Dynamic Asset Allocation with Transaction Costs:
The Importance of Hedging Demands

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Large Literature on 'Frictionless’ Dynamic Portfolio Choice

▶ Markowitz’s (1958) one-period mean-variance efficient (MVE) portfolio choice is still widely popular among practitioners.

▶ Merton (1969, 1971) and Cox-Huang (1989) introduce **Hedging Demand**:

→ In dynamic setting, it may be optimal to deviate from instantaneous MVE portfolio to hedge against future changes of intercept (risk-free rate) and slope (sharpe ratio) of the conditional MVE frontier.

▶ Lots of academic work on **strategic asset allocation** show empirical relevance of hedging demand:


**Q?** Insights from that literature seems to be mostly ignored by practitioners (deliberately?).

→ Realistic asset allocation needs to impose realistic objective function, constraints, transaction and price impact (‘slippage’) costs.
Traditional Literature on Dynamic Portfolio Choice with T-costs

  - Requires sophisticated mathematics (e.g., 'viscosity solutions' Shreve-Soner (1994)) even though limited to one risky asset with i.i.d. returns.
  - Multi-asset analysis in i.i.d. case limited to CARA preferences (Liu (2004)).

- Studies combining predictability in returns and t-costs typically limited to one or two risky assets and use numerical solutions (Balduzzi and Lynch (1999), Lynch and Tan (2011), Longstaff (2001)).

- Optimal execution of fixed size position to trade-off transaction costs and risk (Almgren and Chriss (2000), Grinold and Kahn (chap 16))
The Linear Quadratic Framework

▶ In a Linear-Quadratic (LQ) framework explicit solution for optimal portfolio choice for many stocks, with many predictors, and quadratic t-costs (i.e., linear price impact) obtains (Litterman (2005), Grinold (2006), Garleanu-Pedersen (2013), Collin-Dufresne, Daniel, Saglam (2018))

▶ The LQ framework relies on ad-hoc instantaneous mean-variance objective function which is myopic in the absence of t-costs.

▶ This paper:
  → Standard preferences which nest (micro-found) this LQ-objective function.
  → Explicit portfolio choice solution for (i) general (non-myopic) objective (ii) arbitrary number or assets, (iii) predictable returns.
  → How important are hedging demands for portfolio choice with transaction costs?
Asset allocation with Transaction costs and predictability

- What do (quant) practitioners do?
  - Identify return forecasting factors
  - Identify risk factors
  - Create portfolios with ‘optimal’ exposures to these factors that
  - Maximize expected returns, net of expected trading costs, subject to a risk budget

- As an example consider optimal combination of three signals for large cross-section of individual stocks
  - Short term reversal (REV: half-life of 5 days)
  - Momentum (MOM: half-life 150 days)
  - Value (VAL: long-term reversal with a half-life of 700 days)

- Each stock will have a specific exposure to REV, MOM, and VAL. These factors will decay at different rates and are clearly not independent.

- How do we operationalize the ‘optimal trade-off’ when there are transaction costs?
  - If trade more often, expect to capture more alpha, but pay more transaction costs.
  - If trade less often, may not benefit from fast signals.
    → Should we trade more/less aggressively when signals decay faster/slower?
  - Trading more frequently reduces tracking error, but increases t-costs.
    → Should we trade more/less aggressively when volatility is larger/smaller?
A one-period (practitioner) approach

- A one period problem with quadratic transaction costs:

$$\max_{n_{t+1}} \left\{ n_{t+1}^\top \mu - h \frac{1}{2} (n_{t+1} - n_t)^\top \Lambda (n_{t+1} - n_t) - \frac{\gamma}{2} n_{t+1}^\top \Sigma n_{t+1} \right\}$$

- where
  - $n_t$ is vector of initial stock holdings
  - $\mu$ is vector of expected returns
  - $\Sigma$ is covariance matrix of returns
  - $\Lambda$ is price impact matrix ($\sim$ quadratic transaction costs).

- If $\Lambda = \lambda \Sigma$ the optimal position is

$$n_{t+1}^* = (1 - \tau) n_t + \tau (\gamma \Sigma)^{-1} \mu$$

- Trade-off t-costs vs. tracking error by partially trading towards Markowitz portfolio with trading speed:

$$\tau = \frac{1}{1 + h\lambda/\gamma}$$

- Practitioners typically:
  - target a level of risk with risk-aversion $\gamma$,
  - calibrate to match realistic price impact $\Lambda$ estimates, and
  - optimize over the trading speed parameter ($h$) to maximize backtest performance.
The dynamic model setup and Linear-Quadratic framework

- \(N\)-vector of stock price \(S_t\) has dynamics:
  
  \[
  dS_t = (\mu_0 + \mu x_t) dt + \sigma_s dZ_s(t) \tag{1}
  \]
  
  \[
  dx_t = -\kappa x_t dt + \sigma_x dZ_x(t) + \sigma_{xs} dZ_s(t) \tag{2}
  \]

  N.B.: nests the case where every stock expected return is driven by \(K\) stock-specific predictors with different decay rates (e.g., momentum and value).

- Assuming risk-free \(r = 0\), wealth dynamics with position vector \(n_t\):
  
  \[
  dW_t = n_t^\top dS_t - \frac{1}{2} \theta_t^\top \Lambda \theta_t dt \tag{3}
  \]
  
  \[
  = n_t^\top (\mu_0 + \mu x_t) dt + n_t^\top \sigma_s dZ_s(t) - \frac{1}{2} \theta_t^\top \Lambda \theta_t dt \tag{4}
  \]
  
  \[
  dn_t = \theta_t dt \tag{5}
  \]

- Classic but \textit{ad-hoc} instantaneous mean-variance (LQ) objective function:

  \[
  \max_{\theta} \mathbb{E} \left[ \int_0^T \left\{ dW_t - \frac{1}{2} \gamma dW_t^2 \right\} \right] = \max_{\theta} \mathbb{E} \left[ \int_0^T \left\{ n_t^\top (\mu_0 + \mu x_t) - \frac{1}{2} \theta_t^\top \Lambda \theta_t - \frac{1}{2} \gamma n_t^\top \Sigma n_t \right\} dt \right]
  \]

  where \(\Sigma = \sigma_s \sigma_s^\top\) is the covariance of returns.

The general objective function: finite horizon case

Consider solution \((H_t, \sigma_{H,s}, \sigma_{H,x})\) to the recursive equation:

\[
H_t = E_t \left[ W_T - \int_t^T \left\{ \frac{1}{2} \gamma ||\sigma_{H,s}||^2 + \frac{1}{2} \gamma_x ||\sigma_{H,x}||^2 \right\} du \right]
\]  \(6\)

\(H_t\) is certainty equivalent of source-dependent stochastic differential utility agent, with absolute risk-aversion coefficient \(\gamma\) towards \(Z_s\) and \(\gamma_x\) towards \(Z_x\) shocks.

When \(\gamma_x = \gamma\) it nests CARA expected utility:

\[
H_t = -\frac{1}{\gamma} \log(E_t[e^{-\gamma W_T}]).
\]  \(7\)

When \(\gamma_x \sigma_x \rightarrow 0\) and \(\sigma_{xs} = 0\) it nests instantaneous mean-variance:

\[
H_t = W_t + E_t \left[ \int_t^T \left\{ dW_u - \frac{1}{2} \gamma dW_u^2 \right\} \right].
\]  \(8\)

The instantaneous mean-variance investor is risk-neutral with respect to changes in the investment opportunity set. She only displays risk-aversion towards 'level' shocks.

Q? Compare solution to (7) with solution to (8) to understand impact of non-myopic (e.g., hedging demand) on portfolio choice with transaction costs.
The Optimal Portfolio rule

▶ If $\Lambda = 0$ (no t-costs) the optimal position (Merton 1971):

$$n_t = (\gamma \Sigma)^{-1}(\mu_0 + \mu x_t) - \Sigma^{-1}\Sigma_{sx}(c_1(t) + c_2(t)x)$$

(9)

where $\Sigma = \sigma_s\sigma_s^\top$ and $\Sigma_{sx} = \sigma_s\sigma_{xs}^\top$.

⇒ If zero correlation between returns and expected returns (i.e., $\Sigma_{sx} = 0$) it is optimal to hold the instantaneous mean-variance efficient Markowitz portfolio.

▶ If $\Lambda > 0$ positive definite, then

$$dn_t = \tau_t(aim(x_t, t) - n_t)\,dt$$

(10)

$$\tau_t = \Lambda^{-1}Q(t)$$

(11)

$$aim(x, t) = Q(t)^{-1}(q_0(t) + q(t)^\top x)$$

(12)

where $Q, q, q_0$ solve system of Riccatti-style ODE and $aim(t, x) = \arg\max_n J(n, x, t)$. 

The Importance of Hedging Demands
Insights from the solution: the instantaneous mean-variance case

▶ For an instantaneous mean-variance agent (eq. (8)), the optimal trading speed depends only on the eigenvalues ($\eta_i$) and eigenvectors ($F_i$) of $\gamma \Lambda^{-1} \Sigma \equiv FD\eta F^\top$:

$$\tau_t = FD_h(t)F^\top \quad \text{with} \quad h_i(t) = \sqrt{\eta_i} \frac{1 - e^{-2\sqrt{\eta_i}(T-t)}}{1 + e^{-2\sqrt{\eta_i}(T-t)}}$$

▶ The aim portfolio can be written as $\text{aim}(x, t) = (\gamma \Sigma)^{-1} \int_t^T \omega_t,u \mu_S(t,u)du$, where:

$$\mu_S(t,u) = \frac{1}{dt} E_t[dS_u] = \mu_0 + \mu e^{-\int_t^u \kappa ds} x_t$$

$$\omega_t,u = (\int_t^T e^{-\int_t^z \tau_s^\top ds} dz)^{-1} e^{-\int_t^u \tau_s^\top ds}$$

▶ As in one-period mean-variance benchmark, it is optimal to "trade partially towards aim portfolio", but:

▶ Trading speed is endogenous and depends only on $\gamma \Lambda^{-1} \Sigma$.

▶ Aim portfolio equals the Markowitz MVE portfolio if $\Lambda = 0$ or $\kappa = 0$.

▶ Else, Aim portfolio $\sim$ Markowitz which replaces mean with expected future returns discounted for (i) trading speed ($\tau$), and (ii) signal mean-reversion ($\kappa$).

▶ Allocation is independent of the correlation between returns and signals ($\Sigma_{sx}$).
Results extend to Random Horizon (stationary solution)

Consider solution \((H_t, \sigma_{H,s}, \sigma_{H,x})\) to the recursive equation with random horizon \(T\) (Poisson with intensity \(\rho\)):

\[
H_t = E_t \left[ W_T - \int_t^T \left\{ \frac{1}{2} \gamma \left\lVert \sigma_{H,s} \right\rVert^2 + \frac{1}{2} \gamma_x \left\lVert \sigma_{H,x} \right\rVert^2 + \rho \left( W_s - H_s - \frac{1 - e^{-\gamma_T (W_s - H_s)}}{\gamma_T} \right) \right\} ds \right]
\]

Then \(H_t\) is the certainty equivalent of source-dependent SDU agent with ARA coefficient \(\gamma\) toward \(Z_s\), \(\gamma_x\) towards \(Z_x\), and \(\gamma_T\) towards horizon arrival \(1_{\{T \leq t\}}\).

- When \(\gamma_T = \gamma_x = \gamma\), it nests CARA expected utility:
  \[
  H_t = -\frac{1}{\gamma} \log(E_t[e^{-\gamma W_T}]). \tag{13}
  \]
- When \(\gamma_T = \gamma_x \sigma_x \to 0\) and \(\sigma_{xs} = 0\), it nests the discounted instantaneous mean-variance objective function:
  \[
  H_t = W_t + E_t \left[ \int_t^\infty e^{-\rho(u-t)} \left\{ dW_u - \frac{1}{2} \gamma dW_u^2 \right\} \right]. \tag{14}
  \]

Instantaneous mean-variance investor is risk-neutral with respect to both shocks to expected returns and horizon risk.

In the following we focus on stationary solution case with \(\gamma_T = 0\) (i.e. we ignore horizon risk premium) and compare solution with \(\gamma_x = 0\) to with \(\gamma_x = \gamma > 0\) to investigate role of the hedging demand.

The Importance of Hedging Demands
The Optimal Portfolio rule

▸ If $\Lambda = 0$ (no t-costs) the optimal position (Merton):

$$n_t = \left(\gamma \Sigma\right)^{-1} (\mu_0 + \mu x_t) - \Sigma^{-1} \Sigma_{sx} (c_1 + c_2 x)$$  \hspace{1cm} (15)

Markowitz

HedgingDemand

⇒ If zero correlation ($\Sigma_{sx} = 0$) optimal to hold the instantaneous mean-variance markowitz portfolio:

▸ If $\Lambda > 0$ positive definite, then

$$dn_t = \tau_t (aim(x_t) - n_t) \, dt$$  \hspace{1cm} (16)

$$\tau_t = \Lambda^{-1} Q$$  \hspace{1cm} (17)

$$aim(x, t) = Q^{-1} (q_0 + q^\top x)$$  \hspace{1cm} (18)

where $Q, q, q_0$ solve system of equation and $aim(x) = \arg\max_n J(n, x)$. 

The Importance of Hedging Demands
Insights from the solution: the instantaneous mean-variance case

- For an instantaneous mean-variance agent, the optimal trading speed matrix $\tau = \Lambda^{-1}Q$ is given by:

\[
\tau = F D_h F^\top
\]

\[
h_i = \frac{1}{2} (\sqrt{\rho^2 + 4\eta_i} - \rho)
\]

where $(F_i, \eta_i)$ are eigenfactors and eigenvalues of $\gamma \Lambda^{-1} \Sigma$.

- The optimal aim portfolio

\[
\text{aim}(x_t) = Q^{-1}(q_0 + q^\top x_t)
\]

\[
= (\gamma \Sigma)^{-1} \int_0^\infty \omega_u E_t[\mu_S(x_{t+u})] du
\]

\[
\mu_S(x_t) = \frac{1}{dt} E_t[dS_t] = \mu_0 + \mu x_t
\]

\[
\omega_u = (\rho + \tau^\top) e^{-(\rho + \tau^\top)u}
\]

→ Similar to finite horizon solution, but stationary
Illustration of the trading rule in the one-stock & one factor case

When trading costs are small: Merton hedging demand depends on $\sigma_{xs} \leq 0$

**Figure:** Parameters: $\mu_0 = 0$, $\mu = 1$, $\kappa = 0.1$, $\sigma_s = 0.3$, $\sigma_x = 0.1$, $\Lambda = 2 \times 10^{-11}$, $\gamma_x = 10^{-9}$, $\gamma = 10^{-9}$, $x_0 = 1$, $\rho = 0.8$. 

The Importance of Hedging Demands
For higher t-costs, hedging demands are very important (especially for $\sigma_{xs} < 0$)

For higher t-costs, hedging demands are very important (especially for $\sigma_{xs} < 0$)

**Figure:** Parameters: $\mu_0 = 0$, $\mu = 1$, $\kappa = 0.1$, $\sigma_s = 0.3$, $\sigma_{xs} = -0.1$, $\sigma_x = 0.1$, $\Lambda = 2 \times 10^{-10}$, $\gamma_x = 10^{-9}$, $\gamma = 10^{-9}$, $x_0 = 1$, $\rho = 0.8$. 
When signal is less persistent (high $\kappa$)

**Figure:** Parameters: $\mu_0 = 0$, $\mu = 1$, $\kappa = 0.4$, $\sigma_s = 0.3$, $\sigma_x = 0.1$, $\Lambda = 2 \times 10^{-10}$, $\gamma_x = 10^{-9}$, $\gamma = 10^{-9}$, $x_0 = 1$, $\rho = 0.8$. 
When horizon is longer (low arrival intensity $\rho$)

**Figure:** Parameters: $\mu_0 = 0$, $\mu = 1$, $\kappa = 0.1$, $\sigma_s = 0.3$, $\sigma_x = 0.1$, $\Lambda = 2 \times 10^{-10}$, $\gamma_x = 10^{-9}$, $\gamma = 10^{-9}$, $x_0 = 1$, $\rho = 0.2$. 
Estimation of the return generating process: one stock-one latent factor

Using Kalman filter, we estimate an unobservable state, $x_t$ using time series of S&P 500 index (daily) price changes between 1962-07 and 2018-12:

\[ dS_t = x_t dt + \sigma_s dZ_s(t) \quad (23) \]
\[ dx_t = -\kappa x_t dt + \sigma_x dZ_x(t) + \sigma_{xs} dZ_s(t) \quad (24) \]

Estimates using S&P 500 returns from 1962-07 to 2018-12:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>0.730</td>
<td>0.049</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>0.137</td>
<td>0.008</td>
</tr>
<tr>
<td>$\sigma_{xs}$</td>
<td>-0.076</td>
<td>0.013</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>0.005</td>
<td>0.029</td>
</tr>
</tbody>
</table>

→ Very negative correlation between $dx_t$ and $dS_t$ ($< -0.95$).

For the experiments we set $\gamma = \gamma_x = 10^{-8}$ and $\rho = 0.05$. 

The Importance of Hedging Demands
Estimation of the transaction cost parameters

- Use proprietary institutional money managers from the historical order databases of a large investment bank
- Use executions from top 50 stocks in terms of market capitalization
- Estimate $\Lambda$ by fitting the following panel regression:

$$\Delta P_i = \Lambda \frac{D_i Q_i}{2} + \varepsilon_i$$

where
- $Q_i$ is the number of shares and
- $D_i$ is the direction of the trade.

<table>
<thead>
<tr>
<th>Dependent variable: $10^7 \Delta P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

Note: * $p<0.1$; ** $p<0.05$; *** $p<0.01$
Trading Experiment

- Based on the filtered time series of $x$, the estimated return and t-cost parameters we:
  - Divide the sample into 112 trading intervals of 126 trading days (i.e., six-months).
  - At the beginning of each interval we start with $0$ and trade according to a trading rule (optimal, GP, myopic).
  - Record the stock position and the total accumulated wealth net of t-costs every day.
  - Compute the certainty equivalent, average across all intervals, for the CARA investor of various strategies.

- Results:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Optimal</th>
<th>GP</th>
<th>Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>CE</td>
<td>72,228</td>
<td>52,569</td>
<td></td>
</tr>
<tr>
<td>Avg Util</td>
<td>-0.9993</td>
<td>-0.9995</td>
<td>1.96 \times 10^{-4}</td>
</tr>
<tr>
<td>S.E</td>
<td>1.3 \times 10^{-4}</td>
<td>0.9 \times 10^{-4}</td>
<td>3.8 \times 10^{-5}</td>
</tr>
</tbody>
</table>

- The aim portfolios as a function of $x$. 

The Importance of Hedging Demands
The Importance of Hedging Demands
One Sample path with TC

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Average path without TC

The Importance of Hedging Demands
The Importance of Hedging Demands
Conclusion

▶ We propose a set of preferences based on stochastic differential utility with source-dependent risk-aversion, which nest the widely used instantaneous mean-variance (Litterman (2005), Garleanu-Pedersen (2013)) and CARA expected utility.

▶ We derive an explicit solution for the portfolio choice problem in the presence of quadratic t-costs with arbitrary number of stocks and predictability in returns in terms of an optimal aim portfolio and trading speed.

▶ We show that, for a CARA investor, the hedging demand has large effect on optimal aim portfolio and trading speed, especially when the correlation between stock return and predictor is negative.

▶ In an in-sample experiment where we time the S&P 500 return based on its filtered latent predicted expected return, hedging demands significantly affect strategy performance.

▶ It remains to be seen whether this also holds out-of-sample.
Details on the execution data

- Data contains two trading algorithms:
  - volume weighted average price (VWAP) and
  - percentage of volume (PoV).

- Execution data covers S&P 500 stocks between January 2011 and December 2012.

- Execution duration is greater than 5 minutes but no longer than a full trading day.

- Total number of orders is 81,744 with an average size of approximately $1 million.

- The average participation rate of the order, the ratio of the order size to the total volume realized in the market, is approximately 6%.

- Back to [main](#).