Admissible Surplus Dynamics and the Government Debt Puzzle*

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Abstract

In an influential paper Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2024) argue that financial markets fail to impose a no-arbitrage condition that ties the value of all outstanding US debt to the present value of future government surpluses and that this misvaluation arises because markets do not properly adjust for risk when discounting future (procyclical) surpluses. However, we prove that, in any arbitrage-free model, (i) this *debt-valuation puzzle* arises if and only if the transversality condition fails and (ii) for most *ad-hoc* specifications of surplus processes the transversality condition will fail mechanically for all parameter values. Instead, we propose a simple class of *admissible* surplus processes that match empirical properties of the US government surplus process without generating any debt valuation or risk-premium puzzles.

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1 Introduction

The policy response to the great financial crisis and the COVID pandemic have resulted in unprecedented sovereign debt to GDP ratios. In the US, the level of outstanding debt in 2022 reached 31 Trillion USD, about 96% of GDP. This has rekindled the debate on the sustainability of such debt levels and of the associated fiscal policies. Related, the position of the US Treasury debt as the safe asset 'par excellence' has come into question.

In a recent influential paper, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2024) (henceforth JLNX) argue that financial markets fail to impose a no-arbitrage condition that ties the value of all outstanding government debt (D_t) to the present value of all future government surpluses (V_t). Using a multivariate affine autoregressive model for the surplus fitted to historical data and a realistic (exponential affine) stochastic discount factor calibrated to risk-free yields and stock returns, they find that US sovereign debt is overvalued by a startling 360% of GDP (see JLNX, Figure 9 and the ensuing discussion). According to their fitted no-arbitrage model, the average value of the claim to future surpluses is -320% of GDP, which differs widely from the average market value +39% of GDP observed for the total stock of US Treasury debt (for the 1947 to 2020 period). JLNX claim that this *'valuation gap'* is evidence for mispricing—their *'debt valuation puzzle'*—that arises because markets fail to properly account for the procyclical systematic risk in the surplus process. In turn, JLNX argue that yields on government debt are too low and refer to this observation as *'debt risk-premium puzzle.'*

Their claims are puzzling indeed, since the purported mis-pricing seems dramatic in magnitude (more than USD100 Trillion!) and to affect one of the most liquid and deep markets in the world. Aside from the startling magnitude of the mis-pricing, there also seems to be an inconsistency in their theoretical argument. Under no-arbitrage the total value of government debt must be positive $(D_t \geq 0)$ since outstanding government debt can be seen as a portfolio of zero coupon bonds each with a finite and fixed maturity. If indeed no-arbitrage ties D_t to the present-value of all future surpluses V_t, that is if $D_t = V_t$, then $V_t \geq 0$ is an obvious no-arbitrage requirement *for any parameter choice within a no-arbitrage model*. How can JLNX find a model-implied $V_t = -320\%$ of GDP for some particular set of parameters within their no-arbitrage model?

We provide an explanation for their findings. Specifically, if M_t is an arbitrage free stochastic discount factor (SDF) and if one defines the present value of future government surpluses $(\mathbf{s}_\tau)$ as

$$V_t = V_t(M) := E_t\left[\sum_{\tau=t+1}^{\infty}\frac{M_\tau}{M_t}\mathbf{s}_\tau\right]$$

then our results show that the *'debt valuation equation'* $D_t = V_t$ is satisfied if and only if the transversality condition (TVC)

$$\lim_{T\to\infty} E_t\left[M_T D_T\right] = 0, \qquad \forall t \geq 0,$$

holds relative to same SDF that is used to value the stream of future surpluses. In other words, if one finds that $D_t \neq V_t(M)$ for *some* SDF M then this valuation gap *must* result from a violation of the TVC relative to this particular SDF.[1] This implies that, while D_t must satisfy the government budget constraint as a matter of national accounting, if there is a valuation gap then $V_t(M) \neq D_t$ will not satisfy that accounting identity and, therefore, cannot be considered a valid candidate for the value of government debt!

Furthermore, we show that for most ad-hoc specifications of the surplus process the TVC fails for *any* set of parameters given a particular choice of SDF. This implies that finding a valuation gap, even after careful calibration of such a model to historical data, has no meaningful economic content other than confirming the violation of the TVC for that reduced-form specification of the surplus process and the SDF. In particular, the existence valuation gap does not imply that government debt is mispriced since absent arbitrage total outstanding debt—which has a finite maturity—has a well-defined positive value pinned down by the SDF and its promised future cash-flows. Furthermore, finding that the TVC is violated for an ad-hoc SDF does not imply that it is for all valid SDFs—of which there are many in an incomplete market. So one cannot even conclude that there is some long-run inefficiency in the economy as this would require finding that the TVC fails for *any valid* pricing kernel, a much more difficult proposition.

We illustrate our results by deriving a generic example where the surplus process follows an AR1 process, the SDF is lognormal, and the government balances its budget by issuing one period risk-free debt. We prove analytically that, in this simple arbitrage-

[1] This is at odds with the discussion in JLNX, Section 7.2 who quote our paper arguing that: *"The valuation gap* **can** *result from a violation of the transversality condition"* and that *"In standard infinite-horizon asset pricing models* [...], *the TVC is satisfied."* (emphasis added). This is also at odds with their Footnote 21, where they argue that *"this resolution is premised on the assumption that the* [SDF] *is the correct one."* Our statement holds for *any* valid SDF and certainly must hold within their no-arbitrage model!

free economy, a valuation gap arises for all parameter values. In addition, we show that if the surplus and the SDF of the example are specified in real terms then there is no endogenous (positive) price level dynamics such that the TVC is satisfied. This implies that it is hopeless to fit an AR1 surplus process to historical data and hope to evaluate whether government debt is mispriced based on finding a 'valuation gap,' since such a gap must arise for any parameter choice!

Instead, if we augment the surplus process by including lagged accumulated debt into the dynamics, then under some parameter restrictions the TVC and, thus, the debt-valuation equation can be satisfied.

Making the surplus process path-dependent in the accumulated debt is one way to introduce some stabilizing mechanism that allows the model to enforce the TVC and the long-term solvency of the government, which is reminiscent of the 'S-shaped' process restrictions proposed in Cochrane (2023, Chapter 4). An alternative approach is to allow for path-dependence in the SDF. We note that in the presence of stochastic growth and priced aggregate risk both may be required. In particular, building on Belo, Collin-Dufresne, and Goldstein (2015) we consider a second example with priced aggregate consumption risk, where we derive an *admissible surplus process* consistent with long-term stationary debt to consumption dynamics. We show that the TVC and, hence, the debt valuation equation hold in this case if and only if the risk-premium on consumption risk is sufficiently high.

In a nutshell, our theoretical results and examples show that the approach of specifying a reduced form surplus process and calibrating it to past surplus data in order to price outstanding government debt using a present value relation, by analogy to an approach used for pricing the stock market, is unlikely to deliver debt dynamics that satisfy the TVC. For the stock market it may be reasonable to estimate a stationary model for log-dividend growth using past data and to use those estimates to value the claim to future aggregate dividends. But for a government that is expected to eventually reimburse its debt, surplus dynamics must exhibit some path-dependence that ties it to the accumulated debt in such as way as to insure that the TVC can be satisfied. Furthermore, whether the TVC and the debt valuation equation hold depends on the joint specification of an admissible surplus process and the SDF used to value future surpluses.

We then propose a more realistic empirical model for an admissible government surplus process. We use a multi-factor affine model of the term-structure of interest rates that has been shown to provide a good match of the returns on both risk-free bonds and the portfolio of outstanding government debt (see, e.g., Duffee (2011) and Adrian, Crump, and Moench (2013)). In addition to the state variables extracted from the risk-free yield curve, we calibrate the SDF to also price the value weighted CRSP market portfolio. To do so, we assume that there is one additional state variable driving the market price of risk of equity specific cash-flow and discount rate shocks. We estimate the model to fit the dynamics of risk-free yields, the aggregate stock market dividend process, and the price-dividend ratio of the stock market. Given these estimates we then infer the ratio of the surplus to aggregate consumption from the budget constraint of the government as suggested by Cochrane (2022). Counter to the findings of JLNX, our estimates imply that there is no valuation gap and, thus, no 'debt valuation puzzle' in that the present value of future surpluses, computed using either the calibrated equity-SDF or its projection onto the span of bond returns, is equal to the value of outstanding government debt.

Related literature

There is an extensive literature on the sustainability of sovereign debt and the link to some transversality condition (see, e.g., O'Connell and Zeldes (1988), Bohn (1995), Hansen, Roberds, and Sargent (1991)). Bohn (1995) shows that if the government faces complete markets, then Ponzi schemes are ruled out if the present value of the debt discounted at the unique discount factor goes to zero at infinity. In that case the valuation equation that ties current debt to the present value of futures surpluses also holds.[2] We extend his insight to the case of incomplete markets. Bohn (1998) studies the empirical behavior of US government debt and surpluses and finds empirical evidence that surplus responds to the path of prior debt issuances in a manner consistent with the theoretical intertemporal budget constraint of the government.

Hansen et al. (1991) discuss the econometric restrictions imposed by the combination of the budget balance and a long-term constraint on accumulated debt (such as a transversality condition) on the joint behavior of surplus and debt. They conclude: "[...] *it imposes literally no observable restrictions on the process* (*surplus,debt*)*. This weakness illustrates how difficult it is to verify a present-value budget constraint that restricts the entire (infinite-dimensional) future of the* (*surplus,debt*) *process."* Related, Cochrane (2022) points out that the government budget constraint is an accounting identity that must hold ex-post and,

[2] Bohn (1995) also presents an illuminating example where the equilibrium risk-free rate is smaller than the growth rate of the economy (the ubiquitous $r < g$ constraint), debt grows at g, and the TVC fails when discounting at the risk-free rate but holds if the proper SDF is used.

therefore, uses the standard Campbell-Shiller decomposition to analyze the present value relation between current debt and future surpluses.

Recently, the new macroeconomic environment combining large increases in levels of public debt with persistently high deficits and low interest rates, has rekindled the debate on whether perhaps governments, unlike firms or individuals, are in fact subject to an intertemporal budget constraint. Notably, Blanchard (2019) argues in his AEA presidential lecture that, since $r < g$, public debt may have no fiscal costs and no significant crowding out effects which in turn implies that the optimal level of public debt may be a lot higher than previously thought, even in the presence of persistently negative surpluses. Related, Dumas, Ehling, and Yang (2022) and Brunnermeier, Merkel, and Sannikov (2022a,b) provide incomplete markets models where governments may run persistently negative surpluses, and where the debt valuation equation may fail, because the TVC is violated. In their models, agents want to hold public debt to hedge idiosyncratic risks, adding an extra, non-pecuniary component to its value, which gives rise to the valuation gap. Specifically, Dumas et al. (2022) study an OLG model with no aggregate uncertainty where debt helps to smooth intertemporal consumption and the TVC fails due to presence of a bubble in the sense of Tirole (1985). Instead, in the main models of Brunnermeier et al. (2022a,b) individual agents are infinitely lived risk-averse agents who face idiosyncratic shocks that they can only imperfectly hedge by trading in government bond. Each agent has a different SDF with respect to which their own wealth satisfies a TVC and a valuation gap only arises relative to an artificially constructed representative SDF (an argument reminiscent of Constantinides and Duffie (1996)).

The rest of the paper is organised as follows. In Section 2, we present theoretical results proving that the government debt and related debt beta puzzles arise if and only if the TVC fails relative to the SDF used to value future surpluses. We present a simple example of a model economy where JLNX's debt valuation puzzle arises mechanically as a result of a violation of the TVC and show how modifying the pricing kernel and/or the dynamics of surplus can solve the puzzle. In Section 3 we present an empirically realistic model for an admissible surplus process that can accommodate realistic stochastic aggregate consumption growth and price dynamics. In Section 4, we fit the model to historical data from the US and show that it is consistent with the history of government surpluses and yet does not generate a debt valuation puzzle.

2 Theory

2.1 Financial markets

We consider a discrete-time economy on an infinite horizon. Agents in the economy trade various securities, including government bonds, in frictionless financial markets that are arbitrage free in the following sense:

Assumption 1 *There exists a process* $(M_t)_{t\in\mathbf{N}}$ *with initial value* $M_0 = 1$ *that is strictly positive and such that*

$$E_t\left[\frac{M_{t+1}}{M_t}R_{t+1}\right] = 1$$

at all times and for any traded asset gross return. Any such process is referred to as a stochastic discount factor (SDF).

2.2 Government debt

The government collects taxes according to some process $(T_t)_{t\in\mathbf{N}}$, spends according to another process $(G_t)_{t\in\mathbf{N}}$, and balances its budget by issuing or retiring debt in financial markets. As a result, the market value of all outstanding government debt *mechanically* satisfies the *accounting identity*:

$$D_{t+1} = D_t R^D_{t+1} - \mathbf{s}_{t+1} \tag{AI}$$

where

$$\mathbf{s}_t := T_t - G_t$$

denotes the *government surplus* and R^D_{t+1} denotes the gross return from t to $t+1$ on the portfolio of all government debt outstanding at t.

If the debt issued by the government takes the form of constant coupon bonds—as in the empirical application of Section 4—then outstanding debt can be decomposed into strips as

$$D_t = \sum_{\tau=t+1}^{\infty} Q^\tau_t P^\tau_t$$

where P_t^τ denotes the price at t of a zero-coupon with maturity $\tau \geq t$ and Q_t^τ denotes the amount outstanding at t of payments due at τ. In this case, the gross return on the portfolio of all outstanding government debt satisfies

$$D_t R_{t+1}^D = \sum_{\tau=t+1}^{\infty} Q_t^\tau P_{t+1}^\tau. \tag{1}$$

Therefore, it follows that (AI) can be written as

$$Q_t^{t+1} = \mathbf{s}_{t+1} + \sum_{\tau=t+2}^{\infty} \left(Q_{t+1}^\tau - Q_t^\tau\right) P_{t+1}^\tau$$

which highlights that the government meets its debt payment obligations at date $t+1$ either by drawing from a positive surplus or by issuing zero-coupon bonds.

Our main objective is to study conditions under which the value of government debt equals the present value of future government surpluses

$$V_t(M) := E_t\left[\sum_{\tau=t+1}^{\infty} \frac{M_\tau}{M_t} \mathbf{s}_\tau\right]$$

relative to *some* SDF. For this question to be meaningful one must impose conditions that guarantee that the above conditional expectation is well-defined for some SDF. This is the purpose of the following standing assumption:

Assumption 2 *The surplus process is such that*

$$E\left[\sum_{\tau=1}^{\infty} M_\tau |\mathbf{s}_\tau|\right] < \infty \tag{2}$$

for some SDF $(M_t)_{t\in\mathbf{N}}$.

2.3 The transversality condition

Let $(M_t)_{t\in\mathbf{N}}$ be a SDF that satisfies (2). Since the debt issued by the government is traded we have that

$$E_t\left[\frac{M_{t+1}}{M_t} R_{t+1}^D\right] = 1.$$

Combining this identity with (AI) shows that

$$E_t\left[\frac{M_{t+1}}{M_t}D_{t+1}\right] = E_t\left[\frac{M_{t+1}}{M_t}\left(D_t R^D_{t+1} - \mathbf{s}_{t+1}\right)\right] = D_t - E_t\left[\frac{M_{t+1}}{M_t}\mathbf{s}_{t+1}\right]$$

where the expectation on the left is well-defined as a result of (2). Iterating this relation forward shows that

$$E_t\left[\frac{M_T}{M_t}D_T\right] = D_t - E_t\left[\sum_{\tau=t+1}^{T}\frac{M_\tau}{M_t}\mathbf{s}_\tau\right]$$

for all finite T, and passing to the limit as $T \to \infty$ we deduce that

$$\begin{aligned}\lim_{T\to\infty} E_t\left[\frac{M_T}{M_t}D_T\right] &= D_t - \lim_{T\to\infty} E_t\left[\sum_{\tau=t+1}^{T}\frac{M_\tau}{M_t}\mathbf{s}_\tau\right] \\ &= D_t - E_t\left[\sum_{\tau=t+1}^{\infty}\frac{M_\tau}{M_t}\mathbf{s}_\tau\right] \equiv D_t - V_t(M),\end{aligned} \tag{3}$$

where the second equality follows from (2) and the dominated convergence theorem. This identity immediately leads to the following result:

Theorem 1 *Fix a SDF $(M_t)_{t\in\mathbf{N}}$ such that (2) holds. Then the following conditions are equivalent*

1. *$D_t = V_t(M)$ for all $t \in \mathbf{N}$,*
2. *$D_0 = V_0(M)$ and $(V_t(M))_{t\in\mathbf{N}}$ evolves according to (AI),*
3. *The transversality condition*

$$\lim_{T\to\infty} E_t\left[\frac{M_T}{M_t}D_T\right] = 0 \tag{TVC}$$

holds for all $t \in \mathbf{N}$.

Proof. Equation (3) shows that conditions 1 and 3 are equivalent. On the other hand, if condition 2 holds then (AI) implies that

$$V_t(M) - D_t = (V_0(M) - D_0)\prod_{\tau=1}^{t} R^D_\tau = 0$$

which is condition 1, and the claimed equivalence follows by observing that, since the debt value process satisfies (AI), condition 1 implies condition 2. ■

2.4 The government debt puzzle

The identity $D_t = V_t(M)$ in Theorem 1 is known as the *'debt valuation equation'*. In an attempt to test this restriction JLNX proceed as follows. First, they estimate over the period 1945–2020 a VAR whose components include the logs of U.S. tax revenues and government spending. Second, they calibrate an affine SDF $(M_t^*)_{t \in \mathbf{N}}$ to bond and equity returns over the same period. Third, they use the estimated processes for tax revenues and government spendings to compute the present value of future surpluses relative to their calibrated SDF at all time points in the sample period.

They find that $V_t(M^*)$ is consistently negative and represents on average -320% of GDP over the sample period. In addition, comparing the realized time series of this present value to the time series of outstanding government debt values calculated from historical supply and prices of government bonds they measure a consistently positive government debt valuation gap

$$\mathrm{Gap}_t(M^*) := D_t - V_t(M^*) > 0 \tag{4}$$

that represents on average 360% of GDP over the sample period. Based on these results, JLNX conclude that U.S. government debt is grossly overvalued by financial markets and refer to this finding as the *government debt puzzle*.

We find this conclusion puzzling for two reasons. First, the fact that the present value of future surpluses $V_t(M^*)$ estimated by JLNX is always negative should immediately disqualify this process as a candidate for the value of outstanding government debt as the latter delivers only positive cash flows at finite maturity dates and, thus, must have a positive value to prevent trivial arbitrages. [3] Second, Theorem 1 shows that a non-zero gap in (4) can *only* arise from a failure of the transversality condition. Therefore, if JLNX find a gap, it is necessarily because the debt value process induced by their estimated surplus process through (AI) fails the transversality condition relative to their calibrated SDF. But if that is the case, then regarding the present value of future surpluses $V_t(M^*)$

[3]The model of JLNX and all those we consider below assume the existence of a strictly positive pricing kernel, which is sufficient to rule out simple arbitrage opportunities of that sort.

as a candidate for the value of outstanding government debt makes no sense because it does not satisfy the accounting identity (AI).[4]

The only conclusion that one may draw from the empirical exercise of JLNX is that the value of outstanding debt fails the transversality condition relative to the specific SDF they calibrated. But this *does not mean that government debt is mispriced*. Instead, D_t is still well-defined as the present-value of its promised finite maturity cash-flows relative to any (of the infinitely many) valid SDFs and, of course, satisfies the intertemporal budget constraint (AI) as a matter of national accounting.

The question of whether the debt valuation equation holds *for some pricing kernel* is a statement about whether the infinite horizon dynamic trading strategy that buys all current and future debt issuances at future prevailing market prices generates cash-flows that are repayable in the long-run by the government from its primary surplus. This is a statement about the long-run super-replication properties of that strategy and one can only assert that there is some form of long-run inefficiency if there is a gap *relative to* **all** arbitrage-free SDFs (see, e.g., Santos and Woodford (1997)).

We show below using a simple but generic example that in most cases specifying an ad-hoc surplus process leads to a debt value process that fails to satisfy the TVC relative to a large class of plausible SDFs. Therefore, it is not meaningful to confront such a process to the data as a way to test for the existence of a valuation gap. Instead, we show that under some restrictions *either on the surplus process or on SDF* (or both) it is possible to construct a model where there will be no-gap. Thus, we argue that it is more sensible to specify a surplus process in a class that is 'admissible,' in the sense that for some parameter values the TVC may be satisfied relative to some valid SDF. Arguably, any standard equilibrium macro-finance model should give rise to such an admissible surplus process.

[4]In empirical applications, a gap could also arise because (AI) does not hold, perhaps due to noise in NIPA's primary surplus measure or to time-aggregation. However, within a theoretical model (AI) holds as a matter of identity and in our empirical implementation of Section 4 we enforce it as proposed by Cochrane (2022) by implying the surplus from the accounting identity using the time series of outstanding debt values and returns—which are (more) precisely measured.

2.5 The debt beta puzzle

Assume that government debt only consists of coupon bonds. In that case (1) implies that me may rewrite the accounting identity (AI) as

$$R^D_{t+1} = \frac{D_{t+1} + \mathbf{s}_{t+1}}{D_t} = I_{t+1} R^D_{t+1} + \frac{\mathbf{s}_{t+1}}{D_t}$$

where the random variable

$$I_{t+1} := \sum_{\tau=t+2}^{\infty} Q^{\tau}_{t+1} P^{\tau}_{t+1} \Bigg/ \sum_{\tau=t+1}^{\infty} Q^{\tau}_{t} P^{\tau}_{t+1}$$

captures the issuance or retirement of debt at time $t+1$. This identity implies that the conditional beta of the debt return R^D_{t+1} on aggregate consumption growth g_{t+1} (or any other variable of interest) can be decomposed as

$$\beta_t \left(R^D_{t+1} \middle| g_{t+1} \right) = \beta_t \left(I_{t+1} R^D_{t+1} \middle| g_{t+1} \right) + \beta_t \left(\tfrac{\mathbf{s}_{t+1}}{D_t} \middle| g_{t+1} \right). \tag{5}$$

This relation shows that the return on outstanding government debt can have a low beta, even if the surplus is highly pro-cyclical, provided that aggregate debt issuances are counter-cyclical. In particular, if debt is locally riskfree in the sense that R^D_{t+1} is known at time t then the above identity boils down to

$$\beta_t \left(I_{t+1} | g_{t+1} \right) D_t R^D_{t+1} + \beta_t \left(\mathbf{s}_{t+1} | g_{t+1} \right) = 0$$

which clearly shows that any positive surplus beta (second term) must be perfectly offset by a negative aggregate issuance beta (first term).

Equation (5) only relies on the accounting identity. Therefore, it must hold whether or not the transversality condition holds. But *if the TVC holds* relative to some SDF $(M_t)_{t\in\mathbf{N}}$ then Theorem 1 shows that the debt valuation equation

$$D_t = V_t(M) \equiv E_t \left[\sum_{\tau=t+1}^{\infty} \frac{M_\tau}{M_t} \mathbf{s}_\tau \right]$$

holds at all times. This begs the question of how the return on outstanding government debt can have a low beta (e.g., be riskless) if debt is given by the present value of futures cash flows (surpluses) that are risky (i.e., have non-zero betas). This observation is at the heart of what JLNX coin the *debt beta puzzle*.

If the TVC holds relative to a given $(M_t)_{t\in\mathbf{N}}$ then we may compute the value of outstanding government debt as a sum of future surplus strips relative to that SDF:

$$D_t = V_t(M) = E_t\left[\sum_{\tau=t+1}^{\infty} \frac{M_\tau}{M_t}\mathbf{s}_\tau\right] = \sum_{\tau=t+1}^{\infty} v_t^\tau(M).$$

It immediately follows that we can decompose the debt return and the debt return beta as weighted averages of the returns and return betas of the surplus strips:

$$\begin{aligned} R_{t+1}^D &= \sum_{\tau=t+1}^{\infty}\left[\frac{v_t^\tau(M)}{D_t}\right] R_{t+1}^\tau(M) \\ \beta_t\left(R_{t+1}^D \middle| g_{t+1}\right) &= \sum_{\tau=t+1}^{\infty}\left[\frac{v_t^\tau(M)}{D_t}\right] \beta_t\left(R_{t+1}^\tau(M)\middle| g_{t+1}\right) \end{aligned} \tag{6}$$

where

$$R_{t+1}^\tau(M) := \frac{v_{t+1}^\tau(M)}{v_t^\tau(M)}$$

denotes the return from t to $t+1$ on the strip with maturity $\tau \geq t+1$. In particular, if the return on government debt is locally risk-free, then (6) simplifies to

$$0 = \sum_{\tau=t+1}^{\infty}\left[\frac{v_t^\tau(M)}{D_t}\right] \beta_t\left(R_{t+1}^\tau(M)\middle| g_{t+1}\right)$$

which illustrates that government debt can be locally riskfree and yet satisfy the valuation equation despite the fact that each individual surplus strip is risky. The only property that is required (and indeed must be satisfied if the TVC holds) is that the value-weighted average of betas across surplus strips of different maturities be equal to zero.

2.6 An example with a gap

In this section we present a simple one-factor version of JLNX's model to illustrate how a debt valuation gap can appear mechanically, for any parameter values, due to an ad-hoc reduced form specification of the surplus and the SDF. Then, in the next section, we show that it is possible to modify the specification of the surplus process and/or of the SDF, so as to make the gap disappear for some parameter values. In both cases the gap arises if and only if the TVC is violated.

Assume that uncertainty is generated by a sequence $(\epsilon_\tau)_{\tau=1}^{\infty}$ of iid standard normal random variables and that the surplus evolves according to

$$\mathbf{s}_{t+1} = e^{-\kappa}\mathbf{s}_t + \epsilon_{t+1} \tag{7}$$

for some constant $\kappa > 0$. Assume further that the government only trades one period bonds and that the one period continuously compounded interest rate $r > 0$ is constant so that the accounting identity (AI) writes as

$$D_{t+1} = e^r D_t - \mathbf{s}_{t+1}. \tag{8}$$

Finally, consider the class of SDFs given by

$$-\log \frac{M_{t+1}}{M_t} = r + \frac{1}{2}\lambda^2 + \lambda\epsilon_{t+1} \tag{9}$$

for some constant λ that captures the market price of risk associated with the normal shocks to the government surplus.

Relying on basic properties of normal random variables (see Appendix A below for the details of these calculations) one can show that the above SDF satisfies Assumption 2 and that

$$V_t(M) = E_t\left[\sum_{\tau=t+1}^{\infty} \frac{M_\tau}{M_t}\mathbf{s}_\tau\right] = \frac{\mathbf{s}_t - a}{e^{r+\kappa} - 1} \tag{10}$$

with the constant

$$a := \left(\frac{e^{r+\kappa}}{e^r - 1}\right)\lambda.$$

From this expression we immediately obtain that the *gross return on the claim to future surpluses* is explicitly given by:

$$\frac{V_{t+1}(M) + \mathbf{s}_{t+1}}{V_t(M)} = \frac{e^{r+\kappa}\mathbf{s}_{t+1} - a}{\mathbf{s}_t - a}.$$

In particular, it is clear that this return is random conditional on the information available at date t and, therefore, does not coincide with the constant riskfree return e^r on a one period bond. This immediately implies that $V_t(M)$ *does not* satisfy the accounting identity (8) and it thus follows from Theorem 1 that a non zero debt valuation gap must exist relative to *any* SDF with a constant market price of risk as in (9). In fact, a direct calculation using (8) and (10) shows that this gap is explicitly given by

$$\begin{aligned}\mathrm{Gap}_t(M) = \lim_{T\to\infty} E_t\left[\frac{M_T}{M_t}D_T\right] &= D_t - V_t(M) \\ &= e^{rt}D_0 - \sum_{\tau=1}^{t} e^{r(t-\tau)}\mathbf{s}_\tau - \frac{\mathbf{s}_t - a}{e^{r+\kappa}-1}.\end{aligned}$$

The above arguments ignore inflation and implicitly assume that the surplus, the SDF, and the debt value were all specified in nominal terms. To see that endogenous inflation does not resolve the issue, assume that the government surplus and the SDF in (7) and (9) are specified in real terms and that the government balances its budget by issuing one period *nominal* bonds. In such a model the nominal value of outstanding government debt satisfies the accounting identity

$$D_{t+1} = R_{t,t+1}D_t - \Pi_{t+1}\mathbf{s}_{t+1} \tag{11}$$

where $\Pi_{t+1} > 0$ denotes the price level and $R_{t,t+1} \geq 0$ denotes the predictable gross return on a one period *nominal* bond. Now assume that there exists a strictly positive price level such that the nominal present value

$$\Pi_t V_t(M) = \Pi_t E_t\left[\sum_{\tau=t+1}^{\infty}\frac{M_\tau}{M_t}\mathbf{s}_\tau\right]$$

satisfies the nominal accounting identity (11). Combining this identity with the explicit solution for $V_t(M)$ in (10) then shows that

$$R_{t,t+1}\frac{\Pi_t}{\Pi_{t+1}} = \frac{V_{t+1}(M)+\mathbf{s}_{t+1}}{V_t(M)} = \frac{e^{r+\kappa}\mathbf{s}_{t+1}-a}{\mathbf{s}_t - a}$$

and delivers an immediate contradiction. Indeed, the left handside is nonnegative but the right handside can take any real value since the surplus is normally distributed. This immediately implies that there does not exist any viable price level such that the present nominal value of futures surpluses $\Pi_t V_t(M)$ satisfies the nominal accounting identity (11) and from Theorem 1 we conclude that the nominal debt value fails the TVC relative to any nominal SDF of the form $M_t \frac{\Pi_0}{\Pi_t}$ for some real SDF M_t as in (9).

2.7 Closing the gap

What is missing in the specification (7,8,9) to ensure that the TVC—or, equivalently, the debt valuation equation—holds at least for some parameters is a stabilizing feedback mechanism between accumulated debt and the dynamics of government surpluses. There are at least two natural channels to introduce such a stabilizing mechanism. First, one can introduce debt directly in the physical dynamics of the surplus. After all, it is clear that the government does not decide on taxes and spending independently of the amount of accumulated debt and so it is intuitive that the evolution of the surplus should depend on existing debt. Second, one can allow the market price of risk to depend on the accumulated debt to help satisfy the TVC. As we show below, either of these mechanisms (or a combination) can solve the government debt puzzle.

Consider the same accounting identity as in (8) but assume that we replace the evolutions of the surplus in (7) and the SDF in (9) by:

$$\mathbf{s}_{t+1} = e^{-\kappa}\mathbf{s}_t + \phi_{\mathbf{s}} D_t + \epsilon_{t+1} \tag{12}$$

and

$$-\log \frac{M_{t+1}}{M_t} = r + \frac{1}{2}(\lambda - \phi_M D_t)^2 + (\lambda - \phi_M D_t)\epsilon_{t+1} \tag{13}$$

for some constant $\phi_{\mathbf{s}}, \phi_M$ that capture the feedback of accumulated debt. Note that with $\phi_{\mathbf{s}} = \phi_M = 0$ this model reduces to that of the previous example.

In this specification (8) and (12) form an autoregressive process. Therefore, a standard computation reported in Appendix A show that

$$E_t\left[\frac{M_{t+n}}{M_t}\begin{pmatrix} D_{t+n} \\ \mathbf{s}_{t+n}\end{pmatrix}\right] = \Phi^n \begin{pmatrix} D_t \\ \mathbf{s}_t\end{pmatrix} + e^{-rn}\sum_{k=0}^{n-1}\Phi^k \begin{pmatrix} \lambda \\ -\lambda\end{pmatrix} \tag{14}$$

where the matrix Φ is given by:

$$\Phi := \begin{pmatrix} 1 - e^{-r}(\phi_{\mathbf{s}} + \phi_M) & -e^{-(r+\kappa)} \\ e^{-r}(\phi_{\mathbf{s}} + \phi_M) & e^{-(r+\kappa)} \end{pmatrix}. \tag{15}$$

Since $r > 0$ it immediately follows that the TVC and the debt valuation equation hold relative to M_t if and only if the eigenvalues of Φ are strictly smaller than one in norm. Lemma 1 in Appendix A shows that this property is equivalent to:

$$0 < \phi_{\mathbf{s}} + \phi_M < 2\left(e^r + e^{-\kappa}\right).$$

This result shows that, *if* the market price of risk is constant (i.e., if $\phi_M = 0$), it is necessary that surpluses respond positively to accumulated debt values for the TVC to hold. This property ensures that current deficits are repaid by future surpluses and is closely related to what Cochrane (2022) refers to as *s-shaped* surplus processes.

The result further shows that, *if surplus does not depend on accumulated debt* (i.e., if $\phi_{\mathbf{s}} = 0$), then the market price of risk must depend negatively on accumulated debt to ensure that the TVC is satisfied. The intuition is that when debt levels become very large, without any stabilizing mechanism to ensure that future surpluses will turn positive, it is the risk-premium attached to surplus shocks that needs to adjust in such a way as to put more (less) weight on future high (low) surpluses and, thereby, ensure that the present discounted value of future surpluses remains equal to current debt.

Absent such path-dependence in the surplus and/or the SDF, the TVC is unlikely to hold. Indeed, writing the government debt valuation equation as

$$D_t = E_t\left[\sum_{\tau=t+1}^{\infty} \frac{M_\tau}{M_t}\mathbf{s}_\tau\right]$$

makes it clear that unless the SDF and/or the surplus depend on the accumulated amount of debt there is nothing to guarantee the present value of future surpluses equals the current value of outstanding government debt.

In the next section we consider a general setting with priced consumption risk as well as multiperiod government bonds and show how to construct a surplus process that is *admissible* in the sense that the TVC and, thus, the debt valuation equation holds relative to some arbitrage-free SDF. This more general model demonstrates that admissibility is a joint restriction on the surplus and the SDF.

3 An admissible surplus process

In this section we propose a model for (D_t, M_t) and an *admissible* surplus process $\mathbf{s}_t$ that is sufficiently flexible to match key moments of Treasury bond and equity returns as well as empirical dynamics of the surplus, aggregate consumption, and inflation.

Our approach to the construction of the model proceeds in three steps. First, because we want to accommodate stochastic growth shocks, we specify a process C_t for aggregate consumption and a process

$$\ell_t := \log \frac{D_t}{C_t}$$

for the log of debt over consumption. Second, we specify a process R_t^D for the return on outstanding government debt. Third, following the approach of Belo et al. (2015), we use the accounting identity (AI) to *infer the surplus at* $t+1$:

$$\frac{\mathbf{s}_{t+1}}{C_{t+1}} = e^{\ell_t} \left(\frac{C_t}{C_{t+1}} \right) R_{t+1}^D - e^{\ell_{t+1}}.$$

In the model we take to the data, we will use a multivariate affine SDF calibrated to bond and equity returns and take R_t^D to be the return on a portfolio with fixed proportions of short, medium, and long-term bonds. However, to develop intuition, we first present a simple one-factor version of the model where we can obtain closed-form solutions for all relevant quantities and show that, under appropriate parameter restrictions, the induced government surplus process is indeed admissible.

3.1 A one-factor example

Assume that the government only issues one period debt with return $R_{t+1}^D = e^r$ for some constant $r > 0$ and that the pair (ℓ_t, C_t) evolves according to

$$\ell_{t+1} = (1-\phi)\,\bar{\ell} + \phi \ell_t + \sigma_\ell \epsilon_{t+1}$$

and

$$g_{t+1}^C := \log \frac{C_{t+1}}{C_t} = \mu_C - \frac{1}{2}\sigma_C^2 + \sigma_C \epsilon_{t+1}$$

for some constants such that $\phi \in (0,1)$ where $(\epsilon_t)_{t\in\mathbf{N}}$ is a sequence of iid standard normal random variables. In this simple model the accounting identity of the government (AI) is automatically satisfied by letting

$$\frac{\mathbf{s}_{t+1}}{C_{t+1}} = e^{r-g^C_{t+1}+\ell_t} - e^{\ell_{t+1}}$$

To see that this specification can satisfy the TVC consider the same nominal SDF as in Section 2.6, namely

$$-\log\frac{M_{t+1}}{M_t} = r + \frac{1}{2}\lambda^2 + \lambda\epsilon_{t+1}.$$

For this class of SDFs, a calculation reported in Appendix A shows that

$$\begin{aligned} E_t\left[\frac{M_T}{M_t}D_T\right] &= C_t E_t\left[\frac{M_T C_T}{M_t C_t}e^{\ell_T}\right] \\ &= e^{\ell^*}C_t \exp\left\{-\rho\left(T-t\right) + \phi^{T-t}\left(\ell_t - \ell^*\right) + \frac{1}{2}\sigma_\ell^2\frac{1-\phi^{2(T-t)}}{1-\phi^2}\right\} \end{aligned} \tag{16}$$

with the constants

$$\begin{aligned} \rho &:= r - \mu_C + \lambda\sigma_C \\ \ell^* &:= \overline{\ell} + \sigma_\ell\left(\sigma_C - \lambda\right)/\left(1-\phi\right) \end{aligned}$$

and it follows that the TVC is satisfied if and only if the model parameters of the model are such that $\rho > 0$. In particular, the TVC can be satisfied even if $r < \mu_C$ provided that the market price of risk λ is large enough. Such a parametrization is of special interest, since $r < \mu_C$ and the stationarity of ℓ_t jointly imply that

(i) The unconditional average surplus over consumption is negative as can be seen from the fact that $E\left[\mathbf{s}_t/C_t\right] \approx \left(e^{r-\mu_C}-1\right)e^{\overline{\ell}} < 0$,

(ii) The TVC is violated if one discounts future surpluses at the risk-free rate because surpluses grow on average at the same rate as consumption,

(iii) The TVC is satisfied if one values future surpluses using an SDF with a sufficiently high market price of risk to ensure that $\rho > 0$.

The features of this parametrization are similar to those of the model in Bohn (1995), who shows that a representative agent equilibrium exists as long as the TVC holds relative to the representative agent's SDF even when it fails when discounting debt at the risk-free rate. This example also confirms that, in the presence of stochastic growth, specifying an admissible surplus process imposes joint restrictions on the evolution of the surplus process and the SDF.

3.2 The multi-factor affine model

The multi-factor affine model, that we will take to the data in the next section, extends the above one-factor example to allow for richer dynamics and a more realistic modeling of the return to outstanding government debt.

3.2.1 Macroeconomic state variables

Let $x_t \in \mathbf{R}^{n_x}$ for some $n_x \geq 1$ be a state variable that captures the all the macroeconomic information affecting expected consumption growth, inflation, bond yields, and equity risk-premia. Assume that this vector evolves according to

$$x_{t+1} = \Phi_x x_t + \sigma_x \epsilon_{t+1} \tag{17}$$

where (σ_x, Φ_x) are matrices of appropriate dimensions and $(\epsilon_t)_{t \in \mathbf{N}}$ is a sequence of iid standard normal vectors of dimension $n_\epsilon \geq n_x$. Assume further that

$$\begin{aligned} \ell_{t+1} &= \bar{\ell} + \sum_{i=1}^{L} \phi_i \left(\ell_{t+1-i} - \bar{\ell} \right) + \mu_{\ell x}^\top x_t + \sigma_\ell^\top \epsilon_{t+1} \\ g_{t+1}^C &:= \log \frac{C_{t+1}}{C_t} = \mu_C - \frac{1}{2} ||\sigma_C||^2 + \mu_{Cx}^\top x_t + \sigma_C^\top \epsilon_{t+1} \end{aligned} \tag{18}$$

for some number of lags $L \geq 1$ where $(\phi_i, \mu_j, \mu_{jx}, \sigma_j)$ are scalars and column vectors of appropriate dimensions. Because the Treasury bonds market does not span all the sources of risk driving government debt issuances, we allow for some correlation between ℓ_t, the price level Π_t, and the aggregate stock-market dividend Y_t by setting

$$g^{\Pi}_{t+1} := \log \frac{\Pi_{t+1}}{\Pi_t} = \mu_\Pi - \frac{1}{2}||\sigma_\Pi||^2 + \mu_{\Pi x}^\top x_t + \sigma_\Pi^\top \epsilon_{t+1} \tag{19}$$

$$g^{Y}_{t+1} := \log \frac{Y_{t+1}}{Y_t} = \mu_Y - \frac{1}{2}||\sigma_Y||^2 + \mu_{Yx}^\top x_t + \sigma_Y^\top \epsilon_{t+1} \tag{20}$$

for some scalars and column vectors $(\mu_k, \mu_{kx}, \sigma_k)$ of appropriate dimensions.

3.2.2 SDF, bond prices, and the stock market

We specify the nominal SDF by assuming that the risk free rate and the market price are affine functions of the state vector:

$$-\log \frac{M_{t+1}}{M_t} = r_t + \frac{1}{2}||\Lambda_t||^2 + \Lambda_t^\top \epsilon_{t+1} \tag{21}$$

where

$$(r_t, \Lambda_t) := \left(\rho_0 + \rho_x^\top x_t, \lambda_0 + \lambda_x^\top x_t\right) \tag{22}$$

for some scalars and column vectors $(\rho_0, \rho_x, \lambda_0, \lambda_x)$ of appropriate dimensions. As a result of this affine specification the price at time t of a nominal zero-coupon bond that pays \$1 at time T is explicitly given by:

$$P_t^T := E_t\left[\frac{M_T}{M_t}\right] = e^{-A_t^T - x_t^\top B_t^T} \tag{23}$$

where the deterministic scalar $A_t^T \equiv A^{T-t}$ and vector $B_t^T \equiv B^{T-t}$ depend only on time to maturity and solve a system of difference equations that is standard in exponentially affine term structure theory. See Backus and Zin (1994), Duffie and Kan (1996) for the original arguments and Appendix A for a detailed derivation.[5]

[5] Similarly, (19), (21), and (22) imply that the price at time t of an inflation protected zero-coupon bond that promises one unit of real consumption at time T is given by

$$p_t^T := E_t\left[\frac{M_T}{M_t}\frac{\Pi_T}{\Pi_t}\right] = e^{-a_t^T - x_t^\top b_t^T}$$

for deterministic coefficients a_t^T and b_t^T that solve another system of difference equations. However, because we only have limited data available on real bond yields we will not use this expression for our calibration of the model in the next section.

We further show in Appendix A that, due to (20), (21), and (22), the aggregate value of the stock market admits a closed form as an infinite sum of exponential affine dividend strips that can be approximated using a Campbell-Shiller approximation:

$$\frac{S_t}{Y_t} = E_t \left[\sum_{\tau=t+1}^{\infty} \frac{M_\tau}{M_t} \frac{Y_\tau}{Y_t} \right] \approx e^{\bar{z} + z_x^\top x_t} \tag{24}$$

for some scalar $\bar{z}$ and vector z_x that we derive in Appendix A. In the next section, we will use the fact that zero-coupon yields and the log-price/dividend ratio of the stock market are affine in x_t to identify the relevant latent state variables and the parameters (λ_0, λ_x) that define the market prices for risk.

3.2.3 The surplus process and the return to government debt

As in our one-factor example, the accounting identity of the government (AI) is automatically satisfied by defining the surplus ratio as

$$\frac{\mathbf{s}_{t+1}}{C_t} = e^{\ell_t - g_{t+1}^C} R_{t+1}^D - e^{\ell_{t+1}} \tag{25}$$

However, we now consider the government issues fixed rate coupon bonds of various maturities, so that the value of outstanding government debt can be written as a portfolio of zero-coupon bonds:

$$D_t = \sum_{m \geq t+1} n_t^m P_t^m$$

with return

$$R_{t+1}^D = e^{r_t} + \sum_{m \geq t+1} \frac{n_t^m P_t^m}{D_t} \left(\frac{P_{t+1}^m}{P_t^m} - e^{r_t} \right). \tag{26}$$

For simplicity, we further assume that the portfolio weights

$$\omega_t^m := \frac{n_t^m P_t^m}{D_t} \equiv \omega^m$$

are constant over time and concentrated on rolling maturities ranging from 1 to 20 years. It would be straightforward to allow the weights to be functions $\omega_t^m = \omega^m(t, x_t)$ of time and the state vector but, as we show in the next section, we obtain a good fit with the simpler assumption of constant weights.

Substituting the closed-form expression for risk free zero-coupon bonds from (23) into the return on government debt (26) shows that the evolution of the surplus process in (25) is fully determined by the evolution of the state variables (x_t, ℓ_t) and the vector of shocks ϵ_t. With this specification we can readily check whether the TVC holds. Indeed, a computation provided in Appendix A shows that

$$\begin{aligned} E_t\left[\frac{M_T}{M_t}D_T\right] &= C_t E_t\left[\frac{M_T C_T}{M_t C_t}e^{\ell_T}\right] \\ &= C_t \exp\left\{H_{0t}^T + x_t^\top H_{xt}^T + \sum_{i=1}^{L} H_{it}^T\left(\ell_{t+1-i} - \bar{\ell}\right)\right\} \end{aligned} \tag{27}$$

where the vector H_{xt}^T and scalars $(H_{it}^T)_{i=0}^L$ are given explicitly as solutions to a system of difference equations that depend on the all the parameters of the model. Given this closed form solution it is straightforward to verify whether the TVC

$$\lim_{T\to\infty} E_t\left[\frac{M_T}{M_t}D_T\right] = 0$$

holds for the particular set of parameters under consideration, which then also implies that the debt valuation equation $D_t = V_t(M)$ is satisfied . In the next section we fit this model to US historical data and discuss its empirical predictions.

4 Empirical results

In this section, we calibrate the multifactor affine model of Section 3.2 to historical data from the US. We first discuss the data used and how we construct our empirical measure of the primary surplus. Next, we discuss how to infer the latent state variables x_t in (17) and estimate the coefficients of the SDF from yield curve and stock market data. Then, we explain how to estimate the parameters of the debt to consumption ratio ℓ_t in (18). Finally, we estimate the debt return process R_t^D in (26). We thus obtain an empirical model for the US surplus process that is consistent with the government budget constraint from (25) and can readily test whether the TVC holds using (27).

4.1 Data

The data we use to estimate our model comes from multiple sources. Specifically:

- We use the yield curve data set constructed by Gürkaynak, Sack, and Wright (2007). In this data set 1-year to 7-years yields are available from June 1961 while longer yields from 8 to 20-years are available starting in October 1981.
- We use the time series of the market value of outstanding US debt provided by the website of the Dallas FED (dallasfed.org/research/econdata/govdebt) and which includes monthly observations from January 1942 to October 2022.
- Monthly returns on government debt starting 1790 (Hall, Payne, and Sargent 2018) were obtained from G. Hall's website people.brandeis.edu/ghall.
- Monthly stock market data (dividends and security prices) was obtained from the CRSP value-weighted market portfolio data.
- Yearly data on the accounting surplus of the US government from 1947 to 2022 was obtained from NIPA Table 3.2: *Federal Government Current Receipts and Expenditures* of the Bureau of Economic Analysis (apps.bea.gov/NIPA/Table32)
- Other times series including US GDP, price level, and aggregate consumption were obtained from the FRED database (fred.stlouisfed.org).

For our empirical analysis we take the point of view of the government and thus treat the central bank as an independent creditor that is a pari-pasu claimant to the payments on government debt. Accordingly, we apply the government budget constraint (AI) to the total amount outstanding government debt which consists in both marketable debt (bills, notes, and bonds) and non-marketable debt (long term savings bonds). Alternatively, one could apply the budget constraint to the consolidated balance-sheet of the government by netting out the component of debt held by the central bank and adding the monetary base (coins, paper bills, and reserves) to the liability of the consolidated government. We opt for the first approach as it does not require to value the monetary base.

4.2 Measurement of the primary surplus

We imply the government surplus from the accounting identity (AI) given historical data on the market value D_t and return R_t^D of outstanding government debt. To ascertain the

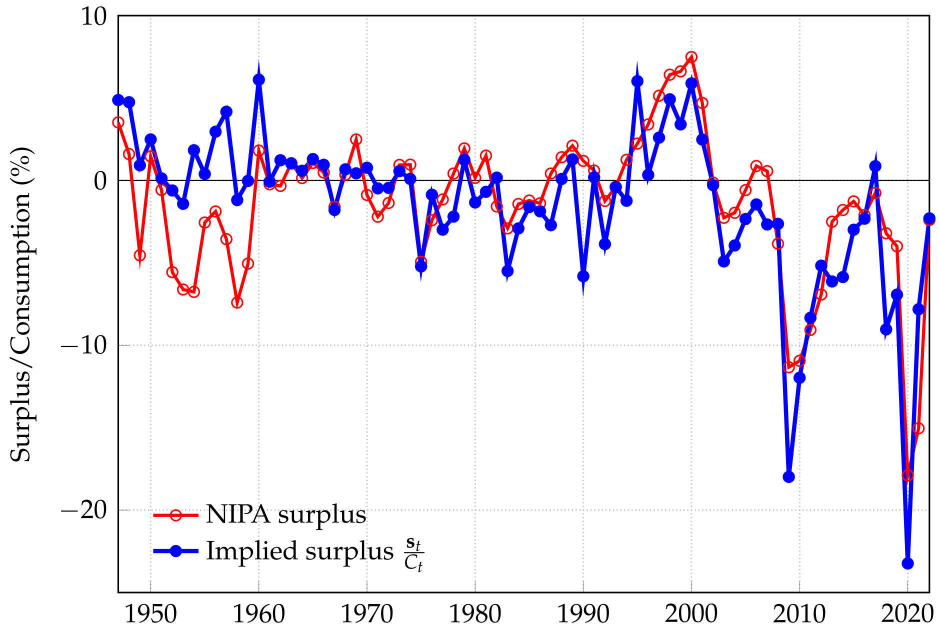

Figure 1: NIPA and Implied surplus

Notes. Comparison of the annual surplus as a percentage of aggregate consumption derived from the NIPA tables and implied from (AI) given data on the market value value and returns to outstanding government debt.

validity of this method we compare the time series of the implied surplus to that of the annual primary surplus computed from NIPA Table 3.2 by adding rows 49 'Net lending or net borrowing' and 33 'Interest payments'. Figure 1 shows that the two time series are closely related but that, as already noted by Cochrane (2022), there are some significant deviations. In particular, while the mean of the implied surplus is close to that of the NIPA surplus (−1.42% vs. −1.36%), its standard deviation is higher (4.8% vs. 4.3%).

These deviations imply that the NIPA surplus fails to satisfy the accounting identity (AI). This is likely due in part to time-aggregation issues of the NIPA numbers. Since debt values and returns are arguably measured more accurately, we follow Cochrane (2022) in using the implied surplus rather than the NIPA surplus as our empirical measure of the true government surplus.

4.3 Estimation of the SDF and identification of the state variables

Here we give a succinct description of how we calibrate the parameters of the multi-factor affine SDF presented in Section 3.2.2 and relegate the details Appendix B. Our goal is to specify a parsimonious model that is able to capture the dynamics of Treasury bond returns and of the aggregate stock market. Because it is plausible that debt issuance and surplus both depend on factors spanned by Treasury bonds as well as equity-specific factors, we proceed in two steps.

First, we assume that there is a set of factors that capture all the bond-specific risks. For the pricing of riskfree nominal bonds our model is a standard exponential affine Gaussian family similar to Joslin, Singleton, and Zhu (2011), Hamilton and Wu (2012), Duffee (2011), Adrian et al. (2013), and Diez de los Rios (2015) among others. Following these papers we use a three-factor model where the state variable can be inferred from the cross-section of yields. Specifically, we use the first three principal components of the yield curve as observable state-variables.[6]

An issue that arise with the dataset constructed by Gürkaynak et al. (2007) is that long maturity yields up to 20 years are available only since 1981. For the earlier period, starting in 1961, only 1- to 7-years yields are available. To use all the available data, we split our sample in two periods, the first one from 1961 to 1980 and the second from 1981 to 2022. For the first period, we compute principal components from the cross section of 1- to 7-years yields. In this period we find that the first three principal components explain 100% of the yield curve variance, but that the third component is almost irrelevant as is only explains 0.03% of the total variance. For the second period, we compute the principal components using 1- to 20-years yields. Here we find that the first three principal components explain 99.98% of the yield curve variance and that the contribution of the third component is around 0.1%. In Figure 2 we show the loadings of yields on the first three principal components in the two periods. From this plot it is clear that, as usual in empirical analysis of the term structure, the first three principal components can be roughly interpreted as 'level','slope' and 'curvature' of the yield curve.

Second, we assume that a single factor h_t drives the pricing of equity-specific risk. The expression for the stock market price-dividend ratio in (24) shows that this factor is observable since it can be inverted from the price-dividend ratio of stock market given the

[6] In an earlier version we used as many as five principal component factors, but found little improvement relative to the three-factor model presented here.

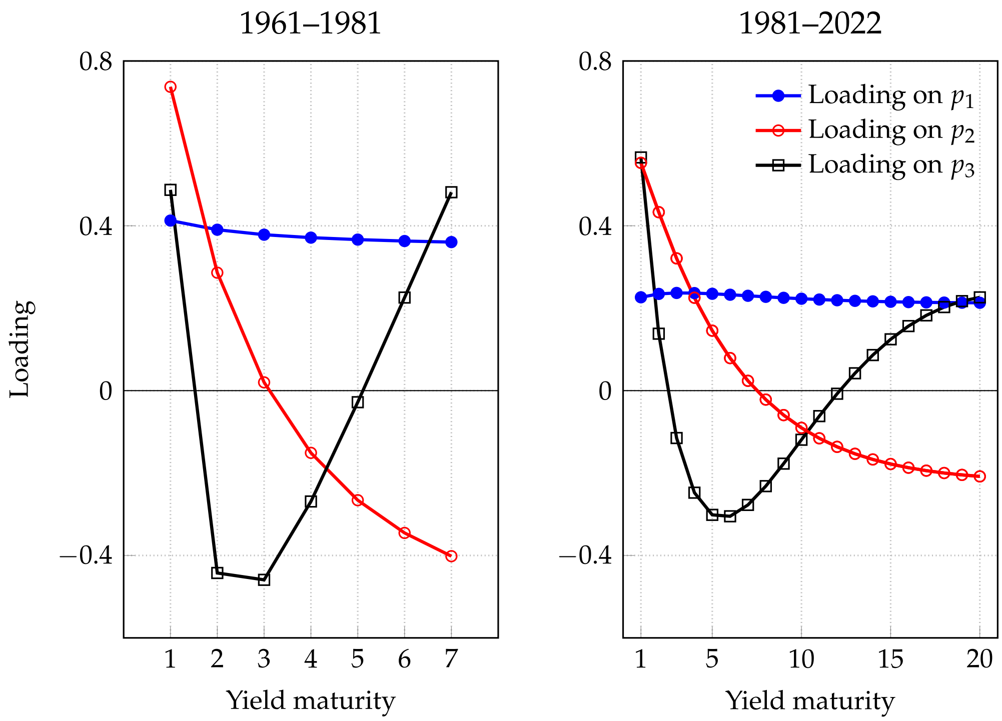

Figure 2: Loadings of yields on principal components

Notes. Loadings of yields on the first three principal components for the period 1961-1981 (left panel) and for the period 1981-2022 (right panel).

term structure factors.[7] In practice, we identify this equity-specific factor as the residual in a linear regression of the demeaned log price-dividend ratio $z_t := \log(S_t/Y_t)$ on the demeaned principal components p_t of the yield curve:

$$z_t - \bar{z} = \beta(\mathrm{p}_t - \overline{\mathrm{p}}) + h_t$$

This two-step procedure identifies our state variable x_t as the four dimensional vector process

$$x_t := \begin{pmatrix} \mathrm{p}_t - \overline{\mathrm{p}} \\ h_t \end{pmatrix}$$

[7]The fact that all the state variables relevant for the state of the economy (inflation, consumption, and dividend growth) can be extracted from the cross-section of asset prices is standard in macro-finance. See for example Bansal and Yaron (2004).

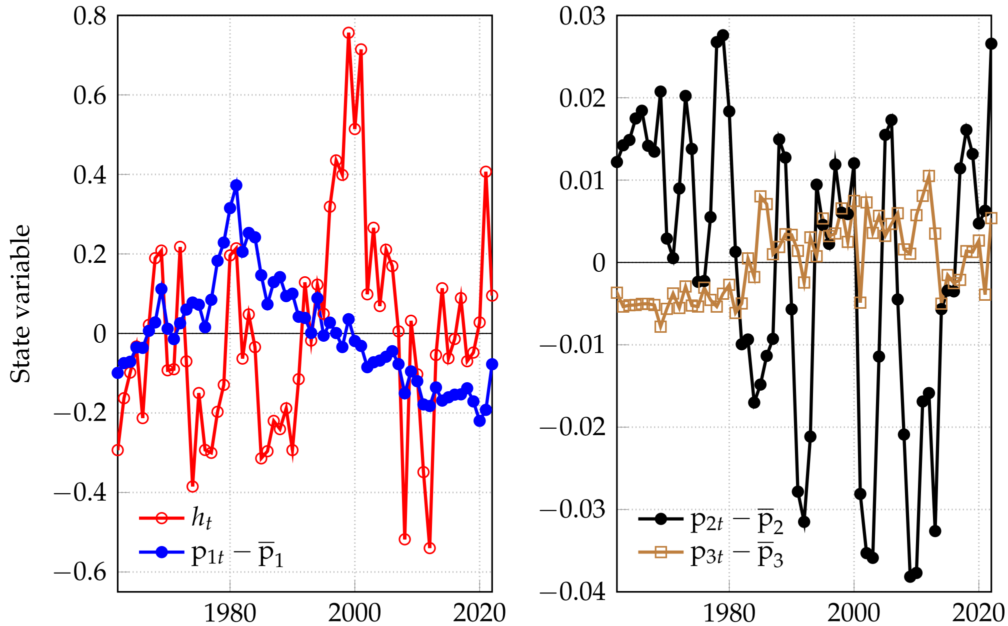

Figure 3: Implied time series of the state variables

Notes. The left panel shows the time series of the equity specific state variable and the first demeaned principal component. The right panel shows the time series on the second and third demeaned principal component.

Given the time series of x_t in Figure 3 we estimate the parameters in (17) using a standard vector auto-regression (VAR) that we write as:

$$\mathrm{p}_{t+1} = \overline{\mathrm{p}} + \Phi_p \left(\mathrm{p}_t - \overline{\mathrm{p}}\right) + \sigma_p \epsilon_{pt+1},$$
$$h_{t+1} = \phi_h h_t + \phi_{hp} \left(\mathrm{p}_t - \overline{\mathrm{p}}\right) + \sigma_{hp}^\top \epsilon_{pt+1} + \sigma_h \epsilon_{h,t+1}.$$

Finally, to estimate the risk-premium parameters (λ_0, λ_x) of the SDF, we use an approach proposed by Adrian et al. (2013) and Diez de los Rios (2015). Specifically, we note that due to the Campbell-Shiller approximation the log price- dividend ratio $z_t = \log S_t / Y_t$ and the zero coupon yields

$$Y_t^{t+n} := -\frac{1}{n} \log P_t^{t+n}, \qquad n = 1, \dots, 20,$$

are affine in x_t with constant loadings given by $(\overline{z}, z_x)$ and $\frac{1}{n}(A^n, B^n)_{n=1}^{20}$. Therefore, since the state-vector is observable, we can estimate these loadings using a cross-sectional

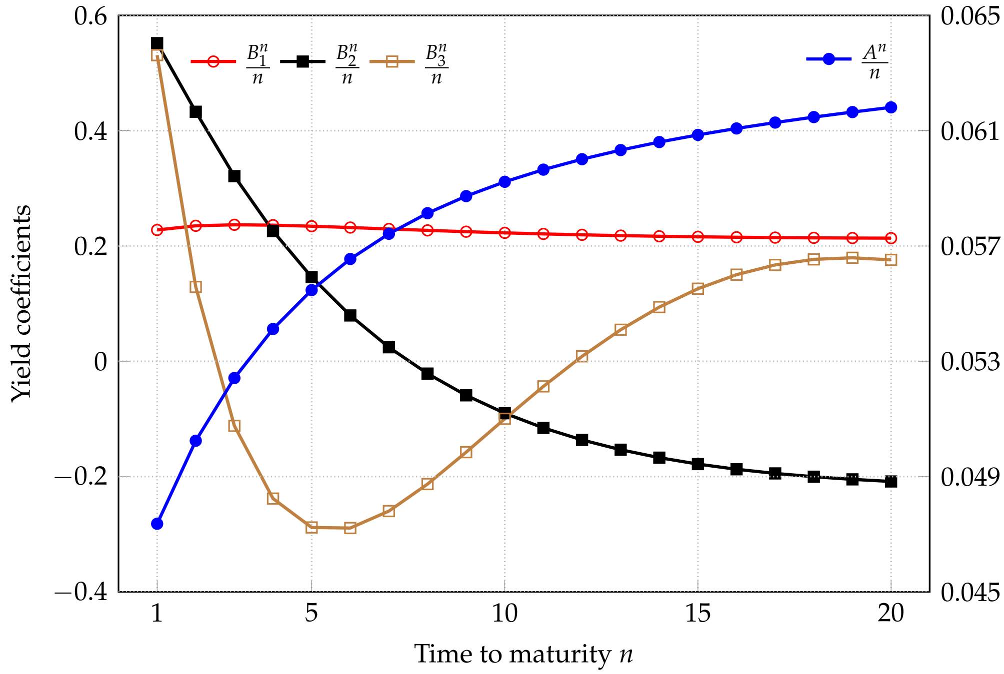

Figure 4: Loading of yields on the state-vector

Notes. The solid line with bullets (scale on the right) represents the loading $\frac{1}{n}A^n$ as a function of time-to-maturity n. The three other (scale on the right) represent the three components of the vector of loadings $\frac{1}{n}B^n$ as functions of time-to-maturity n.

(panel) regression. The resulting estimates of $(\bar{z}, z_x)$ are presented in Table 4 while those of the loadings $\frac{1}{n}(A^n, B^n)_{n=1}^{20}$ are plotted in Figure 4. In particular, the curve labelled $\frac{1}{n}A^n$ gives the unconditional mean of the term structure over the sample.

Within the model (as shown in the appendix), these loadings are nonlinear functions of the risk-premium parameters, given the estimated VAR parameters of the state-vector. We can thus estimate (λ_0, λ_x) to best fit the estimated loadings using the asymptotic nonlinear least squares estimator of Gourieroux, Monfort, and Trognon (1985) as proposed by Diez de los Rios (2015). See Appendix C for further details on the estimation method and Table 5 for the resulting point estimates.

4.4 Dynamics of log Debt to Consumption

Figure 5 plots the yearly time series of the log ℓ_t of outstanding government debt D_t to aggregate consumption C_t from 1942 to 2022. It is clear that this process is slow as

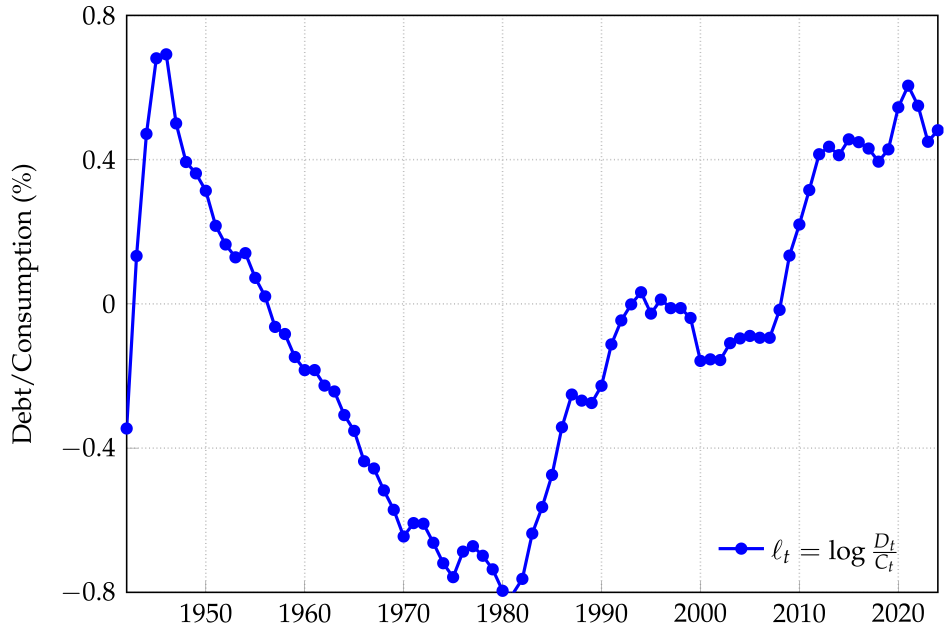

Figure 5: Log of Debt to Consumption, 1942–2022.

well as persistent and that the estimation of its long-run dynamics and mean-reversion properties are very much dependent on the length of the time-series considered. For example, it is mostly decreasing between 1950 and 1980 and mostly increasing after 1980. For our purposes we thus consider the longest time series available.

Due to data limitations—we have debt and aggregate consumption data starting from 1942 but yield curve data only starting from 1961—we cannot directly estimate the VAR specified in (18) for the entire time series. Instead, we use as regressors the yield on a 3-month zero-coupon bond $Y_t^{t+0.25}$ and the log price-dividend ratio z_t, which are both affine in the state-variables. We therefore estimate the process

$$\ell_{t+1} = \overline{\ell} + \sum_{i=1}^{L} \phi_i \left(\ell_{t+1-i} - \overline{\ell} \right) + \mu_{\ell y} \left(Y_t^{t+0.25} - \overline{Y}^{0.25} \right) + \mu_{\ell z} \left(z_t - \overline{z} \right) + u_{t+1}$$

where $\overline{Y}^{0.25}$ is the average T-bill rate. Regardless of the number of lags L, we find that the coefficients $\mu_{\ell y}$ and $\mu_{\ell z}$ are not significantly different from 0. We therefore disregard the

Lags	Estimates ($p-$values)			
	ϕ_1			
$L = 1$	1.012			
	(0.000)			
	ϕ_1	ϕ_2		
$L = 2$	1.455	-0.495		
	(0.0000)	(0.0000)		
	ϕ_1	ϕ_2	ϕ_3	
$L = 3$	1.438	-0.582	0.107	
	(0.0000)	(0.003)	(0.324)	
	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$L = 4$	1.410	-0.521	-0.024	0.098
	(0.0000)	(0.013)	(0.910)	(0.402)

Table 1: Estimation of (28) using different numbers of lags

associated regressors and reestimate the pure autoregressive process:

$$\ell_{t+1} = \overline{\ell} + \sum_{i=1}^{L} \phi_i \left(\ell_{t+1-i} - \overline{\ell} \right) + u_{t+1}. \tag{28}$$

In Table 1 we show the results of this estimation with a number of lags between one and four. As can be seen from the table the first two lag coefficients are strongly significant. Therefore, our final empirical specification for the process is:

$$\ell_{t+1} = \overline{\ell} + \phi_1 \left(\ell_t - \overline{\ell} \right) + \phi_1 \left(\ell_{t-1} - \overline{\ell} \right) + u_{t+1} \tag{29}$$

The basic stationarity conditions for ℓ_t are satisfied given the point estimates in Table 1 since both $\phi_1 \mp \phi_2$ and $|\phi_2|$ are strictly smaller than one. To further explore whether the process is stationary, we perform an augmented Dickey-Fuller (ADF) test by estimating the regression

$$\Delta\ell_{t+1} := \ell_{t+1} - \ell_t = \alpha + \gamma\ell_t + \delta_1 \Delta\ell_t + \tilde{u}_{t+1}. \tag{30}$$

Given the assumed AR(2) dynamics in (29), we have that $\gamma = \phi_1 + \phi_2 - 1$ and $\delta_1 = -\phi_2$. Therefore, we should have $\gamma < 0$ if the process is stationary. The ADF test is a one-sided

Test	γ	$p-$value	$t-$stat	Critical value (95%)	Reject $\gamma \geq 0$
ADF	-0.0309	0.261	-2.082	-2.898	No
ADF-GLS	-0.0308	0.037	-2.082	-1.944	Yes

Table 2: ADF and ADF-GLS stationarity test results

$t-$test of the null hypothesis that $\gamma \geq 0$, using the tabulated distribution of the ADF statistic generated under the null hypothesis of a unit-root process.

As can be seen from the first row of Table 2 the ADF test fails to reject the hypothesis that $\gamma \geq 0$ at 95% confidence level, that is it fails to reject non-stationarity. However, the ADF test is notoriously weak especially in the presence of an unknown mean or time trend and with a short time series. To mitigate this problem, we further implement the more efficient ADF-GLS test developed by Elliott, Rothenberg, and Stock (1996), which consists in running the regression (30) on the demeaned variable while restricting the constant α to be zero. As can be seen from the second row of the Table 2 the augmented test does reject the null hypothesis of non-stationarity at the 95% confidence level.

Overall, the estimation results in this section are consistent with the fact that the process ℓ_t is stationary, although the econometric evidence is not strong. Bohn (1998) argues that standard unit root tests are insufficient to test for stationarity in the log debt to consumption ratio as they likely suffer from a missing variables problem. Instead, he focuses on the behavior of the log surplus to consumption ratio $\log\left(\mathbf{s}_t/C_t\right)$ and finds strong evidence that this process responds in a debt-stabilizing way to shocks affecting ℓ_t. Based on his findings he argues that ℓ_t is a stationary process. Campbell, Gao, and Martin (2023) offer a new perspective on this point. They also document that standard ADF tests do not reject the null of non-stationarity, but breaking down the components of surplus into tax and government spending, they find evidence that government spending reacts in a stabilizing way to debt to GDP shocks.

Finally, to estimate the covariance structure between ℓ_t and the state-vector x_t and, thereby, identify $\epsilon_{\ell,t}$, we regress the residual u_t of the AR(2) process in (29) on the innovations of the state-vector $\hat{\epsilon}_{\mathrm{p},t}$:

$$u_t = \sigma_{\ell\mathrm{p}}\hat{\epsilon}_{\mathrm{p}t} + \sigma_\ell \epsilon_{\ell t}$$

for the period starting in 1961. Given our specification, the innovations ϵ_{p} to the principal component are orthogonal to the innovations, ϵ_h, ϵ_c, ϵ_ℓ, ϵ_y, and ϵ_π to the equity-specific factor h, consumption, debt, dividends and inflation. However, we allow for a general correlation structure for the latter shocks. The estimation results reported in Table 4 show that only three correlation coefficients are significant at a 95% confidence level: $\rho_{c\ell}$, estimated at -69%, ρ_{cy}, estimated at 41%, and $\rho_{c\pi}$, estimated at 44%.

4.5 The return to government debt

As described in Section 3.2, we model government debt as a portfolio with constant weights in zero-coupon bonds of maturities ranging from 1 to 20 years:

$$R^D_{t+1} = e^{r_t} + \sum_{m=1}^{20} \omega_m \left(\frac{P^{t+m}_{t+1}}{P^{t+m}_t} - e^{r_t} \right) \tag{31}$$

where ω^m is the portfolio weight for the m-year bond. In principle, we could estimate the weights by running a time-series regression of the excess return $ER^D_{t+1} := R^D_{t+1} - e^{r_t}$ on the excess returns

$$ER^m_{t+1} := \frac{P^{t+m}_{t+1}}{P^{t+m}_t} - e^{r_t}.$$

of the zero-coupon bonds. However, since the returns on zero-coupon bonds of different maturities are highly correlated, that regression would suffer from a multi-collinearity problem. Noting that bond returns are spanned by the three principal component factors, we instead make the simplifying assumption that the government maintains constant weighs $(\tilde{\omega}_i)^3_{i=1}$ in three factor-portfolios that hold bonds of all maturities with weights equal to the loadings of the corresponding principal component. The excess return on the i-th factor-portfolio is thus defined as

$$ER^{\mathcal{P}_i}_t = \sum_{m=1}^{20} w_{im} ER^m_t$$

where the weight w_{im} is the loading of the i-th principal component on the m-maturity bond. Equipped with this definition we then find the weights $(\tilde{\omega}_i)^3_{i=1}$ by running the

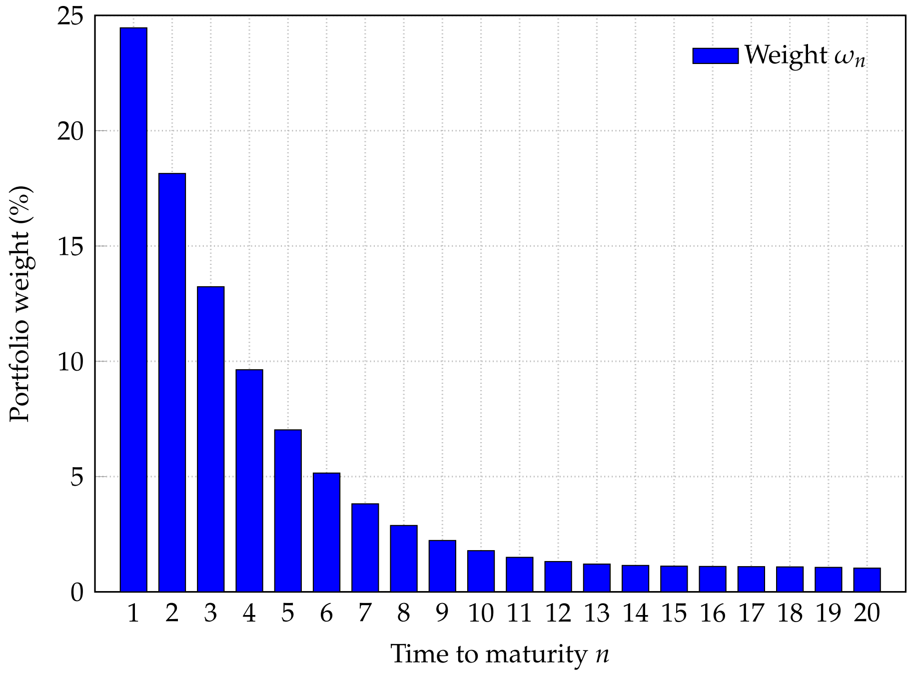

Figure 6: Estimated portfolio weights

regression

$$ER^{D}_{t+1} = \sum_{i=1}^{3} \tilde{\omega}_i ER^{\mathcal{P}_i}_{t+1} + \tilde{u}_{t+1}$$

and finally obtain the weights in (31) as $\omega_m = \sum_{i=1}^{3} \tilde{\omega}_i w_{im}$. The resulting weights are shown in Figure 6 and imply an average maturity of 4.7 years (around 56 months) for outstanding government debt.

4.6 Does the TVC hold?

Having estimated the model we now check whether the TVC is satisfied at our point estimates. To do so, we plot the closed form (27) for the present value of debt

$$G_{t,T} := E_t \left[\frac{M_T}{M_t} D_T \right] = C_t \exp \left\{ H^T_{0t} + x_t^\top H^T_{xt} + \sum_{i=1}^{2} H^T_{it} \left(\ell_{t+1-i} - \bar{\ell} \right) \right\} \tag{32}$$

as a function of the horizon T at the initial time $t = 0$. As can be seen from the top

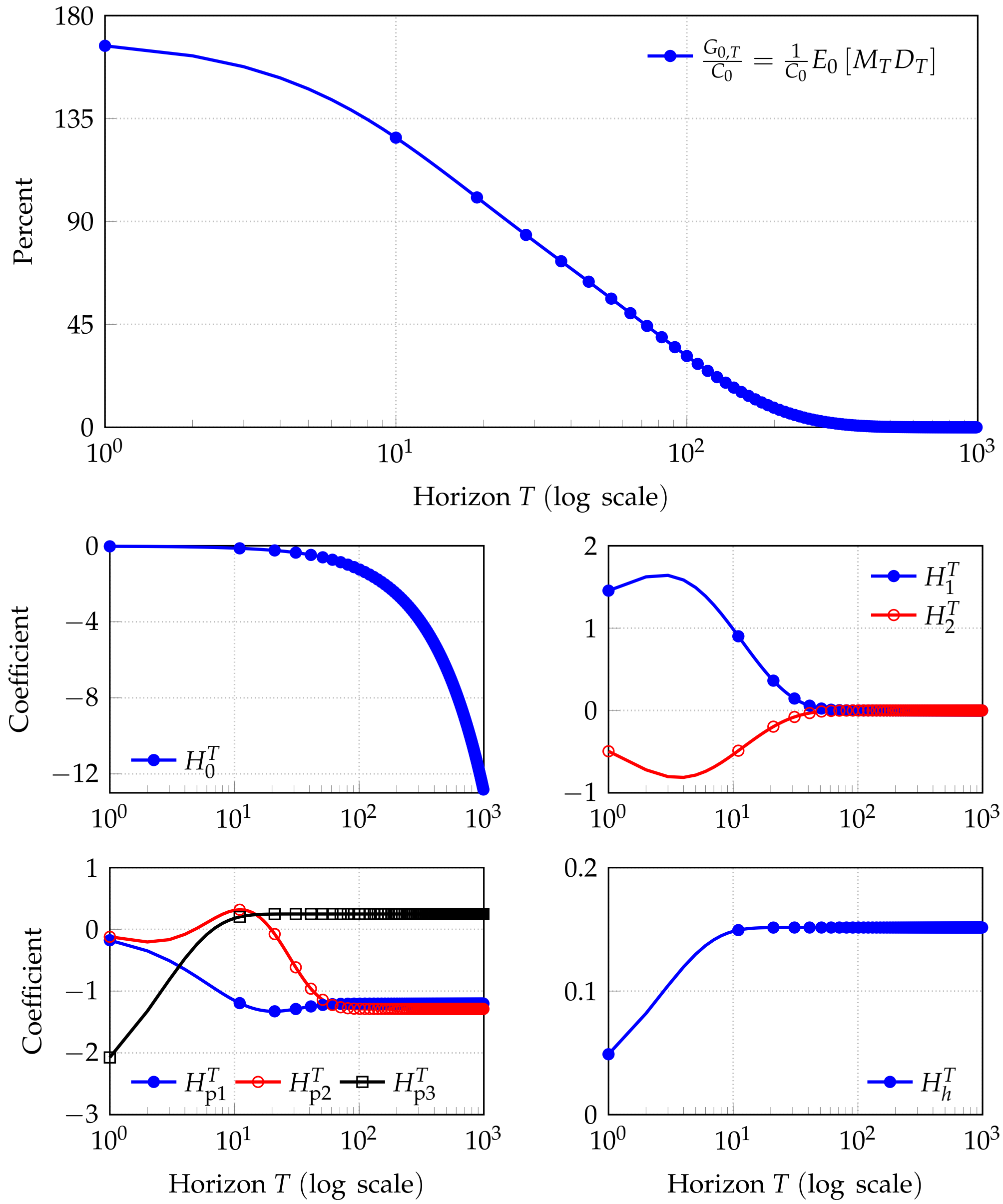

Figure 7: Present value of debt

Notes. The top panel plots the present value of debt scaled by consumption as a function of the horizon on a log scale. The bottom four panels plot the coefficients H_{00}^T, (H_{10}^T, H_{20}^T), H_{p0}^T, and H_{h0}^T in (32) as functions of the horizon on a log scale

panel of Figure 7 this present value clearly decays to zero, albeit relatively slowly, as the horizon increases. Therefore, we conclude that the TVC and, thus also, the debt valuation equation are satisfied in our estimated model. Importantly, this conclusion is independent

T		ℓ_0	$\frac{D_0}{C_0}$	$\sum_{\tau=1}^{T} \frac{\mathbf{s}_\tau}{C_\tau}$	$\sum_{\tau=1}^{T} g_\tau^C$	$\sum_{\tau=1}^{T} \log R_\tau^D$	ℓ_T	$\frac{D_T}{C_T}$
Panel A: All values								
10 Years		0.503	165%	0.040	0.601	0.404	0.230	126%
				(0.001)	(0.000)	(0.000)	(0.001)	
20 Years		0.503	165%	0.014	1.233	0.888	0.075	108%
				(0.001)	(0.001)	(0.001)	(0.001)	
Panel B: Low and High scenarios								
10 Years	Low	0.503	165%	0.499	0.653	0.438	−0.185	83%
	High	0.503	165%	−0.539	0.509	0.392	0.646	191%
20 Years	Low	0.503	165%	0.503	1.322	1.005	−0.390	68%
	High	0.503	165%	−0.612	0.848	0.538	0.457	171%

Table 3: Terms in the decomposition

Notes. The top panel presents the mean and standard deviation of the terms in (33) across 100,000 simulations of the calibrated economy over a ten and a twenty year horizon. The bottom panel presents the mean of the terms in (33) across those simulations where the debt to consumption ratio at the horizon falls in either the bottom quartile (Low) or the top quartile (High).

of the initial value of the state variable. Indeed, and as shown by the bottom panels of the figure, the coefficient $H_{0t}^{T} \equiv H_{00}^{T-t}$ diverges to minus infinity as $T - t \to \infty$ whereas all the other coefficients converge to a finite value.

However, it is important to note that the conclusion strongly depends on the estimates of the risk-premium parameters. In particular, since the estimated average growth rate $\mu_C = 0.065$ is larger than the estimated average short rate $\rho_0 \equiv A^1 \approx 0.04736$ it is clear that the TVC does not hold if risk-premia are set to zero.

4.7 Decomposition implied by the model

Cochrane (2022) (see also Campbell et al. (2023)) derives an approximate decomposition of the current debt to consumption ratio into conditional expected components linked to future consumption growth, debt returns, and surpluses. Specifically, he shows that the following holds approximately ex-post and therefore also ex-ante in expectations (see

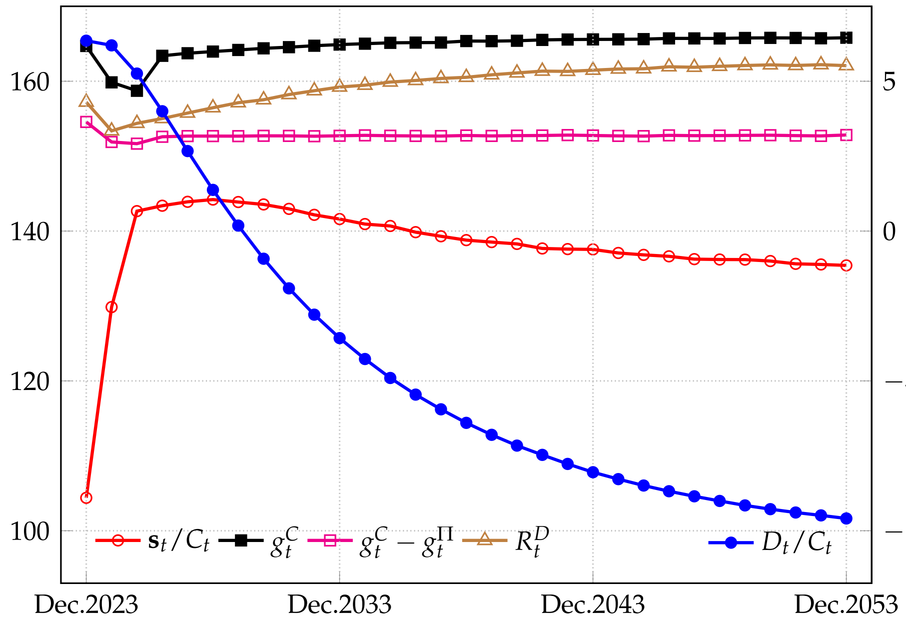

Figure 8: Expected path of the economy

Notes. Expected paths of debt over consumption (left scale in %) and surplus over consumption, nominal and real consumption growth, and government debt return (right scale in %) obtained from 100,000 simulated paths of our calibrated economy over thirty years starting in December 2023.

Appendix A for details):

$$\ell_t \approx \sum_{\tau=t+1}^{T} \left\{ \frac{\mathbf{s}_\tau}{C_\tau} + g_\tau^C - \log R_\tau^D \right\} + \ell_T \tag{33}$$

To illustrate this relation we simulate 100,000 paths of our calibrated economy of our economy over thirty years taking the initial values of all processes equal to those of December 2023. The results are presented in the top panel of Table 3 which reports the summary statistics of each term across all simulations and Figure 8 which plots the path followed by the economy averaged across simulations

Quantitatively, the model predicts that the debt to consumption ratio, which starts from 165% in December 2023, is entering a long phase of decrease that will on average lead it to 126% ten years from now and to 108% within twenty years. In the 'representative path' obtained by averaging across simulations, the surplus to consumption ratio starts from the realized value of −8.2% in 2023 then increases rapidly to become mildly positive

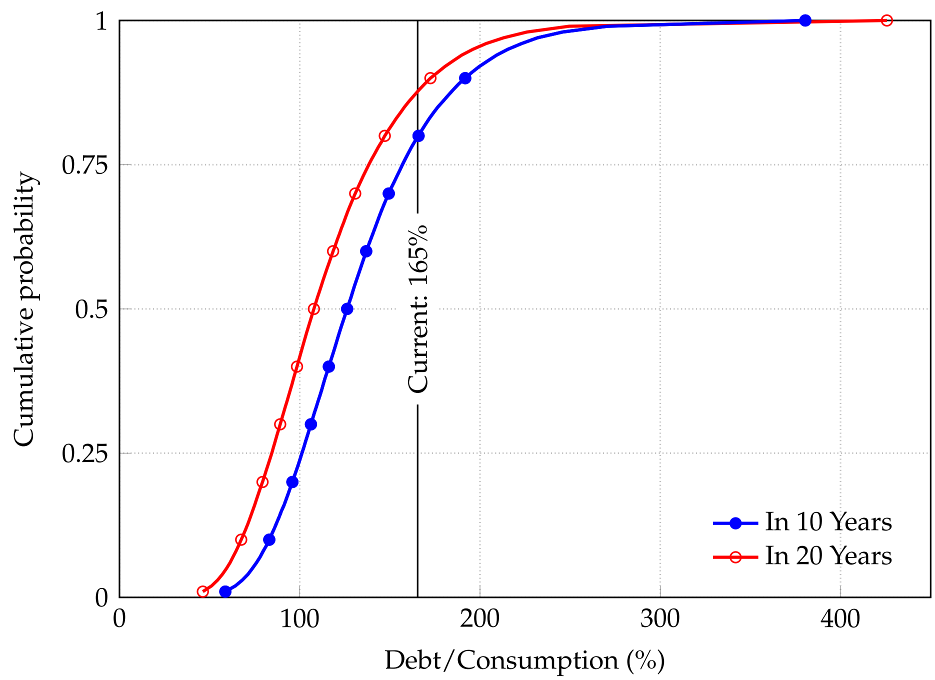

Figure 9: Simulated distribution of debt over consumption

Notes. This figure plots the simulated distribution of the ratio of outstanding government to consumption in ten and twenty years.

for a decade between years 3 and 13, and then decreases towards a negative long term mean of about −1%. The latter scenario of course relies on the assumption that the long-term mean of the nominal consumption growth rate—estimated at about 6.5%—continues to exceed (by around 1% on average) the return on government debt.

Notice that the decomposition (33) actually holds path-by-path. To illustrate how much future debt and surpluses may differ from the average values in shown in the top panel of Table 3 we perform two additional exercises. First, we plot in Figure 9 the simulated distribution of the debt to consumption ratio in ten years and in twenty years. Second, we report in the bottom panel of Table 3 the same terms as in the top panel albeit in two different scenarios: a Low scenario where debt over consumption at the selected horizon falls in the bottom quartile of the simulated distribution and a High scenario where it falls in the top quartile.

5 Conclusion

We show that valuing the stock of outstanding government debt as the risk-adjusted present value of future primary surpluses within a no-arbitrage model is either a tautology (if the TVC is satisfied), or an absurdity (if the TVC is not satisfied). In particular, if the TVC fails relative to a given SDF then the present value of future surpluses relative to that SDF differs from the value of debt by precisely the violation of the TVC. The present value of future surpluses can even turn negative (as in JLNX) which violates the fact, absent trivial arbitrage, that the value of outstanding government is positive and equal to the present value of its future cash flows which are fixed as well as positive and have finite maturities.

In addition, we show that ad-hoc specifications of surplus processes will typically not be admissible, in the sense that the total accumulated government debt (via the accounting identity) will not satisfy the TVC relative to a given valid SDF for any values of the parameters. As a consequence, the approach that consists in specifying reduced form dynamics to fit historical surplus data and evaluating the current stock of government debt using a present value relation, by analogy to valuing a stock as a present value of its future dividends, is almost guaranteed to fail. In particular, we provide a simple example where this approach automatically fails and thus generates a gap between the observed value of debt and the present value of future surpluses. The existence of such a valuation gap is what JLNX refer to as the 'government debt puzzle'.

Some authors have proposed macrofinance models where a violation of the TVC can arise in equilibrium due to some frictions (e.g., Brunnermeier et al. (2022a), Dumas et al. (2022)). Instead, we show that a surplus process that is specified to be admissible, in the sense of being consistent with both the budget constraint of the government and an accumulated debt process that satisfies the TVC, can fit historical data on debt and surplus dynamics while also fitting bonds and equity returns quite well. By construction, such an admissible surplus process does not generate any valuation gap. Simulating the calibrated model forward from the state of the economy in December 2023, predicts that debt to consumption should be expected to decrease over the next 10 years and that surpluses should turn positive for some period of time before returning persistently into negative territory. Thus, the estimated model is consistent with long-term stationary debt to consumption ratios and persistently negative surpluses, because risk-free rates are on average lower than the growth rate of the economy.

The point estimates of the model parameters are consistent with the ratio of debt to consumption being stationary and the TVC being satisfied. However, the econometric evidence for stationarity is weak and the standard errors of the risk-premium parameters are large. This suggests that explicitly modeling and measuring agents' belief uncertainty about the long-term stationarity of the debt to consumption ratio and its interaction with their risk-preferences could be fruitful avenues for future research.

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A Proofs

Proof of Equation (10). Since ϵ_{t+1} is independent from $\mathcal{F}_t$ and follows a standard normal distribution we have that

$$E_t\left[e^{-\lambda_{t+1}\epsilon_{t+1}-\frac{1}{2}\lambda_{t+1}^2}\epsilon_{t+1}\right] = -\lambda_{t+1} \tag{34}$$

for any predictable process $(\lambda_t)_{t\in\mathbf{N}}$. Combining this property with (7) and the definition of the SDF (9) shows that

$$\begin{aligned} E_t\left[\frac{M_{t+1}}{M_t}\mathbf{s}_{t+1}\right] &= E_t\left[\frac{M_{t+1}}{M_t}\left(e^{-\kappa}\mathbf{s}_t+\epsilon_{t+1})\right)\right] \\ &= e^{-(r+\kappa)}\mathbf{s}_t + e^{-r}E_t\left[e^{-\lambda\epsilon_{t+1}-\frac{1}{2}\lambda^2}\epsilon_{t+1}\right] = e^{-(r+\kappa)}\mathbf{s}_t - e^{-r}\lambda. \end{aligned}$$

Iterating this relation gives

$$E_t\left[\frac{M_{t+n}}{M_t}\mathbf{s}_{t+n}\right] = e^{-(r+\kappa)n}\mathbf{s}_t - e^{-rn}\lambda\sum_{m=0}^{n-1}e^{-\kappa m}$$

and summing over n on both shows that

$$\begin{aligned} V_t(M) = \sum_{n=1}^{\infty}E_t\left[\frac{M_{t+n}}{M_t}\mathbf{s}_{t+n}\right] &= \sum_{n=1}^{\infty}e^{-(r+\kappa)n}\mathbf{s}_t - \sum_{n=1}^{\infty}e^{-rn}\lambda\sum_{m=0}^{n-1}e^{-\kappa m} \\ &= \frac{\mathbf{s}_t}{e^{r+\kappa}-1} - \frac{e^{r+\kappa}\lambda}{(e^r-1)(e^{r+\kappa}-1)} \end{aligned}$$

which is the required expression. ■

Proof that Assumption 2 holds in Section 2.6. Equation (7) implies that

$$|\mathbf{s}_{t+1}| \leq e^{-\kappa}|\mathbf{s}_t| + |\epsilon_{t+1}|$$

and therefore

$$
\begin{aligned}
E_t\left[\frac{M_{t+1}}{M_t}|\mathbf{s}_{t+1}|\right] &\leq e^{-(r+\kappa)}|\mathbf{s}_t| + E_t\left[\frac{M_{t+1}}{M_t}|\epsilon_{t+1}|\right] \\
&= e^{-(r+\kappa)}|\mathbf{s}_t| + e^{-\left(r+\frac{1}{2}\lambda^2\right)}\sqrt{\frac{2}{\pi}} + e^{-r}\lambda \mathrm{Erf}\left(\frac{\lambda}{\sqrt{2}}\right) \\
&\leq e^{-(r+\kappa)}|\mathbf{s}_t| + e^{-r}\Lambda
\end{aligned}
$$

where Erf denotes the error function and the equality follows from the fact that ϵ_{t+1} is independent from $\mathcal{F}_t$ and follows a standard normal distribution. Iterating this relation forward gives

$$
E_t\left[\frac{M_{t+n}}{M_t}|\mathbf{s}_{t+n}|\right] \leq e^{-n(r+\kappa)}|\mathbf{s}_t| + e^{-rn}\Lambda\sum_{m=0}^{n-1} e^{-\kappa m}
$$

and the desired result now follows by summing over $n \geq 1$ on both sides and using the fact $r, \kappa > 0$. ■

Proof of Equation (14). The accounting identity (8), the evolution of the surplus (12), the definition of the SDF (13), and (34) jointly imply that

$$
\begin{aligned}
E_t\left[\frac{M_{t+1}}{M_t}\mathbf{s}_{t+1}\right] &= E_t\left[\frac{M_{t+1}}{M_t}\left(e^{-\kappa}\mathbf{s}_t + \phi_{\mathbf{s}}D_t + \epsilon_{t+1})\right)\right] \\
&= e^{-(r+\kappa)}\mathbf{s}_t + e^{-r}\phi_{\mathbf{s}}D_t - e^{-r}\lambda_{t+1} \\
&= e^{-(r+\kappa)}\mathbf{s}_t + e^{-r}\phi_{\mathbf{s}}D_t - e^{-r}(\lambda - \phi_M D_t) \\
&= e^{-(r+\kappa)}\mathbf{s}_t + e^{-r}(\phi_{\mathbf{s}} + \phi_M)D_t - e^{-r}\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
E_t\left[\frac{M_{t+1}}{M_t}D_{t+1}\right] &= E_t\left[\frac{M_{t+1}}{M_t}\left(e^{r}D_t - \mathbf{s}_{t+1}\right)\right] \\
&= D_t - E_t\left[\frac{M_{t+1}}{M_t}\mathbf{s}_{t+1}\right] \\
&= D_t - e^{-(r+\kappa)}\mathbf{s}_t - e^{-r}(\phi_{\mathbf{s}} + \phi_M)D_t + e^{-r}\lambda.
\end{aligned}
$$

Using the definition of the matrix Φ in (15) to rewrite these two equalities in vector form then shows that

$$E_t\left[\frac{M_{t+1}}{M_t}\begin{pmatrix}D_{t+1}\\ \mathbf{s}_{t+1}\end{pmatrix}\right]=\Phi\begin{pmatrix}D_t\\ \mathbf{s}_t\end{pmatrix}+e^{-r}\begin{pmatrix}\lambda\\ -\lambda\end{pmatrix}$$

and the required relation now follows by iterating this relation. ∎

Lemma 1 *The eigenvalues*

$$h_{\mp}:=\frac{1}{2}e^{-(r+\kappa)}\left(\zeta\mp\sqrt{\zeta^2-4e^{r+\kappa}}\right)\Big|_{\zeta=e^r+e^{-\kappa}-\phi}$$

of the matrix Φ *defined in* (15) *are such that* $\|h_{\mp}\|<1$ *if and only if* $0<\phi<2(e^r+e^{-\kappa})$

Proof. The expression of the eigenvalues follows from (15) by direct calculation. For the second part, consider first the region of the parameter space over which

$$\zeta^2<4e^{r+\kappa}\Leftrightarrow\zeta\in\left(-2e^{(r+\kappa)/2},2e^{(r+\kappa)/2}\right).$$

In this region, both eigenvalues are complex and $\|h_{\mp}\|=e^{-(r+\kappa)}<1$ since $r,\kappa>0$. Consider next the region over which

$$\zeta>0 \text{ and } \zeta^2\geq 4e^{r+\kappa}\Leftrightarrow\zeta\in\left[2e^{(r+\kappa)/2},\infty\right).$$

In this region the eigenvalues are real and such that $0<h_-<h_+$. Therefore, we only need to consider the second eigenvalue. A direct calculation shows that over that region the eigenvalue h_+ is strictly increasing in ζ with

$$h_+\left(2e^{(r+\kappa)/2}\right)=e^{-(r+\kappa)/2}<1=h_+\left(1+e^{r+\kappa}\right)<\lim_{\zeta\to\infty}h_+(\zeta)=\infty$$

and it follows that

$$|h_+|<1\Leftrightarrow\zeta\in\left[2e^{(r+\kappa)/2},1+e^{r+\kappa}\right).$$

Finally, consider the region over which

$$\zeta<0 \text{ and } \zeta^2\geq 4e^{r+\kappa}\Leftrightarrow\zeta\in\left(-\infty,-2e^{(r+\kappa)/2}\right].$$

Proceeding as in the previous case albeit with $-h_- = -h_-(\zeta)$ instead of $h_+(\zeta)$ shows that over that region $|h_+| < |h_-|$ and

$$|h_-| < 1 \Leftrightarrow \zeta \in \left(-1 - e^{r+\kappa}, -2e^{(r+\kappa)/2}\right].$$

Combining this with the conclusions over the other two regions shows that the required property holds if and only if $|\zeta| < 1 + e^{r+\kappa}$ and the result now follows from the relation between ζ and ϕ. ■

Proof of Equation (16). From the definition of ℓ_T we have:

$$E_t\left[M_T D_T\right] = E_t\left[M_T C_T e^{\ell_T}\right]$$

To compute this expectation it suffices to find a deterministic sequence $(a_t, b_t)_{t=0}^{T}$ with $a_T = 0$ and $b_T = 1$, such that $C_t M_t e^{a_t + b_t \ell_t}$ is a martingale, since the martingale property and boundary conditions imply that

$$C_t M_t e^{a_t + b_t \ell_t} = E_t[C_T M_T e^{a_T + b_T \ell_T}] = E_t[C_T M_T e^{\ell_T}]$$

The martingale property holds if and only if

$$C_t M_t e^{a_t + b_t \ell_t} = E_t[C_{t+1} M_{t+1} e^{a_{t+1} + b_{t+1}\ell_{t+1}}]$$

or, equivalently,

$$1 = E_t\left[e^{a_{t+1} - a_t + b_{t+1}\ell_{t+1} - b_t \ell_t + g^C_{t+1} + g^M_{t+1}}\right].$$

Since the random variables in the exponential are all normally distributed this condition can be written as

$$\begin{aligned} 0 &= a_{t+1} - a_t - b_t \ell_t + E_t\left[b_{t+1}\ell_{t+1} + g^C_{t+1} + g^M_{t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[b_{t+1}\ell_{t+1} + g^C_{t+1} + g^M_{t+1}\right] \\ &= a_{t+1} - a_t - b_t\ell_t + \mu_C - r + b_{t+1}\left((1-\phi)\,\bar{\ell} + \phi \ell_t\right) \\ &\qquad + \tfrac{1}{2}\left(\sigma_C - \lambda + b_{t+1}\sigma_\ell\right)^2 - \tfrac{1}{2}\left(\sigma_C^2 + \lambda^2\right). \end{aligned}$$

This linear conditions holds for all $t \in \mathbf{N}$ and $\ell_t \in \mathbf{R}$ if and only if

$$\begin{aligned} b_t &= \phi b_{t+1}, \\ a_t &= a_{t+1} + \mu_C - r + b_{t+1}(1-\phi)\,\overline{\ell} + \tfrac{1}{2}\left(\sigma_C - \lambda + b_{t+1}\sigma_\ell\right)^2 - \tfrac{1}{2}\left(\sigma_C^2 + \lambda^2\right). \end{aligned}$$

Combining these recursions with the definitions of (ρ, ℓ^*) and the boundary conditions then shows that

$$\begin{aligned} b_t &= \phi^{T-t} \\ a_t &= a_T - \sum_{\tau=t}^{T-1}\left(a_{\tau+1} - a_\tau\right) \\ &= -\rho\,(T-t) + (1-\phi)\,\ell^* \sum_{\tau=t}^{T-1} \phi^{T-(\tau+1)} + \frac{1}{2}\sigma_\ell^2 \sum_{\tau=t}^{T-1} \phi^{2(T-(\tau+1))} \end{aligned}$$

and the result follows by computing the sums. ■

Proof of Equation (23). Since $P_T^T = 1$ we have that the price of the zero coupon bonds satisfies

$$M_t P_t^T = E_t\left[M_T\right].$$

To compute the expectation it suffices to find a deterministic sequence $\left(A_t^T, B_t^T\right)_{t=0}^T$ with $A_T^T = 0$ and $B_T^T = \mathbf{0}$, such that $e^{-A_t^T - x_t^\top B_t^T} M_t$ is a martingale, since the martingale property and boundary conditions imply that

$$e^{-A_t^T - x_t^\top B_t^T} M_t = E_t\left[e^{-A_T^T - x_T^\top B_T^T} M_T\right] = E_t\left[M_T\right].$$

The desired process is a martingale if and only if

$$\begin{aligned} e^{-A_t^T - x_t^\top B_t^T} &= E_t\left[e^{-A_{t+1}^T - x_{t+1}^\top B_{t+1}^T} \frac{M_{t+1}}{M_t}\right] = E_t\left[e^{-A_{t+1}^T - x_{t+1}^\top B_{t+1}^T + g_{t+1}^M}\right] \\ &= \exp\left\{-E_t\left[A_{t+1}^T + x_{t+1}^\top B_{t+1}^T - g_{t+1}^M\right] + \frac{1}{2}\mathrm{Var}_t\left[x_{t+1}^\top B_{t+1}^T - g_{t+1}^M\right]\right\} \end{aligned}$$

where the last equality follows from the fact that $x_{t+1}^\top B_{t+1}^T - g_{t+1}^M$ is conditionally normal. Using the dynamics of (x_{t+1}, g_{t+1}^M) and the definition of (r_t, λ_t) as affine functions of x_t

then shows that the martingale property holds if and only if

$$A_t^T + x_t^\top B_t^T = \left(\rho_0 + x_t^\top \rho_x\right) + A_{t+1}^T + x_t^\top \Phi^\top B_{t+1}^T \\ - \left(\lambda_0 + \lambda_x x_t\right)^\top \sigma_x^\top B_{t+1}^T - \frac{1}{2}\|\sigma_x^\top B_{t+1}^T\|^2$$

This linear system holds for all $t \in \mathbf{N}$ and $x_t \in \mathbf{R}^4$ if and only if the sequence solves the difference equation

$$B_t^T = \rho_x + \left(\Phi - \sigma_x \lambda_x\right)^\top B_{t+1}^T$$
$$A_t^T = A_{t+1}^T + \rho_0 - \left(\sigma_x \lambda_0\right)^\top B_{t+1}^T - \frac{1}{2}\|\sigma_x^\top B_{t+1}^T\|^2.$$

subject to the boundary conditions $A_T^T = 0$ and $B_T^T = \mathbf{0}$. ■

Proof of Equation (27). From the definition of ℓ_T we have that

$$E_t\left[M_T D_T\right] = E_t\left[M_T C_T e^{\ell_T}\right].$$

Proceeding as the previous proofs shows that in order to compute the expectation on the right it suffices to find a deterministic sequence

$$\left\{\left(H_{0t}^T, H_{1t}^T, \ldots, H_{Lt}^T, H_{xt}^T\right) \in \mathbf{R} \times \mathbf{R} \times \cdots \times \mathbf{R} \times \mathbf{R}^{n_x}\right\}_{t=0}^T$$

with

$$H_{xT}^T = \mathbf{0} \text{ and } H_{iT}^T = \begin{cases} \bar{\ell}, & \text{for } i = 0 \\ 1, & \text{for } i = 1, \\ 0, & \text{for } i > 1. \end{cases} \tag{35}$$

such that the process

$$e^{H_{0t}^T + \sum_{i=1}^L H_{it}^T\left(\ell_{t+1-i} - \bar{\ell}\right) + x_t^\top H_{xt}^T} C_t M_t$$

is a martingale. Since all the variables are conditionally normal, the same argument as in previous proofs shows that this property is equivalent to

$$H_{0t}^T - H_{0t+1}^T + \sum_{i=1}^{L} H_{it}^T \left(\ell_{t+1-i} - \overline{\ell}\right) + x_t^\top H_{xt}^T = E_t\left[J_{t+1}\right] + \frac{1}{2}\mathrm{Var}_t[J_{t+1}] \tag{36}$$

where

$$J_{t+1} := g_{t+1}^M + g_{t+1}^C + \sum_{i=1}^{L} H_{it+1}^T \left(\ell_{t+2-i} - \overline{\ell}\right) + x_{t+1}^\top H_{xt+1}^T.$$

A straightforward calculation shows that

$$\begin{aligned} E_t\left[J_{t+1}\right] &= \mu_C - r_t + \sum_{i=2}^{L} H_{it+1}^T \left(\ell_{t+2-i} - \overline{\ell}\right) + x_t^\top \mu_{Cx} \\ &\quad - \frac{1}{2}\|\lambda_t\|^2 - \frac{1}{2}\|\sigma_C\|^2 + x_t^\top \Phi_x^\top H_{xt+1}^T + H_{1t+1}^T \left\{\sum_{i=1}^{L} \phi_i \left(\ell_{t+1-i} - \overline{\ell}\right) + x_t^\top \mu_{\ell x}\right\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}_t\left[J_{t+1}\right] &= \mathrm{Var}_t\left[g_{t+1}^M + g_{t+1}^C + x_{t+1}^\top H_{xt+1}^T + H_{1t+1}^T \ell_{t+1}\right] = \|\lambda_t\|^2 \\ &\quad + \left(\sigma_C + \sigma_x^\top H_{xt+1}^T + \sigma_\ell H_{1t+1}^T\right)^\top \left(\sigma_C + \sigma_x^\top H_{xt+1}^T + \sigma_\ell H_{1t+1}^T - 2\lambda_t\right). \end{aligned}$$

Substituting these expressions into (36) gives a system of linear equations that holds for all $t \in \mathbf{N}$ and $(x_t, \ell_t) \in \mathbf{R}^{n_x+1}$ if and only if

$$\begin{aligned} H_{it}^T &= H_{i+1t+1}^T + H_{1t+1}^T \phi_i \qquad \text{(for } i \neq 0\text{)}, \\ H_{xt}^T &= \mu_{\ell x} + \mu_{Cx} - \rho_x + \left(\Phi_x - \sigma_x \lambda_x\right)^\top H_{x,t+1}^T - \lambda_x^\top \left(\sigma_C + \sigma_\ell H_{1t+1}^T\right), \\ H_{0t}^T &= H_{0t+1}^T + \mu_C - \rho_0 - \frac{1}{2}\|\sigma_C\|^2 \\ &\quad + \frac{1}{2}\left(\sigma_C + \sigma_x^\top H_{xt+1}^T + \sigma_\ell H_{1t+1}^T\right)^\top \left(\sigma_C + \sigma_x^\top H_{xt+1}^T + \sigma_\ell H_{1t+1}^T - 2\lambda_0\right), \end{aligned}$$

and the desired result follows by solving this system of difference equations subject to the boundary conditions in (35). ∎

Proof of Equation (24). We first derive the exact solution for the stock price and then the log-linear approximation. Specifically

$$\frac{S_t}{Y_t} = E_t\left[\sum_{\tau=t+1}^{\infty} \frac{M_\tau}{M_t}\frac{Y_\tau}{Y_t}\right] = \sum_{\tau=t+1}^{\infty} e^{-\mathcal{A}_t^\tau - x_t^\top \mathcal{B}_t^\tau} \approx e^{\bar{z}+z_x^\top x_t}$$

For the former we need to compute

$$E_t\left[\frac{M_n Y_n}{M_t Y_t}\right]$$

and then sum over $n \geq t+1$ to obtain the price-dividend ratio. As in previous proofs, this can be achieved find a deterministic sequence

$$\{(\mathcal{A}_t^n, \mathcal{B}_t^n) \in \mathbf{R} \times \mathbf{R}^{n_x}\}_{t=0}^n$$

with boundary conditions $\mathcal{A}_n^n = 0$ and $\mathcal{B}_n^n = \mathbf{0}$ such that $e^{-\mathcal{A}_t^n - x_t^\top \mathcal{B}_t^n} M_t Y_t$ is a martingale on $[0, n]$. Relying on the conditional normality of the state variables then shows that this property holds if and only if

$$\mathcal{A}_t^n + x_t^\top \mathcal{B}_t^n = \mathcal{A}_{t+1}^n + \left(E_t + \frac{1}{2}\mathrm{Var}_t\right)\left[x_{t+1}^\top \mathcal{B}_{t+1}^n - g_{t+1}^Y - g_{t+1}^M\right]$$

and the same separation of variables argument as in previous proofs shows that the desired coefficients must solve

$$\begin{aligned}
\mathcal{B}_t^T &= \rho_x - \mu_{Yx} + \lambda_x^\top \sigma_Y + (\Phi_x - \sigma_x \lambda_x)^\top \mathcal{B}_{t+1}^T \\
\mathcal{A}_t^T &= \mathcal{A}_{t+1}^T + \rho_0 - \mu_Y + \frac{1}{2}||\sigma_Y||^2 - \frac{1}{2}\left(\sigma_Y - \sigma_x^\top \mathcal{B}_{t+1}^T\right)^\top \left(\sigma_Y - \sigma_x^\top \mathcal{B}_{t+1}^T - 2\lambda_0\right)
\end{aligned}$$

subject to the boundary conditions $\mathcal{A}_n^n = 0$ and $\mathcal{B}_n^n = \mathbf{0}$.

To derive the log-linear approximation reported in (24) of the main text we start from the no-arbitrage restriction

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{S_{t+1}+Y_{t+1}}{S_t}\right] = 1$$

which we rewrite in terms of the log price-dividend ratio $z_t := \log S_t/Y_t$ as

$$E_t\left[e^{g^M_{t+1}+g^Y_{t+1}-z_t}\left(e^{z_{t+1}}+1\right)\right]=1. \tag{37}$$

Following Campbell and Shiller (1988) we then approximate $e^{z_{t+1}}+1 \approx e^{\bar{c}+cz_{t+1}}$ where the constants

$$c = e^{\bar{z}}/\left(1+e^{\bar{z}}\right), \tag{38}$$

$$\bar{c} = -c\bar{z}+\log\left(1+e^{\bar{z}}\right). \tag{39}$$

With this approximation the no-arbitrage condition (37) reads

$$E_t\left[e^{-z_t+\bar{c}+cz_{t+1}+g^M_{t+1}+g^Y_{t+1}}\right]=1.$$

Guessing that $z_t = \bar{z}+x_t^\top z_x$ and using the conditional normality of the state variable together with arguments similar to those of the previous proofs then shows that the coefficients $(\bar{z}, z_x)$ in (24) solve

$$0 = \mu_{Yx}-\rho_x-z_x+c\left(\Phi_x-\sigma_x\lambda_x\right)^\top z_x-\lambda_x^\top\sigma_Y,$$
$$0 = \mu_Y-\rho_0-\frac{1}{2}||\sigma_Y||^2+\bar{c}+\bar{z}\left(c-1\right)+\frac{1}{2}\left(\sigma_Y+c\sigma_x^\top z_x\right)^\top\left(\sigma_Y+c\sigma_x^\top z_x-2\lambda_0\right),$$

subject to (38) and (39). ■

Proof of the decomposition (33). The accounting identity (AI) implies that

$$\log\left(\hat{\mathbf{s}}_{t+1}+e^{\ell_{t+1}}\right)=\ell_t-g^C_{t+1}+r^D_t$$

where we have set

$$\left(\hat{\mathbf{s}}_t, r^D_t\right) := \left(\mathbf{s}_t/C_t, \log R^D_t\right).$$

Taylor expanding the left-hand side around $(\hat{\mathbf{s}}_t, \ell_t) = (\mathbf{s}^*, \ell^*)$ shows that

$$q_0+q_\ell\ell_{t+1}+q_s\hat{\mathbf{s}}_{t+1}=\ell_t-g^C_{t+1}+r^D_{t+1}$$

with

$$q_0 = \log\left(\mathbf{s}^* + e^{\ell^*}\right) - q_\ell \ell^* - q_s \mathbf{s}^*$$
$$q_\ell = \frac{e^{\ell^*}}{\mathbf{s}^* + e^{\ell^*}}$$
$$q_s = \frac{1}{\mathbf{s}^* + e^{\ell^*}},$$

and iterating this relation forward we conclude that

$$\ell_t = \frac{q_0}{1-q_\ell} + \sum_{\tau=t+1}^{T} q_\ell^{\tau-t-1}\left(q_s \hat{\mathbf{s}}_\tau + g_\tau^C - r_\tau^D\right) + q_\ell^{T-t}\ell_T.$$

This expression shows that the current level of debt to consumption can be decomposed into a constant and four components related to the future expected paths of surpluses, consumption growth, debt returns, and future debt. Cochrane picks the points $\mathbf{s}^* = 0$ and $\ell^* = 0$ so that $q_0 = 0$, $q_\ell = q_s = 1$ to obtain the expression (33) in the main text. Note that in that case, it is not appropriate to take the limit at $T \to \infty$ as, for most parametrization, the sums on the right-hand-side will diverge. Still, the decomposition should hold approximately and be informative for finite horizons. ■

B Calibration of the model

Here we give detailed explanation about how the estimation of the parameters of the general multivariate affine model that we use in Section 4.

Because we want to allow for factors driven by priced shocks outside of the Treasury bond market, we parametrize the state-vector and the pricing kernel using a two-stage approach. First, we assume that a set of factors capture all bond-specific risks. We follow a large literature on yield-curve factor models and use the first three principal components of the yield curve as the observable term structure factors. Second, we assume that there is one equity-specific factor (h_t) that drives the pricing of stock specific risk. This factor is identified as the residual in a regression of the log-price to dividend ratio onto the term structure factors. Whence it is also directly observable.

B.1 Model specification

Let $\hat{\mathrm{p}}_t := \mathrm{p}_t - \bar{\mathrm{p}}_t$ denote the vector of demeaned principal components of the term structure. The specification of the general affine model that we take to the data is given by

$$\log \frac{M_{t+1}}{M_t} = m_{\mathrm{p}t+1} + m_{Yt+1}$$

where

$$\begin{aligned}
m_{\mathrm{p}t+1} &= -r_t - \frac{1}{2}\|\lambda_{\mathrm{p}t}\|^2 - \lambda_{\mathrm{p}t}^{\top}\epsilon_{\mathrm{p}t+1}, \\
r_t &= \rho_0 + \rho_x^{\top}\hat{\mathrm{p}}_t, \\
\lambda_{\mathrm{p}t} &= \sigma_{\mathrm{p}}^{-1}\lambda_{\mathrm{p}0} + \left(\sigma_{\mathrm{p}}^{-1}\lambda_{\mathrm{p}1}\right)\hat{\mathrm{p}}_t, \\
m_{Yt+1} &= -\frac{1}{2}\lambda_{Yt}^2 - \lambda_{Yt}\epsilon_{Yt+1}, \\
\lambda_{Yt} &= \lambda_{Y0} + \lambda_{Y1}h_t,
\end{aligned}$$

and

$$\hat{\mathrm{p}}_{t+1} = \Phi_{\mathrm{p}}\hat{\mathrm{p}}_t + \sigma_{\mathrm{p}}\epsilon_{\mathrm{p}t+1}, \tag{40}$$

$$h_{t+1} = \Phi_{h\mathrm{p}}\hat{\mathrm{p}}_t + \Phi_h h_t + \sigma_{h\mathrm{p}}^{\top}\epsilon_{\mathrm{p}t+1} + \sigma_h\epsilon_{ht+1}. \tag{41}$$

The above SDF is the product of the one period discount factor e^{-r_t} and two orthogonal positive martingales that price, respectively, bond-specific and stock-specific risks. This parametrization is nested in the general model presented in Section 3.2 but introduces the simplification that term structure factors 'granger-cause' the equity factor in the sense that h_t does not affect the dynamics of Treasury yields. This simplification is convenient as it allows for a two-step estimation procedure of the term structure and equity-market blocks of our asset pricing model.

The dynamics of the log debt to consumption ratio ℓ_t, consumption C_t, stock market dividends Y_t, and the price level Π_t are specified as:

$$\ell_{t+1} = \overline{\ell} + \sum_{i=1}^{2}\phi_i\left(\ell_{t+1-i} - \overline{\ell}\right)\sigma_{\ell\mathrm{p}}^{\top}\epsilon_{\mathrm{p}t+1} + \sigma_\ell\epsilon_{\ell t+1}$$

and

$$
\begin{aligned}
g^C_{t+1} &:= \log \frac{C_{t+1}}{C_t} = \mu_C - \tfrac{1}{2} V_C + \mu_{C\mathrm{p}}^\top \hat{\mathrm{p}}_t + \sigma_{C\mathrm{p}}^\top \epsilon_{\mathrm{p}t+1} + \sigma_C \epsilon_{Ct+1} \\
g^Y_{t+1} &:= \log \frac{Y_{t+1}}{Y_t} = \mu_Y - \tfrac{1}{2} V_Y + \mu_{Y\mathrm{p}}^\top \hat{\mathrm{p}}_t + \sigma_{Y\mathrm{p}}^\top \epsilon_{\mathrm{p}t+1} + \sigma_Y \epsilon_{Yt+1} \\
g^\Pi_{t+1} &:= \log \frac{\Pi_{t+1}}{\Pi_t} = \mu_\Pi - \tfrac{1}{2} V_\Pi + \mu_{\Pi\mathrm{p}}^\top \hat{\mathrm{p}}_t + \sigma_{\Pi\mathrm{p}}^\top \epsilon_{\mathrm{p}t+1} + \sigma_\Pi \epsilon_{\Pi t+1}
\end{aligned}
\tag{42}
$$

with the constants defined by

$$
V_J := \|\sigma_{J\mathrm{p}}\|^2 + \|\sigma_J\|^2, \qquad J \in \{C, Y, \Pi\}.
$$

We assume that the scalar normal shocks $\epsilon_C, \epsilon_Y, \epsilon_\Pi, \epsilon_\ell, \epsilon_h$ are independent from the vector ϵ_p of normal shocks driving the principal components of the term structure, but we allow them to have a potentially full correlation matrix.

B.2 Asset pricing formulae

In the above model, the prices of zero-coupon bonds and of the claim to the aggregate stock market dividend can be obtained in closed form as special cases of the formulae derived in Appendix A. For completeness, we provide here full details of the pricing formulae for this special case.

B.2.1 Zero-coupon bond prices

The price at date $t \geq 0$ of a zero-coupon bond paying \$1 at date T satisfies:

$$
P_t^T = E_t \left[\frac{M_{t+1}}{M_t} P_{t+1}^T \right]
$$

for all $t \leq T-1$ and $P_T^T = 1$. In particular, with $T = t+1$ we obtain that

$$
-\log P_t^{t+1} = r_t = \rho_0 + \hat{\mathrm{p}}_t^\top \rho_\mathrm{p}.
$$

On the other hand, denoting by

$$
rx_{t+1}^{n-1} = \log P_{t+1}^{t+n} - \log P_t^{t+n} - r_t
$$

the log-excess return on an n period bond between dates t and $t+1$, we can rewrite the no-arbitrage condition as:

$$1 = E_t\left[e^{m_{\text{p}t+1}+m_{Yt+1}+r_t+rx^{n-1}_{t+1}}\right].$$

To compute this expectation we conjecture (and later will verify) that

$$-\log P^{t+n}_t = A^n + \hat{\text{p}}_t^\top B^n$$

for all $n > 0$ and some deterministic sequence $(A^n, B^n) \in \mathbf{R}\times\mathbf{R}^3$. For $n = 1$ this conjecture requires that

$$(A^1, B^1) = \left(\rho_0, \rho_{\text{p}}\right) \tag{43}$$

To derive the equations satisfied by the coefficients for higher values of n we use the fact that $m_{Y,t}$ is independent of p_t together with the normality of $m_{\text{p}t+1}$ and

$$r_t + rx^{n-1}_{t+1} = A^n + \hat{\text{p}}_t^\top B^n - \left(A^{n-1} + \hat{\text{p}}_{t+1}^\top B^{n-1}\right)$$

to rewrite the no-arbitrage condition as

$$\begin{aligned}0 &= E_t\left[m_{\text{p},t+1} + r_t + rx^{n-1}_{t+1}\right] + \frac{1}{2}\text{Var}_t\left[m_{\text{p},t+1} + rx^{n-1}_{t+1}\right]\\ &= A^n - A^1 - A^{n-1}\\ &+ \hat{\text{p}}_t^\top\left(B^n - B_1 - \Phi_{\text{p}}^\top B^{n-1}\right) + \frac{1}{2}\|\sigma_{\text{p}}^\top B^{n-1}\|^2 + \left(\lambda_{\text{p}0} + \lambda_{\text{p}1}\hat{\text{p}}_t\right)^\top B^{n-1}.\end{aligned}$$

This linear equation holds simultaneously for all $n \geq 1$ and $\hat{\text{p}}_t \in \mathbf{R}^3$ if and only the parameter satisfy the difference equations

$$B^n = B^1 + (\Phi_{\text{p}} - \lambda_{\text{p}1})^\top B^{n-1}, \tag{44}$$

$$A^n = A^1 + A^{n-1} - \lambda_{\text{p}0}^\top B^{n-1} - \frac{1}{2}\|\sigma_{\text{p}}^\top B^{n-1}\|^2. \tag{45}$$

for all $n \geq 2$ subject to (43).

Given these expressions our estimation of the term structure block of the model runs in four steps. First, we extract the path of the demeaned principal components from the cross-section of yields. Second, we run a VAR on the principal components to estimate the parameters $(\overline{p}, \Phi_{\mathrm{p}}, \sigma_{\mathrm{p}})$. Third, we use a panel regression of yields on the principal components to estimate the coefficients $(A^n, B^n)_{n=1}^{20}$. Fourth, we use the asymptotic least squares (ALS) method of Gourieroux, Monfort, and Trognon (1985) as implemented by Diez de los Rios (2015) to estimate the rik premia parameters $(\lambda_{\mathrm{p}0}, \lambda_{\mathrm{p}1})$ by exploiting that the coefficient $(A^n, B^n)_{n=1}^{20}$ must satisfy (44)–(45) given the estimated dynamics of the principal components. See Appendix C for a description of this method.

B.2.2 The price-dividend ratio of the stock market

Following Campbell and Shiller (1988) we approximate the log return on stock market as

$$ry_{t+1} := \log\left(1 + \frac{Y_t}{S_t}\right) \approx g^Y_{t+1} - z_t + \bar{c} + c\, z_{t+1}$$

for some normally distributed scalar variable z_t to be determined. Given this approximation, the no-arbitrage condition applied to the stock market portfolio writes as

$$\begin{aligned}
0 &= \log E_t\left[e^{m_{Yt+1} + m_{\mathrm{p}t+1} + ry_{t+1}}\right] \\
&= E_t\left[e^{m_{Yt+1} + m_{\mathrm{p}t+1} + ry_{t+1}}\right] + \frac{1}{2}\mathrm{Var}_t\left[e^{m_{Yt+1} + m_{\mathrm{p}t+1} + ry_{t+1}}\right] \\
&= E_t\left[ry_{t+1}\right] - r_t + \frac{1}{2}\mathrm{Var}_t\left[ry_{t+1}\right] + \mathrm{Cov}_t\left[m_{Yt+1} + m_{\mathrm{p}t+1}, ry_{t+1}\right] \qquad (46)
\end{aligned}$$

Because of the Campbell-Shiller approximation, we conjecture that there exists an affine solution to this equation of the form:[8]

$$z_t = \overline{z} + z_{\mathrm{p}}^{\top} \hat{\mathrm{p}}_t + h_t$$

[8]The Campbell-Shiller approximation and the exponential affine SDF jointly imply that the price dividend ratio is exponential affine. Without this approximation, the price can be solved exactly as an infinite sum of exponential affine dividend strips. Therefore, we can assess the accuracy of the approximation by comparing it to the true solution for the parameter values that we estimate. It would be interesting to use data on dividend strips rather than only the price-dividend ratio, to get a better understanding of how the pricing of dividend strips lines up with that of zero-coupon bonds, effectively adding a maturity dimension to the pricing of the stock market. We leave this for future research.

for some coefficients $(\bar{z}, z_{\mathrm{p}})$ to be determined. Due to the log-linear approximation we have that

$$\begin{aligned} ry_{t+1} &\approx g^{Y}_{t+1} - z_t + \bar{c} + c\, z_{t+1} \\ &= g^{Y}_{t+1} + \bar{z}\,(c-1) + z_{\mathrm{p}}^{\top}\,(c\hat{\mathrm{p}}_{t+1} - \hat{\mathrm{p}}_t) + ch_{t+1} - h_t. \end{aligned}$$

Combining this with (40), (41), and (42) to compute the moments, substituting into (46), and separating variables shows that our conjecture holds if and only if

$$(\sigma_Y + c\,\rho_{Yh}\sigma_h)\,\lambda_{y1} = c\,\Phi_h - 1 \tag{47}$$

and

$$\left(\sigma_{Y\mathrm{p}}^{\top} + c\,z_{\mathrm{p}}^{\top}\sigma_{\mathrm{p}} + c\,\sigma_{h\mathrm{p}}^{\top}\right)\sigma_{\mathrm{p}}^{-1}\lambda_{\mathrm{p}1} = \mu_{Y\mathrm{p}} + z_{\mathrm{p}}^{\top}\left(c\,\Phi_{\mathrm{p}} - I\right) + c\,\Phi_{h\mathrm{p}} - B^{1} \tag{48}$$

$$\begin{aligned} \left(\sigma_{Y\mathrm{p}}^{\top} + c\,z_{\mathrm{p}}^{\top}\sigma_{\mathrm{p}} + c\,\sigma_{h\mathrm{p}}^{\top}\right)\sigma_{\mathrm{p}}^{-1}\lambda_{\mathrm{p}0} = {} & \mu_Y - \tfrac{1}{2}\|\sigma_{Y\mathrm{p}}\|^2 + \bar{c} + \bar{z}\,(c-1) - A^{1} \\ & + c\rho_{Yh}\sigma_Y\sigma_h + \tfrac{1}{2}\|\sigma_{Y\mathrm{p}} + c\,\sigma_{\mathrm{p}}^{\top} z_{\mathrm{p}} + c\,\sigma_{h\mathrm{p}}\|^2 + \tfrac{1}{2}\|c\,\sigma_h\|^2 \\ & - (\sigma_Y + c\,\rho_{Yh}\sigma_h)\,\lambda_{y0} \end{aligned} \tag{49}$$

subject to (38) and (39) where $A^1 = \rho_0$ and $B^1 = \rho_{\mathrm{p}}$.

These equations provide no-arbitrage restrictions on the parameters of the Campbell Shiller approximation. We will use these restrictions to discipline the choice of the equity risk-premium parameters $(\lambda_{y0}, \lambda_{y1})$. Specifically, we first estimate $(\bar{z}, z_{\mathrm{p}})$ and the path of h_t from a time series regression of z_t onto p_t; then we identify the parameters of the processes (h_t, Y_t) by estimating the VAR in (41)–(42) given the path of p_t; and we finally apply the asymptotic least squares method to estimate the equity risk-premium parameters by exploiting the above no-arbitrage restrictions.

B.3 Parameter estimates

B.3.1 Principal components, consumption, dividends, and inflation

We start by extracting the principal components p_t from the interest rate data and estimating a VAR on the resulting time series to identify the parameters $(\overline{\mathrm{p}}, \Phi_{\mathrm{p}}, \sigma_{\mathrm{p}})$ and the yield curve shocks $\epsilon_{\mathrm{p}t+1}$. Taking as given the results of this first step we execute the following sequence of estimations:

1. For $J \in \{C, Y, \Pi\}$ we regress g_t^J onto $(\hat{\mathrm{p}}_t := \mathrm{p}_t - \overline{\mathrm{p}}, \epsilon_{\mathrm{p}t+1})$ to estimate the parameters $(\mu_J, \sigma_J, \mu_{J\mathrm{p}}, \sigma_{J\mathrm{p}})$ and identify the sequence of specific shocks ϵ_{Jt+1}

2. We regress z_t onto $\hat{\mathrm{p}}_t$ to estimate the parameters $(\overline{z}, z_{\mathrm{p}})$ and identify the time series of the stock-market specific state variable h_t

3. We regress h_{t+1} onto $(h_t, \hat{\mathrm{p}}_t, \epsilon_{\mathrm{p},t+1})$ to estimate the parameters $(\sigma_h, \sigma_{h\mathrm{p}}, \Phi_h, \Phi_{h\mathrm{p}})$ and identify the sequence of equity discount rate shocks $\epsilon_{h,t+1}$

4. We regress ℓ_{t+1} onto (ℓ_t, ℓ_{t-1}) to estimate the parameters $(\phi_1, \phi_2, \overline{\ell})$ and identify a sequence of residuals u_{t+1} that we then regress on $\epsilon_{\mathrm{p}t+1}$ to estimate $(\sigma_\ell, \sigma_{\ell\mathrm{p}})$ and identify the sequence of specific shocks $\epsilon_{\ell t+1}$.

5. We estimate the correlations between the residuals $(\epsilon_{Ct}, \epsilon_{Yt}, \epsilon_{\Pi t}, \epsilon_{ht}, \epsilon_{\ell t})$.

This procedure delivers estimates for *all* the parameters of the model. However, some of these estimates turn out to be statistically insignificant in the sense that the associated p-value is below 0.05. To obtain the most parsimonious model, we exogenously set all such parameters to zero and re-execute the above set of estimations under this constraint. The results of this second estimation are presented in Table 4 where we explicitly indicate which parameters were set to zero.

B.3.2 Bond loadings and risk-premium parameters

To estimate the term structure factor loadings (A^n, B^n) we regress the n-year zero coupon bond yield onto the principal components, i.e. is we estimate

$$Y_t^{t+n} = \frac{1}{n}\left(A^n + \mathrm{p}_t^\top B^n\right) + \epsilon_{nt}.$$

We run this regression for $n \in \{1, \ldots, 20\}$ to obtain a vector $(A^n)_{n=1}^{20}$ and a matrix $(B^n)_{n=1}^{20}$. Importantly, and consistent with the selection of the first three principal components as factors, we find that the R^2 of these regressions always exceeds 99.9%. We plot in Figure 4 the estimated average yield curve A^n/n and the estimated vector of factor loadings B^n/n as functions of maturity n.

Process	Parameter	Estimate	Standard errors
p_t	$\overline{\mathrm{p}}^\top$	$\begin{pmatrix}0.258 & -0.025 & 0.005\end{pmatrix}$	$\begin{pmatrix}0.003 & 0.0004 & 0.0001\end{pmatrix}$
	Φ_p	$\begin{pmatrix}0.884 & 0 & 0\\ 0 & 0.657 & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}0.060 & * & *\\ * & 0.130 & *\\ * & * & *\end{pmatrix}$
	σ_p	$\begin{pmatrix}0.046 & 0 & 0\\ 0 & 0.014 & 0\\ 0 & 0 & 0.004\end{pmatrix}$	$\begin{pmatrix}0.004 & * & *\\ * & 0.001 & *\\ * & * & 0.0003\end{pmatrix}$
g_t^C	μ_C	0.065	0.003
	σ_C	0.021	0.002
	$\mu_{C\mathrm{p}}^\top$	$\begin{pmatrix}0.089 & 0 & -1.524\end{pmatrix}$	$\begin{pmatrix}0.021 & * & 0.644\end{pmatrix}$
	$\sigma_{C\mathrm{p}}^\top$	$\begin{pmatrix}0 & 0.008 & 0\end{pmatrix}$	$\begin{pmatrix}* & 0.003 & *\end{pmatrix}$
g_t^Y	μ_Y	0.074	0.008
	σ_Y	0.059	0.005
	$\mu_{Yp}^\top$	$\begin{pmatrix}0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}* & * & *\end{pmatrix}$
	$\sigma_{Yp}^\top$	$\begin{pmatrix}0 & 0.033 & 0\end{pmatrix}$	$\begin{pmatrix}* & 0.008 & *\end{pmatrix}$
g_t^Π	μ_Π	0.033	0.002
	σ_Π	0.016	0.002
	$\mu_{\Pi\mathrm{p}}^\top$	$\begin{pmatrix}0.083 & 0 & -1.213\end{pmatrix}$	$\begin{pmatrix}0.017 & * & 0.504\end{pmatrix}$
	$\sigma_{\Pi\mathrm{p}}^\top$	$\begin{pmatrix}0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}* & * & *\end{pmatrix}$
h_t	Φ_h	0.677	0.084
	σ_h	0.161	0.015
	$\Phi_{h\mathrm{p}}^\top$	$\begin{pmatrix}0 & 0 & 15.334\end{pmatrix}$	$\begin{pmatrix}* & * & 5.207\end{pmatrix}$
	$\sigma_{h\mathrm{p}}^\top$	$\begin{pmatrix}0.077 & 0 & -0.091\end{pmatrix}$	$\begin{pmatrix}0.024 & * & 0.023\end{pmatrix}$
z_t	$\overline{z}$	3.653	0.034
	z_p	$\begin{pmatrix}-1.719 & 0 & 27.31\end{pmatrix}$	$\begin{pmatrix}0.271 & * & 7.928\end{pmatrix}$
ℓ_t	ϕ	$\begin{pmatrix}1.455 & -0.495\end{pmatrix}$	$\begin{pmatrix}0.091 & 0.091\end{pmatrix}$
	σ_ℓ	0.064	0.012
	$\sigma_{\ell\mathrm{p}}^\top$	$\begin{pmatrix}0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}* & * & *\end{pmatrix}$
	$\rho_{C\ell}$	-0.695	0.130
	ρ_{CY}	0.406	0.109
	$\rho_{C\pi}$	0.444	0.105

Table 4: Parameter estimates

Notes. An estimate of 0 associated with a standard error equal to $*$ indicates that the parameter was not significantly different from zero in a first estimation including all parameters and was set to zero before re-estimating the model. In the bottom panel, only the correlation coefficients that are statistically significant are reported.

Parameter	Estimates	Standard errors
λ_{p0}	$\begin{pmatrix} -0.013 \\ -0.005 \\ -0.001 \end{pmatrix}$	$\begin{pmatrix} 0.005 \\ 0.002 \\ 0.002 \end{pmatrix}$
λ_{p1}	$\begin{pmatrix} 0 & 0.431 & 0 \\ 0 & -0.148 & 0 \\ 0 & -0.056 & 0 \end{pmatrix}$	$\begin{pmatrix} * & 0.006 & * \\ * & 0.131 & * \\ * & 0.164 & * \end{pmatrix}$
λ_{y0}	1.404	2.888
λ_{y1}	−5.797	2.423

Table 5: Risk-premium parameter estimates

Notes. Following the literature (e.g. Duffee (2011), Adrian et al. (2013), and Diez de los Rios (2015)) we impose that the only non-zero entries of λ_{p1} be in the second column.

Next, we estimate the risk-premium parameters λ_{p0}, λ_{p1}, λ_{y0} and λ_{y1} from the no arbitrage restrictions (44)–(45) and (47)–(48)–(49) by using the nonlinear asymptotic least squares methodology of Gourieroux, Monfort, and Trognon (1985) reviewed in the next section. Following the literature (see, e.g., Duffee (2011), Adrian et al. (2013), and Diez de los Rios (2015)) and to reduce the number of parameters, we impose that the only non-zero entries of λ_{p1} be in second column. This simplification implies that risk-premia—and thus bond excess returns—are driven by a single factor related to the slope of the yield curve, which is consistent with empirical findings (see Fama and Bliss (1987), Cochrane and Piazzesi (2005)). The results of this estimation are presented in Table 5.

C Non-linear asymptotic least squares estimation

Here we review the nonlinear asymptotic least squares methodology of Gourieroux, Monfort, and Trognon (1985) (henceforth GMT) and its application to the estimation of our risk premium parameters.

C.1 Theory

GMT consider a setting where a *parameter of interest* $a \in \mathbf{R}^K$ is to be estimated from an *auxiliary parameter* $b \in \mathbf{R}^H$ via a set of $G \geq K$ *nonlinear equations* $g(b, a) = \mathbf{0}$ that are known to be satisfied by the true parameters (a_0, b_0).

Assume that the parameter $b \in \mathbf{R}^H$ is estimated from a time series $(X_1, \ldots, X_n)$ using an estimator $\hat{b}_n$ that is assumed to converge almost surely to b_0 and to be asymptotically normal with mean $\mathbf{0}$ and variance-covariance matrix $\Omega_0 \in \mathbf{R}^{H\times H}$. Now, pick a positive definite symmetric matrix $S_0 \in \mathbf{R}^{G\times G}$. Under natural conditions on the function g (see H1, H3-H5 p.96) GMT show that the estimator

$$\hat{a}_n = \underset{a\in\mathbf{R}^K}{\operatorname{argmin}} \left\{ g(a, \hat{b}_n)^\top S_0 g(a, \hat{b}_n) \right\}$$

converges almost surely to a_0 and is asymptotically normal with mean $\mathbf{0}$ and variance covariance matrix

$$\Sigma_0 = \left[\frac{\partial g}{\partial a}^\top S_0 \frac{\partial g}{\partial a} \right]^{-1} \frac{\partial g}{\partial a}^\top S_0 \frac{\partial g}{\partial b} \Omega_0 \frac{\partial g}{\partial b}^\top S_0 \frac{\partial g}{\partial a} \left[\frac{\partial g}{\partial a}^\top S_0 \frac{\partial g}{\partial a} \right]^{-1} \tag{50}$$

where the matrices $\frac{\partial g}{\partial a} = (\frac{\partial g_i}{\partial a_j}) \in \mathbf{R}^{G\times K}$ and $\frac{\partial g}{\partial b} = (\frac{\partial g_i}{\partial b_j})\mathbf{R}^{G\times H}$ are evaluated at the true parameters (a_0, b_0). The choice of the weighting matrix is unconstrained. For simplicity, we take it to be the identify matrix in our application, but GMT also address the issue of optimally choosing this matrix. In particular, they show that

$$S_0^* = \left[\frac{\partial g}{\partial b} \Omega_0 \frac{\partial g}{\partial b}^\top \right]^{-1}$$

minimizes the asymptotic variance of the resulting estimator.

C.2 Application

To estimate the risk premium parameters we apply the ALS methodology sequentially to different specifications of the triple (a, b, g).

First, we estimate the intercept $\lambda_{\mathrm{p}0}$ of the bond-specific risk premia by exploiting the restrictions (45):

$$\forall n \in \{2, \ldots, 20\} : \; A^n - A^{n-1} - A^1 + \lambda_{\mathrm{p}0}^\top B^{n-1} + \frac{1}{2} \|\sigma_{\mathrm{p}}^\top B^{n-1}\|^2 = 0. \tag{51}$$

Given the previously estimated parameter $b = (A, B, \sigma_{\mathrm{p}})$ we have set of 19 restrictions to estimate the parameter of interest $a = \lambda_{\mathrm{p}0}$. We stack these restrictions into a vector $g(a, \hat{b})$

then estimate a by solving

$$\min_a \left\| g(a, \hat{b}) \right\|^2,$$

and compute the variance covariance matrix of the estimator from (50) taking as given the previously computed variance covariance matrix of the estimator of b. Next, we turn to the matrix $\lambda_{\mathrm{p}1}$ that we estimate using the restrictions (44):

$$\forall n \in \{2, \ldots, 20\} : \; B^n - \left(\Phi_{\mathrm{p}} - \lambda_{\mathrm{p}1}\right)^\top B^{n-1} - B^1 = \mathbf{0}.$$

Here, the previously-estimated auxiliary parameter is $b = (A, B, \Phi_{\mathrm{p}})$, the parameters of interest is $a \in \mathbf{R}^3$ such that $\lambda_{\mathrm{p}1} = (\mathbf{0}|a|\mathbf{0}) \in \mathbf{R}^{3\times 3}$, and the function $g(a, b)$ is obtained by stacking up the 57 equations that are implicit in the above restrictions. With these elements in place we can proceed to estimate a by solving (51) and using (50) to compute the variance covariance matrix of the estimator.

Finally, we estimate the slope λ_{y0} and intercept λ_{y1} of the equity-specific risk premium by separately solving (47) and (49) taking as previously estimated parameter

$$b = [\sigma_Y, \bar{z}, \rho_{Yh}, \sigma_h, \Phi_h]$$

for the first equation, and

$$b = \left[\sigma_Y, \sigma_{Y\mathrm{p}}, \bar{z}, z_{\mathrm{p}}, \sigma_{\mathrm{p}}, \sigma_{h\mathrm{p}}, \lambda_{\mathrm{p}0}, \rho_{Yh}, \sigma_h, \mu_Y, A\right]$$

for the second equation; and then using (50) to compute the variance of each estimator. The results of this sequential procedure are reported in Table 5.