>
<td>0.124 (0.0006)</td>
<td>115%</td>
</tr>
<tr>
<td>20 years</td>
<td>0.484</td>
<td>162%</td>
<td>0.096 (0.001)</td>
<td>1.256 (0.0005)</td>
<td>0.884 (0.0006)</td>
<td>-0.011 (0.0007)</td>
<td>99%</td>
</tr>
</tbody>
</table>
Decomposition: Quartiles

<table>
<thead>
<tr>
<th>T</th>
<th>Quartile</th>
<th>$\ell_0$</th>
<th>$\frac{D_0}{C_0}$</th>
<th>$\sum_{t=1}^T \hat{s}(t)$</th>
<th>$\sum_{t=1}^T g_C(t)$</th>
<th>$\sum_{t=1}^T r_D(t)$</th>
<th>$\ell_T$</th>
<th>$\frac{D_T}{C_T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 years</td>
<td>1</td>
<td>0.484</td>
<td>162%</td>
<td>0.392</td>
<td>0.697</td>
<td>0.462</td>
<td>-0.101</td>
<td>90%</td>
</tr>
<tr>
<td>10 years</td>
<td>4</td>
<td>0.484</td>
<td>162%</td>
<td>-0.105</td>
<td>0.566</td>
<td>0.372</td>
<td>0.348</td>
<td>109%</td>
</tr>
<tr>
<td>20 years</td>
<td>1</td>
<td>0.484</td>
<td>162%</td>
<td>0.389</td>
<td>1.366</td>
<td>1.017</td>
<td>-0.271</td>
<td>76%</td>
</tr>
<tr>
<td>20 years</td>
<td>4</td>
<td>0.484</td>
<td>162%</td>
<td>-0.242</td>
<td>1.148</td>
<td>0.753</td>
<td>0.245</td>
<td>128%</td>
</tr>
</tbody>
</table>
Conclusion

“That’s all folks!”
No-arb and AI ⇒ Debt valuation

- (AI) and (M) imply:

\[
\sum_{n=t+1}^{T} E_t \left[ \frac{M_n}{M_t} S_n \right] = \sum_{n=t+1}^{T} E_t \left[ \frac{M_n}{M_t} (D_{n-1} R_D(n) - D_n) \right]
\]

\[
= \sum_{n=t+1}^{T} \left\{ E_t \left[ \frac{M_{n-1}}{M_t} D_{n-1} E_{n-1} \left[ \frac{M_n}{M_{n-1}} R_D(n) \right] \right] - E_t \left[ \frac{M_n}{M_t} D_n \right] \right\}
\]

\[
= D_t - E_t \left[ \frac{M_T}{M_t} D_T \right]
\]

- This holds for any valid SDF!
Derivation for example 2: how to close the gap

\[ V_{t,N}^S = \sum_{n=1}^{N} E_t \left[ \frac{M_{t+n}}{M_t} S_{t+n} \right] \] (6)

\[ = \sum_{n=1}^{N} E_t \left[ \frac{M_{t+n}}{M_t} (D_{t+n-1} e^{r_0} - D_{t+n}) \right] \] (7)

\[ = \sum_{n=1}^{N} E_t [E_{t+n-1} \left( \frac{M_{t+n}}{M_t} D_{t+n-1} e^{r_0} \right)] - \sum_{n=1}^{N} E_t \left[ \frac{M_{t+n}}{M_t} D_{t+n} \right] \] (8)

\[ = \sum_{n=1}^{N} E_t \left[ \frac{M_{t+n-1}}{M_t} D_{t+n-1} \right] - \sum_{n=1}^{N} E_t \left[ \frac{M_{t+n}}{M_t} D_{t+n} \right] \] (9)

\[ = D_t - E_t \left[ \frac{M_{t+N}}{M_t} D_{t+N} \right] \] (10)
Log-linearization of the (AI)

▶ Use the AI to write:

\[
\ln(S_{t+1} + e^{\ell_{t+1}}) = \ell_t - gC(t+1) + r_D(t+1)
\]

where \( S_t = S_t/C_t, \ell_t = \ln D_t/C_t, r_D(t) = \ln R_D(t). \)

▶ Taylor expand LHS around \( S_t = \bar{S} \) and \( \ell_t = \bar{\ell} \) to get:

\[
\rho_0 + \rho_{\ell}\ell_{t+1} + \rho_s S_{t+1} = \ell_t - gC(t+1) + r_D(t+1)
\]

\[
\rho_0 = \ln(\bar{S} + e^{\bar{\ell}}) - \rho_{\ell}\bar{\ell} - \rho_s \bar{S}
\]

\[
\rho_{\ell} = \frac{e^{\bar{\ell}}}{\bar{S} + e^{\bar{\ell}}}
\]

\[
\rho_s = \frac{1}{\bar{S} + e^{\bar{\ell}}}
\]

▶ Iterate forward:

\[
\ell_t = \frac{\rho_0}{1 - \rho_{\ell}} + \sum_{n=t+1}^T \rho_{\ell}^{n-t-1} \left\{ \rho_s S_n + gC(n) - r_d(n) \right\} + \rho_{\ell}^{T-t} \ell_T
\]

▶ Pick \( \bar{S} = \bar{\ell} = 0 \) to get the Cochrane decomposition.
### Parameter Estimates

<table>
<thead>
<tr>
<th>Process</th>
<th>Parameter</th>
<th>Estimated Value</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \bar{p} )</td>
<td>(0.240, -0.036, 0.006, 0.001, 0.0001)</td>
<td>(0.003, 0.0004, 0.0001, 0.000, 0.000)</td>
</tr>
</tbody>
</table>
| \( \Phi \) | | \[
\begin{pmatrix}
0.854 & -0.120 & 0.872 & 6.960 & -0.158 \\
-0.120 & 0.677 & 0.113 & 0.239 & -2.456 \\
0.872 & -0.803 & 0.113 & -0.070 & -0.033 \\
6.960 & 1.920 & 0.239 & 0.409 & -0.047 \\
-0.158 & -0.506 & -2.456 & 0.014 & -0.005
\end{pmatrix}
\times 10^{-5}
\]
| \( \sigma_p \) | | | |
| \( gY \) | \( \mu_Y \) | 0.087 | 0.015 |
| \( \mu_{Yp} \) | | (0.146, -0.584, 0.302, 18.61, -41.00) | (0.143, 0.662, 4.571, 10.10, 26.18) |
| \( \sigma_{Yp} \) | | (-0.009, 0.013, -0.010, 0.063, 0.031) | (0.015, 0.015, 0.015, 0.016, 0.016) |
| | \( \sigma_Y \) | 0.107 | 0.002 |
| \( gc \) | \( \mu_C \) | 0.065 | 0.003 |
| \( \mu_{ Cp} \) | | (0.089, 0.524, -1.220, 4.027, -0.285) | (0.025, 0.117, 0.804, 1.784, 4.605) |
| \( \sigma_{ Cp} \) | | (0.007, 0.005, 0.000, 0.000, -0.001) | (0.003, 0.003, 0.003, 0.003, 0.003) |
| | \( \sigma_C \) | 0.019 | 6 \times 10^{-5} |
| \( z \) | \( \bar{z} \) | -2.533 | 0.010 |
| \( z_p \) | | (-0.087, -0.069, -3.185, -9.523, 4.714) | (0.095, 0.439, 3.031, 6.853, 17.896) |
| \( h \) | \( \bar{h} \) | 0 | 0 |
| \( \Phi_{hp} \) | | (0.009, -0.019, 0.753, 1.152, 12.20) | (0.075, 0.347, 2.394, 5.311, 13.71) |
| \( \Phi_h \) | | -0.300 | 0.1162 |
| \( \sigma_{hp} \) | | (-0.001, -0.012, 0.002, 0.003, -0.004) | (0.008, 0.008, 0.008, 0.008, 0.008) |
| \( \sigma_{h\bar{Y}} \) | | -0.053 | 0.008 |
| \( \sigma_h \) | | 0.055 | 5.6 \times 10^{-4} |
| \( M \) | \( \lambda_{0p} \) | (0.0140, -0.001, 0, 0, 0) | (0.0001, 0.0004, 0, 0, 0) |
| \( \lambda_{1p} \) | | \[
\begin{pmatrix}
-0.055 & 1.140 & 0 & 0 & 0 \\
-0.018 & -0.137 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
| | | (0.0002, 0.022, 0.000, 0.000, 0.000) | (0.000, 0.0008, 0.000, 0.000, 0.000) |