Heterogeneous Beliefs Recovery*

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Abstract

In a standard continuous-time economy with heterogeneous beliefs and constant relative risk aversion, equilibrium prices reveal the cross-sectional distribution of wealth and consumption shares across beliefs. Specifically, we establish a novel recovery theorem showing that the equilibrium paths of the risky asset price and the interest rate determine the evolution of these distributions. Motivated by this finding, we develop an optimization-based method to approximate the implied distribution of consumption shares across beliefs, given discrete time series of prices and interest rates. We confirm the accuracy of this method on simulated data and illustrate the versatility of our approach by providing extensions of our basic recovery theorem that allow for learning and multidimensional beliefs.

JEL classification: D53, D83, G10, G12

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1 Introduction

Heterogenous beliefs and their asset pricing implications have been a central theme in financial economics for decades.[1] While the way in which an exogenous specification of investors' beliefs shapes equilibrium prices and expected returns is by now well understood from a theoretical perspective, relatively little is known about the actual distribution of beliefs among market participants. To elicit information regarding this distribution, a simple method consists in conducting surveys of analysts forecasts but this approach suffers from significant drawbacks.[2] Indeed, the opinions of analysts may not accurately represent the beliefs of market participants and, even if they did, the limited statistical power of such surveys has for consequence that existing studies focus almost exclusively on the first two moments of the distribution. To circumvent these difficulties we propose an alternative structural approach that relies only on market prices. Specifically, we show that within a standard continuous-time general equilibrium model the observation of realized asset prices and interest rates fully reveals the distributions of consumption and wealth shares across beliefs.

Our baseline model is a continuous-time Lucas (1978) economy with an infinite horizon[3] and a continuum of investors who have homogeneous CRRA utility over intermediate consumption but heterogenous dogmatic beliefs about the growth rate of aggregate consumption. These beliefs are distributed across investors according to an arbitrary measure on a bounded support. This economy admits a unique equilibrium that we characterize in closed-form using as state variables aggregate consumption and the *endogenous* distributions of consumption and wealth shares across beliefs. In line with existing results (e.g., Cvitanić et al. 2011) our characterization implies that the continuous observation of the equilibrium market price of risk and interest

[1]One strand of the literature studies speculative trading models in which each investor fully internalizes her impact on prices and other investors. Example of contributions to this literature include Miller (1977), Harrison and Kreps (1978), Harris and Raviv (1993), Kandel and Pearson (1995), and Scheinkman and Xiong (2003). A paralel strand of the literature studies competitive general equilibrium models in which price-taking investors agree to disagree about the interpretation of public information. Examples of theoretical contributions to this vast literature include Dothan and Feldman (1986), Feldman (1989), Abel (1989), Detemple and Murthy (1994), Basak (2005), David (2008), Gallmeyer and Hollifield (2008), Dumas, Kurshev, and Uppal (2009), Cvitanić, Jouini, Malamud, and Napp (2011), Jouini and Napp (2007), Bhamra and Uppal (2014), Atmaz and Basak (2018), and Martin and Papadimitriou (2022). The empirical implications of competitive models with heterogenous beliefs have been explored by Buraschi and Jiltsov (2006), Xiong and Yan (2009), Beber, Breedon, and Buraschi (2010), Buraschi, Trojani, and Vedolin (2014), and Cujean and Hasler (2017) among others.

[2]See, for example, Diether, Malloy, and Scherbina (2002), Johnson (2004), Yu (2011), or Greenwood and Shleifer (2014).

[3]The assumption of an infinite horizon simplifies the presentation as it avoids having to keep track of time as a state variable, but it is in no way necessary for the validity of our results.

rate reveals the paths of the first two moments of the cross-sectional distribution of consumption shares across beliefs. Our key contribution is to show that the diffusion coefficient of the nth moment of this cross-sectional distribution depends linearly on itself and the $(n+1)$th moment. Therefore, computing the quadratic covariation between the observed paths of aggregate consumption and the second moment reveals the path of the third moment, and repeating this procedure allows to recursively construct the paths of *all* the moments of the equilibrium distribution of consumption shares across beliefs. Well-known results on the Haussdorff moment problem[4] then guarantee that the distribution of consumption shares across beliefs and it support are uniquely determined by its moment sequence and, thus, can be recovered from the continuous observation of the market price of risk and interest rate. Furthermore, we show that the distribution of *wealth shares* across beliefs is equivalent to the distribution of consumption shares across beliefs with an observable density and thus can also be uniquely recovered from the observation of market prices.[5]

The economic intuition behind our recovery theorem lies in the stochastic evolution of wealth and consumption shares: Investors whose beliefs are vindicated by realized aggregate consumption accumulate more wealth and consume more, thus shifting the equilibrium distribution of consumption shares across beliefs. The key feature in this mechanism is that changes in wealth trigger changes in consumption that generate time variation in the first two moments of the distribution of consumption shares across beliefs. In that respect it is crucial that investors have constant relative risk aversion. Indeed, if instead investors have constant absolute risk aversion then the distribution of consumption shares across beliefs is constant over time which prevents us from using successive quadratic covariation to recover moments of order larger than two. Another crucial ingredient for our result is that prices are *continuously* observed: In an otherwise equivalent discrete-time model, only a small fraction of the distribution is revealed in each period, requiring an infinite amount of time for complete recovery. By contrast, Brownian motion compacts an infinite number of time steps into an arbitrarily small interval and, thus, allows for complete recovery over an arbitrarily small interval. Importantly, our recovery theorem relies solely on *realized*

[4]The Haussdorf moment problem consists in finding necessary and sufficient conditions for a given sequence of numbers to be moment sequence of Borel measure on a given interval. See Schmüdgen (2017) for a survey of results on this and other moment problems.

[5]The underlying *unweighted* distribution of beliefs cannot be recovered from the observation of market prices alone. However, it is clear that the economically meaningful distributions to study are the distribution of consumption and wealth shares across beliefs as these distributions inform us on the economic characteristics of investors who share a given belief.

prices and *subjective beliefs* and, thus, requires neither inference about unrealized states (as opposed to Ross 2015) nor the existence of an objective probability.

The recovery theorem relies on the continuous observations of prices to compute successive quadratic covariations. In practice, however, prices are only observed at discrete points and, while one can approximate a single quadratic covariation using finite sums, it is unfeasible to reliably approximate *successive* quadratic covariations from discrete time series. To circumvent these difficulties, we develop a practical implementation of our main theorem that reformulates the recovery problem as a pathwise optimization problem which minimizes the discrepancy between realized market prices and those implied by a given distribution of consumption shares across beliefs. The latter optimization problem can be discretized in time to accommodate discrete observations and we prove that the solution to the discretized optimization problem converges to the real distribution as the sampling frequency increases. We illustrate the accuracy of our approach on simulated data using both discrete and continuous underlying distributions of beliefs. In the former case, we show that our procedure successfully determines the number of investor types and their respective beliefs. In particular, when the data comes from an homogeneous economy we correctly recover this feature and the belief of the single investor type.

Our baseline model assumes that there is a single risky asset and that investors have dogmatic beliefs about the growth rate of aggregate consumption. In the last part of the paper we establish the generality of our approach by showing that both of these assumptions can be relaxed. First, we consider an extension that includes imperfectly correlated risky assets that each represent a claim to an exogenous stream of dividends. In this setting, an investor's type is a vector whose components capture the dogmatic beliefs of the investor regarding the growth rates of each individual dividend process. As in the scalar case, the continuous observation of the interest rate and the vector of market prices of risk reveals the paths of some key moments of the multidimensional distribution of consumption shares across beliefs and the paths of all further moments can be recovered by computing successive quadratic covariations with aggregate consumption.

For our second extension we go back to the scalar case but relax the assumption of dogmatic beliefs by considering a model in which beliefs about the growth rate of aggregate consumption are stochastic but indexed by parameters taking values in a compact set. Importantly, this setting nests standard models in which investors have heterogenous priors and use Bayesian updating to learn about the growth rate of aggregate consumption. We show that recovery is possible in this extended model under a technical condition that holds in the classical cases where investors perform Gaussian filtering starting from heterogenous prior means or variance (as in, e.g. Gallmeyer and Hollifield 2008, Dumas et al. 2009) or learn about the state of an unobserved Markov chain (as in, e.g. Veronesi 2000, David 2008, Veronesi 2015) starting from heterogenous prior probabilities.

Our work contributes to the recent literature on the recovery of investor beliefs from observable market data. Extending Hansen and Scheinkman (2009), Ross (2015) shows that in a discrete-time model driven by an irreducible Markov process with finitely many states the beliefs of the representative investor can be inferred from the risk-neutral probability transition matrix derived from a rich cross-section of option prices. Borovicka, Hansen, and Scheinkman (2016) argue that the assumptions of Ross (2015) are restrictive. In particular, they show recovery fails with recursive preferences or permanent macroeconomic shocks. Carr and Yu (2012), Dubynskiy and Goldstein (2013), Qin and Linetsky (2016), and Walden (2017) study conditions for the validity of the Ross recovery theorem in a continuous-time model with a possibly unbounded state space. Notably, it follows from Walden (2017) that Ross's recovery approach fails if the state variable follows a geometric Brownian motion and/or the utility of the representative agent is CRRA as in the homogeneous version of our model where our pathwise approach remains sucessful. Jackwerth and Menner (2020) shows that the empirical probability measure recovered using Ross' approach fails to predict future returns. This finding does not necessarily refute the theorem, rather it may reflect a mismatch between the beliefs of the representative investor and the objective probability measure.

Ross (2015) and its extensions rely on the risk-neutral transition matrix to achieve recovery and, thus, require a large cross-section of option prices. Alternative recovery methods, including our pathwise approach, use fewer assets but require more detailed knowledge of their dynamics. Bick (1990), He and Leland (1993), and Wang (1993) provide conditions on the stochastic differential equations followed by asset prices for the utility function of the representative investor to be identifiable. In particular, He and Leland (1993) express the utility function as the solution to an ordinary differential equation. Cuoco and Zapatero (2000) extend these results by providing sufficient conditions for the recovery of both the preferences and the beliefs of the representative agent. Unlike our approach, these methods necessitate the knowledge of the drift and diffusion of the state variable over the entire domain and thus may require some inference about unrealized states. From a methodological standpoint, the closest paper to ours is Dybvig and Rogers (1997) who recover the preferences and beliefs of a single investor from the continuous observation of a *single path* of the risky asset price and the investor's optimal wealth process.

The literature on recovery in models with heterogeneous investors is new and sparse. Ghosh, Korteweg, and Xu (2022) propose a method to recover the preferences and beliefs of heterogeneous types of investors. However, they do not characterize the weights, wealth share, or consumption share of each type in the market. By contrast, we precisely specify the preferences of investors and focus on the determination of the beliefs present in the market and the consumption/wealth shares allocated to each such belief. Pazarbaşı, Schneider, and Vilkov (2024) propose a non-parametric method to recover from observed prices an ex-ante bound on the dispersion of beliefs across market participants. Egan, MacKay, and Yang (2022) use data on demand for leveraged index funds to characterize the empirical distribution of beliefs among retail investors under the assumption that perceived expected returns proportional to ETF leverage. By contrast, our methodology does not require any such assumption, only uses publicly available market prices, and is not limited to retail investors.

The remainder of the paper is organized as follows. Section 2 presents the baseline model. Section 3 provides a detailed characterization of the equilibrium and proves its existence. Section 4 establishes our baseline recovery theorem, outlines a practical implementation of the recovery procedure, and confirms its efficacy on simulated data. Section 5 discusses extensions and Section 6 concludes. Appendices provide additional theoretical results, proofs and methodological details.

2 Baseline model

2.1 Fundamental uncertainty

We consider a continuous-time economy on an infinite time horizon. There is a single non-storable good available for consumption at every date. Its supply $D_t > 0$ follows a continuous-paths process with quadratic variation given by

$$\langle D \rangle_t = \int_0^t (vD_u)^2 \, du$$

for some constant volatility $v > 0$. We denote by $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ the filtration generated by the continuous observation of aggregate consumption and implicitly assume that all relevant processes are adapted to this filtration.

2.2 Investors

The economy is populated by a *continuum* of heterogeneous investors whose *types* are distributed according to a finite measure $\mu(dx)$ on a bounded interval $\mathcal{X} := [\underline{x}, \overline{x}]$ of the real line. No assumption about the structure of $\mu(dx)$ is required for our analysis. In particular, this measure can be discrete, continuous, or a mixture of discrete and continuous components. In the degenerate case where this measure is concentrated on a single point, the model nests the classical Lucas (1978) model with a continuum of ex-ante identical investors.

Investors continuously observe aggregate consumption as well as the prices of traded assets, but have different perceptions of the underlying dynamics. Formally, we assume that investors of type $x \in \mathcal{X}$ are endowed with a subjective probability measure $\mathbf{P}^x$ relative to which the process

$$B_t^x := \int_0^t \frac{1}{v} \left(\frac{dD_u}{D_u} - x du \right) \tag{1}$$

is a Brownian motion. As a result, investors of type x believe that

$$\frac{1}{\Delta} \mathbf{E}_t^x \left[\int_t^{t+\Delta} \frac{dD_u}{D_u} \right] = x, \qquad \Delta > 0,$$

and it follows that an investor's type captures his belief, or degree of optimism, about the growth rate of aggregate consumption. In particular, the evolution of aggregate consumption on which investors of type x base their decisions is given by

$$dD_t = D_t \left(v dB_t^x + x dt \right). \tag{2}$$

Investors have identical CRRA preferences conditional on their beliefs and we assume that all investors of a given type $x \in \mathcal{X}$ are endowed with initial wealth $W_0^x > 0$ so that the probability measure

$$\mathcal{W}_0(A) := \frac{\int_A W_0^x \mu(dx)}{\int_{\mathcal{X}} W_0^x \mu(dx)}, \qquad A \in \text{Borel}(\mathcal{X}), \tag{3}$$

captures the initial *distribution of wealth shares across beliefs*. Each individual investor of course knows his initial wealth but none of them directly observes the cross-sectional distribution $\mathcal{W}_0$. The goals of this paper are to show that this distribution and its temporal evolution are revealed by market prices, and to propose a method to recover them from the time series of observed prices.

Remark 1. Equation (3) remains valid if investors of a given type are endowed with different initial wealths but in this case $W_0^x > 0$ should be understood as the average initial wealth of investors of type x. All the results of the paper continue to hold in this slightly more general setting due to the assumption of CRRA utility which implies that agents optimally invest and consume proportionally to their wealth.

Remark 2. To simplify the exposition and the proofs, the model is cast with a constant volatility of aggregate consumption. However, most of our results, including our main result Theorem 1, still hold under the weaker assumption that the volatility v_t is nonnegative and uniformly bounded away from zero.

2.3 Traded assets

The financial market operates in continous-time and consists in two assets: A locally riskless asset in zero net supply and a risky asset in positive supply of one unit. The price of the riskless asset evolves according to

$$dP_{0t} = r_t P_{0t} dt \tag{4}$$

for some interest rate process r_t that is to be determined in equilibrium. On the other hand, the risky asset is a claim to aggregate consumption and its price evolves according to

$$P_t = P_0 + \int_0^t (r_u P_u - D_u)\, du + \int_0^t P_u \sigma_u \left(dB_u^x + \theta_u^x du\right)$$

where the initial value $P_0 > 0$, the volatility σ_t, and the collection of type-specific perceived market prices of risk

$$\theta_t^x = \theta_t^y + \left(\frac{x-y}{v}\right) t, \qquad (x,y) \in \mathcal{X}^2, \tag{5}$$

are to be determined endogenously in equilibrium.

2.4 Equilibrium definition

Under the usual self-financing condition, the wealth of an investor of type x evolves according to the linear equation

$$dW_t^x = (r_t W_t^x - s_t D_t)\, dt + \pi_t \sigma_t P_t \left(dB_t^x + \theta_t^x dt\right) \tag{6}$$

where $s_t \geq 0$ and $\pi_t \in \mathbf{R}$ capture, respectively, her consumption as a fraction of aggregate consumption and the number of shares of the risky asset that she holds. The optimization problem of such an investor is given by

$$\sup_{(s,\pi)} \left\{ \mathbf{E}^x \left[\int_0^\infty e^{-\rho t} \frac{(s_t D_t)^{1-\gamma}}{1-\gamma} dt \right] \text{ subject to (6) and } \inf_{t\geq 0} W_t^x \geq 0 \right\} \tag{7}$$

for some subjective discount rate $\rho > 0$ and constant relative risk aversion $\gamma > 0$ that are common to all investors.

Definition 1. *An equilibrium is a list*

$$\mathcal{E} = \langle r_t, \sigma_t, \theta_t^x, P_0 \rangle$$

such that (5) holds; $\sigma_t \neq 0$ almost everywhere; the optimal strategy (s_t^x, π_t^x) in (7) exists for all types; and the market clearing conditions

$$\int_{\mathcal{X}} s_t^x \mu(dx) \overset{(a)}{=} 1, \qquad \int_{\mathcal{X}} \pi_t^x \mu(dx) \overset{(b)}{=} 1, \qquad \int_{\mathcal{X}} W_t^x \mu(dx) \overset{(c)}{=} P_t$$

hold almost everywhere.

If there are finitely many types of investors then $\mu(dx)$ is a discrete measure. In that case, the above is the standard definition of an equilibrium of prices, plans, and expectations (see, e.g., Cvitanić et al. (2011)). When there are infinitely many types of investors the above definition is slightly more general as it replaces the sums by integrals, but the intuition remains the same: Investors choose optimal consumption and portfolio policies taking prices as given, and an equilibrium is attained if these choices clear the market in the sense that investors collectively consume the available supply of the good (a), hold the whole supply of the risky asset (b), and have a zero net position in the riskless asset (c). By Walras's law one of the three market clearing conditions is redundant given the two others. Taking advantage of this simplification we will characterize equilibria by enforcing (a) and (c) only.

2.5 The recovery problem

The problem that we study in this paper can be loosely stated as follows: can an investor within the model recover the distribution of wealth across beliefs from the observation of prices and dividends? More formally, we ask whether having access to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and knowing the structure of the equilibrium model one can reconstruct the family of distributions $\{\mathcal{W}_t\}_{t\geq 0}$. Importantly, the recovery is to be performed knowing that the support of this distribution is contained in $\mathcal{X}$ but without any information about its structure. As a result, part of the recovery effectively consists in determining the cardinality of the set of investor types.

To show how this recovery can be performed we proceed in three steps. First, we construct the unique equilibrium of our economy in terms of the endogenous distributions of consumption shares across beliefs $\{\mathcal{S}_t\}_{t\geq 0}$ (defined in (12) below). Second, we show that these distributions *can be recovered* from the observation of prices and dividends. Third, and finally, we show that the distribution of wealth across beliefs, $\mathcal{W}_t$, is absolutely continuous with respect to the distribution of consumption shares across beliefs, $\mathcal{S}_t$, with an observable density process and, thus, can be recovered. The underliyng distribution of beliefs across the population $\mu(dx)$ *cannot be recovered* but we do not view this as a problem since that distribution is a lot less interesting from the economic point of view.

3 Equilibrium

3.1 Individual optimality

In the eyes of any given investor, there are two traded assets and a single Brownian motion. Therefore, markets are dynamically complete for all investors provided that the risky asset volatility $\sigma_t \neq 0$ almost everywhere and this is why we require this condition as part of our equilibrium definition. Under this condition, classical results (see, e.g., Karatzas and Shreve (1998), Duffie (2010)) show that the optimal consumption share of an investor of type x satisfies

$$s_t^x = s_0^x \left(e^{\rho t} H_t^x\right)^{-\frac{1}{\gamma}} \left(\frac{D_0}{D_t}\right) \tag{8}$$

where

$$H_t^x := \exp\left(-\int_0^t r_u du - \frac{1}{2}\int_0^t |\theta_u^x|^2 du - \int_0^t \theta_u^x dB_u^x\right) \tag{9}$$

is the unique state price density relative to $\mathbf{P}^x$ and the initial share $s_0^x \in (0,1)$ is determined by saturating the static budget constraint

$$\mathbf{E}^x\left[\int_0^\infty H_t^x s_t^x D_t dt\right] = W_0^x$$

which guarantees that, starting from W_0^x, the investor can afford to consume according to (8) while maintaining nonnegative wealth at all times.

3.2 Instantaneous pricing of risk and time

Equation (1) implies that the Brownian motions perceived by investors of different types are related by

$$B_t^x = B_t^y + \int_0^t \left(\frac{y-x}{v}\right) du, \qquad (x,y) \in \mathcal{X}^2.$$

Therefore, Girsanov theorem and (5) imply that their subjective probability measures are locally equivalent and that their state price densities are related by

$$Z_t^x H_t^x = Z_t^y H_t^y = H_t^0 \tag{10}$$

where the process

$$Z_t^x := \left.\frac{d\mathbf{P}^x}{d\mathbf{P}^0}\right|_{\mathcal{F}_t} = \exp\left[-\frac{1}{2}\int_0^t \left(\frac{x}{v}\right)^2 du + \int_0^t \left(\frac{x}{v}\right) dB_u^0\right] \tag{11}$$

is the density of the subjective probability of an investor of type x relative to that of an investor of type 0. Whether or not such investors are actually present in the market does not matter for our purpose as we only use $\mathbf{P}^0$ and the associated state price density as references to express the probabilities, consumption shares, and state price densities of investors who are actually present.

To aggregate the optimal consumptions of individual investors, we use as a state variable the positive measure

$$\mathcal{S}_t(A) := \int_A s_t^x \mu(dx) = \int_A \left(e^{\rho t} H_t^x\right)^{-\frac{1}{\gamma}} \left(\frac{D_0}{D_t}\right) \mathcal{S}_0(dx), \qquad A \in \text{Borel}(\mathcal{X}), \tag{12}$$

that captures the equilibrium *distribution of consumption shares across beliefs*. In terms of this object, the goods market clearing condition can be written as

$$1 = \mathcal{S}_t(\mathcal{X}), \qquad t \geq 0$$

and thus requires that the process $t \mapsto \mathcal{S}_t$ takes values in the set $\mathcal{P}$ of probability measures defined on $\mathcal{X}$. Substituting (10) into (12) at $A = \mathcal{X}$ shows that this condition can be equivalently stated as

$$1 = \left(e^{\rho t} H_t^0 \big/ \Phi_t\right)^{-\frac{1}{\gamma}} \left(\frac{D_0}{D_t}\right) \tag{13}$$

with the auxiliary process defined by

$$\Phi_t := \left[\int_{\mathcal{X}} (Z_t^x)^{\frac{1}{\gamma}} \mathcal{S}_0(dx)\right]^{\gamma}. \tag{14}$$

Next, solving (13) for the state price density gives

$$H_t^0 = e^{-\rho t} \Phi_t \left(\frac{D_t}{D_0}\right)^{-\gamma} \tag{15}$$

and substituting back into (12) shows that the equilibrium distribution of consumption shares across beliefs satisfies

$$\mathcal{S}_t(A) = \frac{\int_A (Z_t^x)^{\frac{1}{\gamma}} \mathcal{S}_0(dx)}{\int_{\mathcal{X}} (Z_t^x)^{\frac{1}{\gamma}} \mathcal{S}_0(dx)} = \int_A \left(\frac{Z_t^x}{\Phi_t}\right)^{\frac{1}{\gamma}} \mathcal{S}_0(dx). \tag{16}$$

Finally, differentiating (9) and (15) and matching terms delivers explicit expressions for the equilibrium interest rate and market prices of risk.

Proposition 1. *In equilibrium*

$$\theta_t^x = \gamma v + \frac{1}{v}(x - M_{1t}) \tag{17}$$

and

$$r_t = \rho + \gamma M_{1t} - \frac{\gamma}{2}(1+\gamma)v^2 + \frac{1}{2v^2}\left(1 - \frac{1}{\gamma}\right)\left(M_{2t} - M_{1t}^2\right) \tag{18}$$

where the process

$$M_{nt} := \int_{\mathcal{X}} x^n \mathcal{S}_t(dx) = \int_{\mathcal{X}} x^n \left(\frac{Z_t^x}{\Phi_t}\right)^{\frac{1}{\gamma}} \mathcal{S}_0(dx)$$

tracks the nth non central moment of the probability measure $\mathcal{S}_t$.

Proof. Since $\mathcal{X}$ is bounded it is clear that

$$\begin{aligned}\sup_{x\in\mathcal{X}} \mathbf{E}^0\left[\left(\frac{x}{\gamma v}\right)^2 (Z_t^x)^{\frac{2}{\gamma}}\right] &= \sup_{x\in\mathcal{X}}\left\{\left(\frac{x}{\gamma v}\right)^2 e^{(2-\gamma)\left(\frac{x}{\gamma v}\right)^2 t}\mathbf{E}^0\left[e^{-\frac{1}{2}\left(\frac{2x}{\gamma v}\right)^2 t + \frac{2x}{\gamma v}B_t^0}\right]\right\} \\ &= \sup_{x\in\mathcal{X}}\left\{\left(\frac{x}{\gamma v}\right)^2 e^{(2-\gamma)\left(\frac{x}{\gamma v}\right)^2 t}\right\} \leq be^{bt}\end{aligned}$$

for some $b > 0$ where the second equality follows from the fact the process below the expectation is a martingale on any finite time interval. Combining this estimate with Tonelli's theorem shows that

$$\mathbf{E}^0 \int_0^t \left\{\int_{\mathcal{X}} \left(\frac{x}{\gamma v}\right)^2 (Z_s^x)^{\frac{2}{\gamma}} \mathcal{S}_0(dx)\right\} ds \leq \int_0^t be^{bs} ds \leq e^{bt}$$

and it thus follows from Fubini's theorem for stochastic integrals (see Protter 2003, Theorem 65) that the auxiliary process satisfies

$$\begin{aligned}\Phi_t^{\frac{1}{\gamma}} - 1 &= \int_{\mathcal{X}}\left((Z_t^x)^{\frac{1}{\gamma}} - 1\right)\mathcal{S}_0(dx) \\ &= \int_{\mathcal{X}}\int_0^t\left[\frac{1-\gamma}{2}\left(\frac{x}{\gamma v}\right)^2 ds + \frac{x}{\gamma v} dB_s^0\right](Z_s^x)^{\frac{1}{\gamma}}\mathcal{S}_0(dx) \\ &= \int_0^t\left[\frac{1-\gamma}{2}\int_{\mathcal{X}} x^2 (Z_s^x)^{\frac{1}{\gamma}}\mathcal{S}_0(dx)\right]\frac{ds}{(\gamma v)^2} + \int_0^t\left[\int_{\mathcal{X}} x (Z_s^x)^{\frac{1}{\gamma}}\mathcal{S}_0(dx)\right]\frac{dB_s^0}{\gamma v} \\ &= \int_0^t \Phi_s^{\frac{1}{\gamma}}\left[\frac{1-\gamma}{2(\gamma v)^2}M_{2s}ds + \frac{M_{1s}}{\gamma v}dB_s^0\right].\end{aligned} \tag{19}$$

Combining this result with Itô's lemma then shows that the state price density of reference investors evolves according to

$$
\begin{aligned}
-\frac{dH_t^0}{H_t^0} &= \theta_t^0 dB_t^0 + r_t dt = -\frac{D_t^\gamma}{e^{-\rho t}\Phi_t} d\left(\frac{e^{-\rho t}\Phi_t}{D_t^\gamma}\right) \\
&= \left(\gamma v - \frac{M_{1t}}{v}\right) dB_t^0 \\
&+ \left(\rho + \gamma M_{1t} - \frac{\gamma}{2}(1+\gamma) v^2 - \frac{1-\gamma}{2\gamma v^2}\left(M_{2t} - M_{1t}^2\right)\right) dt
\end{aligned}
$$

and the desired result now follows from (5) by matching terms. ■

Equation (17) shows that the market price of risk perceived by an investor a type x is the sum of two terms. The first is the market price of risk $\theta_t^* := \gamma v$ that would prevail absent beliefs heterogeneity and the second

$$
\theta_t^x - \theta_t^* = \frac{1}{v}(x - M_{1t})
$$

is the volatility-scaled difference between the investor's perceived growth rate x and the consumption-weighted average belief. In particular,

$$
\int_{\mathcal{X}} \theta_t^x \mathcal{S}_t(dx) = \theta_t^*
$$

so that the market price of risk perceived by an hypothetical investor endowed with the consumption-weighted average belief M_{1t} is the average market price of risk. On the other hand, equation (18) shows that the equilibrium interest rate is equal to the sum of the interest rate

$$
r_t^* := \rho + \gamma M_{1t} - \frac{\gamma}{2}(1+\gamma) v^2
$$

that would prevail if all investors shared the consumption-weighted average belief M_{1t} and a correction term

$$
r_t - r_t^* = \frac{1}{2v^2}\left(1 - \frac{1}{\gamma}\right)\left(M_{2t} - M_{1t}^2\right)
$$

that is proportional to the consumption-weighted variance of beliefs. Importantly, the sign of this correction is determined by the investors' relative risk aversion. When $\gamma < 1$, the correction is negative, meaning that greater heterogeneity in beliefs lowers the interest rate. Conversely, when $\gamma > 1$, the correction is positive, causing the interest rate to rise. In the special case where investors are myopic ($\gamma = 1$), the correction term vanishes, implying that only the first moment of the consumption weighted distribution of beliefs influences the instantaneous pricing of risk and time. To see why this occurs, assume without loss of generality that $\gamma < 1$. When belief dispersion increases, both deposit and loan demands rise. However, because investors with an elasticity of intertemporal substitution $1/\gamma > 1$ are more inclined to lend than to borrow, the increase in deposit demand is stronger. As a result, an excess demand for deposits emerges, which drives the interest rate down to restore market equilibrium.

3.3 The distribution of wealth

Along the optimal path, the wealth of an investor of type x is given by the risk-adjusted present value of his future consumption:

$$W_t^x = \mathbf{E}_t^x \left[\int_t^\infty \frac{H_u^x}{H_t^x} s_u^x D_u du \right] = \mathbf{E}_t^0 \left[\int_t^\infty \frac{H_u^0}{H_t^0} s_u^x D_u du \right]$$

where the second equality follows from (10) and (11). Substituting (8) and (15) into the right-hand side of this identity and integrating both sides of the resulting equation against $\mu(dx)$ shows that the equilibrium distributions of wealth and consumption shares across beliefs are related by

$$\mathcal{W}_t(A) = \int_A \frac{W_t^x}{P_t} \mu(dx) = \int_A \left(\frac{D_t}{P_t} \right) \Lambda_t^x \mathcal{S}_t(dx), \qquad A \in \text{Borel}(\mathcal{X}), \tag{20}$$

where the density process

$$\Lambda_t^x := \frac{W_t^x}{s_t^x D_t} = \mathbf{E}_t^0 \left[\int_t^\infty e^{-\rho(u-t)} \left(\frac{D_u}{D_t} \right)^{1-\gamma} \left(\frac{\Phi_u}{\Phi_t} \right)^{1-\frac{1}{\gamma}} \left(\frac{Z_u^x}{Z_t^x} \right)^{\frac{1}{\gamma}} du \right] \tag{21}$$

represents the wealth-consumption ratio of an investor of type x. In terms of these objects, the riskless asset market clearing condition may be stated as

$$\frac{P_t}{D_t} = \int_{\mathcal{X}} \Lambda_t^x \mathcal{S}_t(dx), \tag{22}$$

and changing the order of integration on the right hand side delivers a forward looking representation of the equilibrium price dividend ratio:

Proposition 2. *In equilibrium*

$$\frac{P_t}{D_t} = \mathbf{E}_t^0 \left[\int_t^\infty e^{-\rho(u-t)} \left(\frac{D_u}{D_t} \right)^{1-\gamma} \frac{\Phi_u}{\Phi_t} du \right] \tag{23}$$

with the auxiliary process Φ_t *defined by equation* (14).

Proof. Inserting (21) into (22) and changing the order of integration—which is licit here as all terms are nonnegative—we obtain

$$\frac{P_t}{D_t} = \mathbf{E}_t^0 \left[\int_t^\infty e^{-\rho(u-t)} \left(\frac{D_u}{D_t} \right)^{1-\gamma} \left(\frac{\Phi_u}{\Phi_t} \right)^{1-\frac{1}{\gamma}} \left\{ \int_{\mathcal{X}} \left(\frac{Z_u^x}{Z_t^x} \right)^{\frac{1}{\gamma}} \mathcal{S}_t(dx) \right\} du \right]$$

and the conclusion follows by observing that

$$\int_{\mathcal{X}} \left(\frac{Z_u^x}{Z_t^x} \right)^{\frac{1}{\gamma}} \mathcal{S}_t(dx) = \int_{\mathcal{X}} \left(\frac{Z_u^x}{\Phi_t} \right)^{\frac{1}{\gamma}} \mathcal{S}_0(dx) = \left(\frac{\Phi_u}{\Phi_t} \right)^{\frac{1}{\gamma}} \tag{24}$$

as a result of (16). ■

In view of Proposition 2 it is clear that for an equilibrium to exist one needs the conditional expectation in (23) to be well-defined. A necessary condition for this property is that

$$\lim_{u\to\infty} \mathbf{E}_t^0 \left[e^{-\rho(u-t)} \left(\frac{D_u}{D_t} \right)^{1-\gamma} \frac{\Phi_u}{\Phi_t} \right] = 0, \qquad t \geq 0. \tag{25}$$

This transversality condition is automatically satisfied if agents have log utility since in that case

$$\mathbf{E}_t^0 \left[e^{-\rho(u-t)} \left(\frac{D_u}{D_t} \right)^{1-\gamma} \frac{\Phi_u}{\Phi_t} \right] = \int_{\mathcal{X}} \mathbf{E}_t^0 \left[e^{-\rho(u-t)} Z_t^x \right] \mathcal{S}_0(dx) = e^{-\rho(u-t)}.$$

If agents are non myopic ($\gamma \neq 1$) then (25) may fail but sufficient conditions are easily established. In particular, we show in Appendix A that the transversality condition holds provided that

$$\lambda := \min_{x\in\mathcal{X}} \lambda^x > 0 \tag{26}$$

where

$$\lambda^x := -\frac{1}{T}\log \mathbf{E}^x\left[e^{-\rho t} D_T^{1-\gamma}\right] = \rho - (1-\gamma)x + \frac{\gamma}{2}(1-\gamma)v^2.$$

This condition is intuitive as it is equivalent to requiring that for any given $x \in \mathcal{X}$ the transversality condition holds for the homogeneous economy where all agents are of type x. It is also minimal as it cannot be improved upon without imposing restrictions on the initial distribution of consumption/wealth across beliefs.

We close this section by providing an explicit representation of the equilibrium asset volatility in terms of the first moments of the wealth and consumption-weighted distribution of beliefs. The result, which we believe is novel in the literature, explicits the impact of disagreement on the asset volatility and confirms that in a model with heterogenous beliefs the asset volatility differs from that of the fundamental unless agents are myopic in which case the distributions of wealth and consumption shares across beliefs coincide at all times.

Proposition 3. *In equilibrium*

$$\sigma_t = v + \frac{1}{v}\int_{\mathcal{X}} x\left(\mathcal{W}_t(dx) - \mathcal{S}_t(dx)\right). \tag{27}$$

Proof. Let $\mathrm{diff}_t(Y)$ denote the diffusion coefficient of an Itô process Y_t and observe that since

$$\sigma_t = \frac{\mathrm{diff}_t(P)}{P_t} = v + \frac{\mathrm{diff}_t(P/D)}{P_t/D_t}, \tag{28}$$

it suffices to compute the diffusion coefficient of the price-dividend ratio. Applying the Clark-Ocone formula (see Nualart (2006, Chapter 1)) to (23) shows that this object can be represented as

$$\begin{aligned}\mathrm{diff}_t(P/D) &= \mathbf{E}_t^0\left[\int_t^\infty e^{-\rho(u-t)}\mathcal{D}_t\left(\left(\frac{D_u}{D_t}\right)^{1-\gamma}\frac{\Phi_u}{\Phi_t}\right)\right] \\ &= \mathbf{E}_t^0\left[\int_t^\infty e^{-\rho(u-t)}\left(\left(\frac{\Phi_u}{\Phi_t}\right)\mathcal{D}_t\left(\frac{D_u}{D_t}\right)^{1-\gamma} + \left(\frac{D_u}{D_t}\right)^{1-\gamma}\mathcal{D}_t\left(\frac{\Phi_u}{\Phi_t}\right)\right)du\right]\end{aligned}$$

where $\mathcal{D}_t$ denotes the Malliavin derivative and the second equality follows from the chain rule. Since the ratio D_u/D_t only depends on the increment of the Brownian motion we have that the first derivative $\mathcal{D}_t\left(D_u/D_t\right)^{1-\gamma} = 0$. On the other hand, the chain rule, (14), and (29) imply that

$$\frac{\Phi_t}{\Phi_u}\mathcal{D}_t\left(\frac{\Phi_u}{\Phi_t}\right) = \frac{\mathcal{D}_t\Phi_u}{\Phi_u} - \frac{\mathcal{D}_t\Phi_t}{\Phi_t} = \gamma\Phi_u^{-\frac{1}{\gamma}}\int_{\mathcal{X}}\mathcal{D}_t\left(Z_u^x\right)^{\frac{1}{\gamma}}\mathcal{S}_0(dx) - \frac{M_{1t}}{v}$$
$$= \int_{\mathcal{X}}\left(\frac{Z_u^x}{\Phi_u}\right)^{\frac{1}{\gamma}}\left(\frac{x}{v}\right)\mathcal{S}_0(dx) - \frac{M_{1t}}{v}.$$

Plugging this expression into (28), changing the order of integration, and using (16) and (20) then gives

$$\text{diff}_t(P/D) = \int_{\mathcal{X}}\Lambda_t^x\left(\frac{x}{v}\right)\mathcal{S}_t(dx) - \frac{P_t}{vD_t}M_{1t}$$

and the conclusion follows from (20). ∎

Proposition 3 shows that the difference between the asset volatility σ_t and the fundamental volatility v is proportional to the difference between the averages of the wealth-weighted and consumption-weighted distributions of beliefs. In particular, the model generates excess volatility if and only if the latter difference is positive. To grasp the intuition behind this result, note that using (20) we may rewrite (27) as

$$\text{diff}_t\left(\frac{P_t}{D_t}\right) = (\sigma_t - v)\left(\frac{P_t}{D_t}\right) = \int_{\mathcal{X}}\left(\frac{x - M_{1t}}{v}\right)\Lambda_t^x\mathcal{S}_t(dx).$$

For the expression on the right hand side to be positive it is necessary that investors who are more optimistic than average tend to have larger wealth consumption ratios. If that is the case then a positive fundamental shock will increase the asset demand of these more optimistic investors more than it decreases that of less optimistic investors, thus triggering a larger positive price response than in the homogeneous economy where the asset volatility equals that of the fundamental.

Remark 3. Itô's lemma and (19) imply that

$$\frac{d\Phi_t}{\Phi_t} = \frac{M_{1t}}{v}dB_t^0 - \frac{1}{2v^2}\left(1 - \frac{1}{\gamma}\right)\left(M_{2t} - M_{1t}^2\right)dt. \tag{29}$$

Since $|M_{1t}/v|$ is bounded, it follows from Girsanov's theorem that the equilibrium asset price in (23) can be written as

$$\frac{P_t}{D_t} = \mathbf{E}_t^*\left[\int_t^\infty e^{-\int_t^u \rho_s^* ds}\left(\frac{D_u}{D_t}\right)^{1-\gamma}du\right]$$

where the expectation is relative to the probability $\mathbf{P}^*$ under which the growth rate of aggregate consumption is M_{1t} and

$$\rho_t^* := \rho + \frac{1}{2v^2}\left(1 - \frac{1}{\gamma}\right)\left(M_{2t} - M_{1t}^2\right).$$

This alternative representation shows that the equilibrium asset price in our economy *cannot be obtained* within a standard representative agent economy unless ρ_t^* is constant which occurs if and only if investors have homogeneous beliefs $(M_{2t} - M_{1t}^2 = 0)$ or logarithmic utility $(\gamma = 1)$. See Jouini and Napp (2007) and Cvitanić et al. (2011) for related results in economies with finitely many investor types.

4 Main results

4.1 The recovery theorem

The continuous observation of the market price of risk implies that the first moment of the consumption weighted distribution of beliefs

$$M_{1t} \equiv \gamma v^2 - v\theta_t^0$$

is continuously observable. The following theorem constitutes our main result. It shows that, in equilibrium, *all further moments* can be recovered recursively from the continuous observation of M_{1t} and combines this property with well-known results on the Hausdorff moment problem (e.g. Schmüdgen 2017, Chapter 4.1) to recover the entire consumption weighted distribution of beliefs. To state the result, let

$$Q_t(X) := \langle X, \log D \rangle_t$$

stand for the quadratic covariation between an Itô process X_t and the log of aggregate consumption and denote by $\dot{Q}_t(X)$ its time derivative.

Theorem 1. *In equilibrium*

$$M_{n+1t} = M_{1t}M_{nt} + \gamma \dot{Q}_t(M_n), \qquad n \in \mathbf{N}. \tag{30}$$

In particular, if $\tau > 0$ then the consumption weighted distributions of beliefs $\{\mathcal{S}_t\}_{t \le \tau}$ can be recovered from the continuous observation of $\{(r_t, P_t, D_t)\}_{t \le \tau}$.

Proof. Using the absolute continuity relation (16) between $\mathcal{S}_t$ and $\mathcal{S}_0$ together with (19), Itô's lemma and arguments similar to those of the proof of Proposition 1 shows that the moments evolve according to

$$
\begin{aligned}
dM_{nt} &= \int_{\mathcal{X}} x^n d\left(\frac{Z_t^x}{\Phi_t}\right)^{\frac{1}{\gamma}} \mathcal{S}_0(dx) = \int_{\mathcal{X}} x^n \left(\frac{Z_t^x}{\Phi_t}\right)^{-\frac{1}{\gamma}} d\left(\frac{Z_t^x}{\Phi_t}\right)^{\frac{1}{\gamma}} \mathcal{S}_t(dx) \\
&= \int_{\mathcal{X}} \left\{ \varphi_{nt}^x dt + \frac{1}{\gamma v}\left(x^{n+1} - x^n M_{1t}\right) dB_t^0 \right\} \mathcal{S}_t(dx) \\
&= \hat{\varphi}_{nt} dt + \frac{1}{\gamma v}\left(M_{n+1t} - M_{nt}M_{1t}\right) dB_t^0
\end{aligned}
$$

with the drift terms

$$
\begin{aligned}
(\gamma v)^2 \varphi_{nt}^x &:= x^n M_{1t}^2 - x^{n+1} M_{1t} + \frac{1}{2}(1-\gamma)\left(x^{n+2} - x^n M_{2t}\right), \\
(\gamma v)^2 \hat{\varphi}_{nt} &:= M_{nt} M_{1t}^2 - M_{n+1t} M_{1t} + \frac{1}{2}(1-\gamma)\left(M_{n+2t} - M_{nt} M_{2t}\right).
\end{aligned}
$$

In particular, we have that

$$
\gamma \dot{Q}_t(M_n) = \frac{d\langle M_n, \log D\rangle_t}{dt} = M_{n+1t} - M_{nt}M_{1t}
$$

and rearranging this expression delivers the recursion of the statement. To complete the proof, let $\tau > 0$ be fixed and recall that if two Itô processes (X, Y) are continuously observed on the interval $[0, \tau]$ then their quadratic covariation and the corresponding density process can be recovered almost everywhere on $[0, \tau] \times \Omega$ as

$$
\langle X, Y \rangle_t = \lim_{|\Pi| \to 0} \sum_{n=1}^{N} \left(X_{t_n} - X_{t_{n-1}}\right)\left(Y_{t_n} - Y_{t_{n-1}}\right)
$$

and

$$
\tfrac{d}{dt}\langle X, Y \rangle_t = \lim_{\Delta \to 0} \frac{\langle X, Y \rangle_{t+\Delta} - \langle X, Y \rangle_t}{\Delta}
$$

where $\Pi = \{0 = t_0, t_1, \ldots, t_N = t\}$ is a partition of $[0, t]$ and $|\Pi|$ denotes its mesh. As a result, the continuous observation of aggregate consumption on the interval $[0, \tau]$ implies that the volatility of aggregate consumption

$$
v = \dot{Q}_t(\log D)^{\frac{1}{2}} > 0
$$

is observable. Since the stock price is continuously observable on $[0,\tau]$ this in turn implies that its volatility

$$\sigma_t = \frac{1}{v}\dot{Q}_t(\log S) \neq 0$$

is continuously observable on $[0,\tau]$. Combining this property with the continuous observability of the interest rate then shows that the market price of risk θ_t^0 and the first moment

$$M_{1t} = v\left(\gamma v - \theta_t^0\right) = v\left(\gamma v - \frac{1}{dt}\left[\frac{dP_t}{\sigma_t P_t} - \frac{dD_t}{vD_t}\right] + \frac{r_t}{\sigma_t}\right)$$

are both continuously observable on $[0,\tau]$. The moment recursion (30) then implies that all the successive moments of the consumption weighted distribution of beliefs are continuously observable on $[0,\tau]$ and, since $\mathcal{X}$ is a bounded interval, the result follows from the determinacy of the Hausdorff moment problem. ■

Corollary 1. *If $\tau > 0$ then the wealth-weighted distributions of beliefs $\{\mathcal{W}_t\}_{t\leq\tau}$ can be recovered from the continuous observation of $\{(r_t, P_t, D_t)\}_{t\leq\tau}$.*

Proof. This follows from Theorem 1 and the absolute continuity relation

$$\frac{\mathcal{W}_t(dx)}{\mathcal{S}_t(dx)} = \frac{\Lambda_t^x D_t}{P_t}$$

since the density process Λ_t^x defined in equation (21) can be computed given the initial consumption-weighted distribution of beliefs. ■

The above results raise the question of whether the recovery of the consumption or wealth-weighted distributions of beliefs is in fact possible for all utility functions. The answer is *negative*. Intuitively, the key feature that permits the recovery in our setting is that, as a result of the wealth effects induced by our preference specification, the moments of the consumption weighted distribution of beliefs are stochastic processes that covary with aggregate consumption. If instead investors' preferences are such that these moments are constant then one would naturally expect that recovery is impossible because no information can be obtained from their successive quadratic covariation with aggregate consumption.

To confirm this intuition, consider a version of the model where investors have exponential utility with absolute risk aversion $a > 0$ and aggregate consumption evolves according to

$$dD_t = v dB_t^0 = x dt + v dB_t^x$$

for some constant $v > 0$. The derivation of the equilibrium for such a model is standard, at least with finitely many types. In particular, if

$$b := \rho + am_1 + \frac{1}{2}a^2v^2 - \frac{m_2 - m_1^2}{2v^2} > 0$$

where

$$m_n := \int_{\mathcal{X}} x^n \mu(dx)$$

denotes the nth moment of the (unweighted) beliefs distribution, then the interest rate, the market price of risk, and the asset price are given by

$$\left(r, \theta^0, P_t\right) = \left(b, av - \frac{m_1}{v}, \frac{D_t}{b} + \frac{m_1 - av^2}{b^2}\right).$$

Within such an economy the observation of aggregate consumption, the asset price, and the interest rate reveals the first two moments of the probability measure μ. But, since these moments are always constant, it is no longer possible to recover further moments by computing iterated quadratic covariations. By contrast, the only way that r_t and θ_t^0 in our economy with constant relative risk aversion is that agents share a common belief given by the first moment.

Remark 4. The moment recursion (30) can also written in terms of the cumulants of the consumption weighted distribution of beliefs. Indeed, a straightforward but tedious calculation combining (30) with the relation

$$\kappa_{nt} = M_{nt} - \sum_{i=1}^{n-1} \binom{n-1}{i} \kappa_{n-it} M_{it}$$

between cumulants and moments shows that the sequence of cumulants satisfies the simpler recursion given by $\kappa_{n+1t} = \gamma \dot{Q}_t(\kappa_n)$ subject to $\kappa_{1t} = M_{1t}$. A similar relation between successive cumulants has recently been exploited by Dew-Becker, Giglio, and Molavi (2024) in a Bayesian filtering environment.

Remark 5. The assumption that μ has compact support plays a crucial role in the proof of our recovery theorem as it guarantees that the distributions $\{\mathcal{S}_t\}$ are uniquely determined by their sequence of moments. If we relax this assumption to allow for eiher $\mathcal{X} = \mathbf{R}_+$ or $\mathcal{X} = \mathbf{R}$ then this property is lost in general but can be recovered by imposing appropriate technical conditions. In particular, if we assume that the *initial* distribution satisfies *Carleman's condition*:

$$\sum_{n=1}^{\infty} M_{an,0}^{-\frac{1}{2n}} = \infty$$

with $a = 1$ for $\mathcal{X} = \mathbf{R}_+$ and $a = 2$ for $\mathcal{X} = \mathbf{R}$, then $\mathcal{S}_0$ is uniquely determined by its moments and, thus, can be recovered from the observation of these moments which in turn implies that $\mathcal{S}_t$ at later dates can be recovered through the absolute continuity relation (16). See Schmüdgen (2017, Chapter 4.2–3) for a discussion of sufficient conditions for determinacy of the moment problem.

4.2 An optimization-based approach

The result of Theorem 1 relies on the *continuous* observation of (r_t, D_t, P_t) over the interval $[0, \tau]$ to reconstruct the paths of all the moments of the consumption-weighted distribution of beliefs. In practice, however, the fact these processes are only observed at discrete points effectively prevents one from using the recursion (30) as a way of recovering the moments. To see this, assume that we are given observations of the pair (D_t, M_{1t}) over a sequence of partitions of $[0, \tau]$ such that

$$\Pi_N = \{0 = t_{0,N}, t_{1,N}, \ldots, t_{N,N} = \tau\} \subseteq \Pi_{N+1} \subseteq \Pi_{N+2} \cdots \to [0, \tau].$$

Suppose that for each $t = t_{j_0,N_0} = t_{j_N,N} \in \Pi_{N_0}$ with $N \geq N_0$ we approximate the first required quadratic covariation as

$$Q_t(M_1) \approx Q_{t,N}(M_1) := \sum_{i=1}^{j_N} \Delta M_{1t_{i,N}} \Delta \log D_{t_{i,N}}$$

and then use a finite difference scheme to approximate its time derivative and thereby obtain an approximation of the second moment

$$M_{2t} = M_{1t}^2 + \gamma \dot{Q}_t(M_1) \approx M_{1t}^2 + \gamma \dot{Q}_{t,N}(M_1).$$

Well-known results ensure that this simple approximation converges as $N \to \infty$, but the approximation error is unbounded since the quadratic covariation is not twice continuously differentiable. As a result, iterating this approximation to reconstruct the path of sufficiently many moments generates evergrowing errors that seriously impair the recovery process. In particular, there is no guarantee that the induced sequence of approximate moments actually represents the moment sequence of some well-defined distribution on the type space.

To circumvent this difficulty one could try to refine the approximation to achieve a better control of the errors. Such an approach may allow to reconstruct the path of the second moment M_{2t} but it is unlikely to remain efficient after two or more successive applications of the moment recursion (30). Instead, we show in Theorem 2 below that the model is *perfectly identified* in the sense that the equilibrium interest rate and the asset price are injective when viewed as mappings from the set of *initial* consumption weighted distributions of beliefs to the set of paths; and use this result to formulate the recovery problem as an optimization problem.

Before stating the results, let us first fix some notation. For an arbitrary probability measure $\mathfrak{S} \in \mathcal{P}$ we let

$$\Phi_t^{\frac{1}{\gamma}}(\mathfrak{S}) := \int_{\mathcal{X}} (Z_t^x)^{\frac{1}{\gamma}} \mathfrak{S}(dx), \tag{31a}$$

$$M_{nt}(\mathfrak{S}) := \int_{\mathcal{X}} x^n \left(\frac{Z_t^x}{\Phi_t(\mathfrak{S})} \right)^{\frac{1}{\gamma}} \mathfrak{S}(dx), \tag{31b}$$

and denote by

$$P_t(\mathfrak{S}) := D_t \mathbf{E}_t^0 \left[\int_t^\infty e^{-\rho(u-t)} \left(\frac{D_u}{D_t} \right)^{1-\gamma} \frac{\Phi_u(\mathfrak{S})}{\Phi_t(\mathfrak{S})} du \right], \tag{31c}$$

$$\sigma_t(\mathfrak{S}) := \frac{1}{v} \dot{Q}_t(\log P(\mathfrak{S})), \tag{31d}$$

and

$$\begin{aligned} r_t(\mathfrak{S}) := \rho + \gamma M_{1t}(\mathfrak{S}) - \frac{\gamma}{2}(1+\gamma) v^2 \\ + \frac{1}{2v^2}\left(1 - \frac{1}{\gamma}\right)\left(M_{2t}(\mathfrak{S}) - M_{1t}^2(\mathfrak{S})\right) \end{aligned} \tag{31e}$$

the risky asset price, the risky asset volatility, and the interest rate that obtain when the *initial* consumption-weighted distribution of beliefs is set to $\mathcal{S}_0 = \mathfrak{S}$. The next proposition provides an integral representation of the price-dividend ratio induced by an initial consumption-weighted distribution of beliefs and uses the representation to establish the existence and uniqueness of the equilibrium.

Proposition 4. *If condition (26) holds then*

$$\frac{P_t(\mathfrak{S})}{D_t} = \int_0^\infty d\tau \int_{\mathbf{R}} \mathbf{n}(dz) \left\{ \frac{\int_{\mathcal{X}} \xi\left(\tau, x, z\right) \psi\left(t, x, \frac{D_t}{D_0}\right) \mathfrak{S}(dx)}{\int_{\mathcal{X}} \psi\left(t, x, \frac{D_t}{D_0}\right) \mathfrak{S}(dx)} \right\}^{\gamma} \tag{32}$$

for any $\mathfrak{S} \in \mathcal{P}$ *where*

$$\begin{aligned} \psi\left(t, x, \delta\right) &:= \delta^{\frac{x}{\gamma v^2}} e^{\frac{x}{2\gamma}\left(1 - \frac{x}{v^2}\right)t} \\ \xi\left(\tau, x, z\right) &:= e^{\left\{\frac{1}{2}\left(1-\frac{1}{\gamma}\right)v^2 - \frac{\rho}{\gamma} - \frac{1}{2}\frac{x^2}{\gamma v^2}\right\}\tau + \left\{\frac{x}{\gamma v} + \frac{1}{\gamma} - 1\right\} v\sqrt{\tau} z} \end{aligned} \tag{33}$$

and $\mathbf{n}(dz)$ *denotes the standard gaussian density. Furthermore,* $\sigma_t(\mathfrak{S}) \neq 0$ *almost everywhere and it follows that there exists a unique equilibrium for any initial consumption-weighted distribution of beliefs.*

Proof. See Appendix B. ■

Theorem 2. *Assume that (26) holds. If* $(\mathfrak{S}, \mathfrak{T}) \in \mathcal{P}^2$ *are distinct probability measures on the type space then the event*

$$E_\tau := \left\{ \omega \in \Omega : \max_{t \in [0,\tau]} \left| (r_t(\mathfrak{S}), P_t(\mathfrak{S})) - (r_t(\mathfrak{T}), P_t(\mathfrak{T})) \right| = 0 \right\}$$

has zero measure under $\mathbf{P}^0$ *for any* $\tau > 0$*. In particular, if* $(\mathfrak{S}, \mathfrak{T}) \in \mathcal{P}^2$ *then*

$$(\mathfrak{S} = \mathfrak{T}) \Leftrightarrow \int_0^\tau \phi\left((r_t(\mathfrak{S}), P_t(\mathfrak{S})) - (r_t(\mathfrak{T}), P_t(\mathfrak{T})) \right) dt = 0$$

for any $\tau > 0$ *and any norm* $\phi : \mathbf{R}^2 \to \mathbf{R}_+$*.*

Proof. Fix some $\tau > 0$ and some $(\mathfrak{S}, \mathfrak{T}) \in \mathcal{P}^2$ such that $\mathfrak{S} \neq \mathfrak{T}$. To prove the first part of the statement assume towards a contradiction that the set E_τ has strictly positive measure under $\mathbf{P}^0$. By definition

$$e^{-\int_0^t r_u(\mathfrak{S}) du} P_t(\mathfrak{S}) = e^{-\int_0^t r_u(\mathfrak{T}) du} P_t(\mathfrak{T})$$

for almost every $(t,\omega) \in [0,\tau] \times E_\tau$. Applying Itô's lemma to both sides of this equality and rearranging the result then shows that

$$\int_0^t e^{-\int_0^s r_u(\mathfrak{S})du} P_s(\mathfrak{S}) \left(\sigma_t(\mathfrak{S}) - \sigma_t(\mathfrak{T})\right) dB_s^0 \\ = \int_0^t e^{-\int_0^s r_u(\mathfrak{S})du} P_s(\mathfrak{S}) \left(\sigma_t(\mathfrak{S})\theta_s^0(\mathfrak{S}) - \sigma_t(\mathfrak{T})\theta_s^0(\mathfrak{T})\right) ds$$

for almost every $(t,\omega) \in [0,\tau] \times E_\tau$. Because the left hand side is singular and the right hand side is absolutely continuous, it is clear that both sides must equal zero. In particular, we have that $\sigma_t(\mathfrak{S}) = \sigma_t(\mathfrak{T})$ almost everywhere and combining this property with the fact that $\sigma_t(\mathfrak{S}) \neq 0$ almost everywhere we deduce that

$$\gamma v - \frac{1}{v} M_{1t}(\mathfrak{S}) = \theta_s^0(\mathfrak{S}) = \theta_s^0(\mathfrak{T}) = \gamma v - \frac{1}{v} M_{1t}(\mathfrak{T})$$

for almost every $(t,\omega) \in [0,\tau] \times E_\tau$. Applying the moment recursion (30) to both sides of this equality then shows that

$$M_{nt}(\mathfrak{T}) = M_{nt}(\mathfrak{S}), \qquad n \geq 1,$$

for almost every $(t,\omega) \in [0,\tau] \times E_\tau$. In particular, $M_{n0}(\mathfrak{S}) = M_{n0}(\mathfrak{T})$ for all $n \geq 1$ and the required contradiction now follows from the fact that measures on a bounded set are uniquely determined by their moments. The second part of the statement follows immediately from the first. ■

In view of Theorem 2, a natural way to recover $\mathcal{S}$ from the continuous observation of the induced interest rate and asset price over a time interval $[0,\tau]$ is to pick a norm ϕ on $\mathbf{R}^2$ and solve

$$\mathcal{L}(\mathcal{S}) := \min_{\mathfrak{T}\in\mathcal{P}} \int_0^\tau \phi\left((r_t(\mathcal{S}), P_t(\mathcal{S})) - (r_t(\mathfrak{T}), P_t(\mathfrak{T}))\right) dt.$$

In practice, this approach is not directly feasible as we typically only have access to discrete observations of these processes. To accommodate this feature we consider instead the discretized optimization problem given by

$$\begin{aligned} \mathcal{L}(\mathcal{S};N) &:= \min_{\mathfrak{T}\in\mathcal{P}} \mathcal{L}\left(\mathcal{S},\mathfrak{T};N\right) \\ &= \min_{\mathfrak{T}\in\mathcal{P}} \sum_{n=1}^N \phi\left(\left(r_{t_{n,N}}(\mathcal{S}), P_{t_{n,N}}(\mathcal{S})\right) - \left(r_{t_{n,N}}(\mathfrak{T}), P_{t_{n,N}}(\mathfrak{T})\right)\right) \Delta_{n,N} \end{aligned} \tag{34}$$

where $t_{n,N} \in \Pi_N$ denote the observation times and $\Delta_{n,N} := t_{n,N} - t_{n-1,N}$. The next result justifies this approach by showing that any solution to the discretized problem provides an approximation of the underlying $\mathcal{S}$. Since any continuous distribution is the weak limit of a sequence of discrete distributions, it also shows that in our framework the economy with a continuum of types can be realized as the limit of a sequence of economies with finitely many types.

Proposition 5. *Assume that condition (26) holds and that $\{\mathfrak{S}_N^*\} \subseteq \mathcal{P}$ is a sequence of probability measures such that*

$$\mathfrak{S}_N^* \in \operatorname*{argmin}_{\mathfrak{S}\in\mathcal{P}} \mathcal{L}\left(\mathcal{S}, \mathfrak{S}; N\right), \qquad N \geq 1.$$

Then the sequence $\{\mathfrak{S}_N^*\}$ converges weakly to $\mathcal{S}$ as $N \to \infty$.

Proof. See Appendix C. ■

4.3 Implementation

In this section we illustrate the approach put forth in Theorem 2 and Proposition 5 by implementing the recovery process on discrete time-series of equilibrium interest rates and asset prices simulated from the model.

Investors. We assume that investors discount the future at rate $\rho = 0.03$ and have relative risk aversion $\gamma = 2$. From Mehra and Prescott (1985) we know that aggregate consumption grows at a yearly rate of $m = 2\%$ with a standard deviation of about $v = 4\%$. With these figures in mind we set

$$\mathcal{X} := [m - v, m + v] = [-2\%, 6\%]$$

so that the type space only includes beliefs that lie within one standard deviation of the empirical mean. Note with these parameter choices we have $\lambda = 0.0084 > 0$ which shows that condition (26) is satisfied.

Test cases. For the initial consumption-weighted distributions of beliefs $\mathcal{S}_0$ to be recovered we use six distributions on $\mathcal{X}$. As illustrated in Figure 1 these include:

1. A degenerate distribution supported on $\{0.02\}$

2. A discrete distribution supported on $\{-0.01, 0.02, 0.05\}$

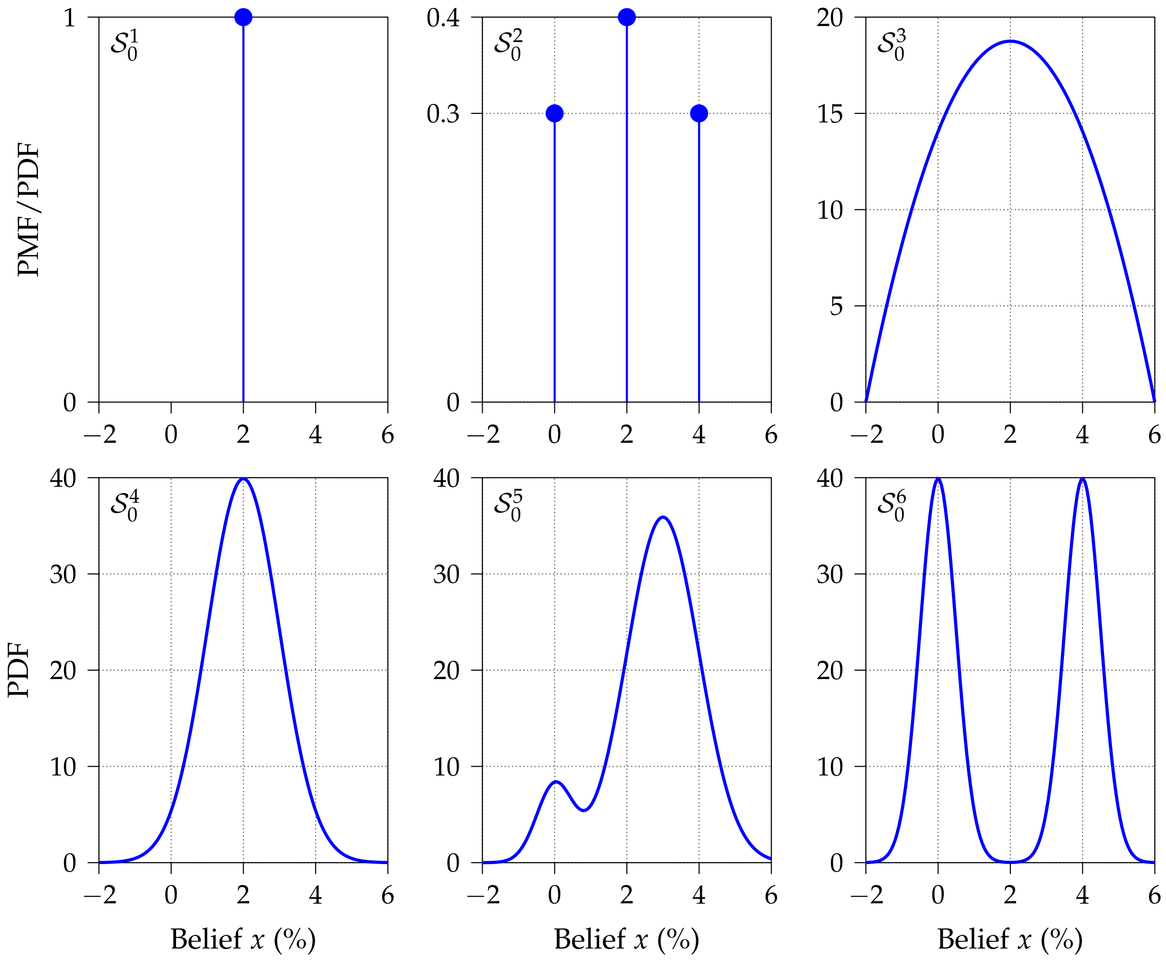

Figure 1: Test cases

Notes. This figure plots the probability mass/density functions (PMF/PDF) of the different initial consumption-weighted distributions of beliefs to be recovered. The first two are discrete while the last four are continuous and correspond to, respectively, a rescaled $\beta(2,2)$ distribution, a truncated normal distribution, and two different truncated mixtures of two normal distributions.

3. A rescaled $\text{Beta}(2,2)$
4. A truncated normal with parameters $(0.02, 0.01)$
5. A truncated mixture of two normals with parameters $(0, 0.01)$ and $(0, 0.005)$
6. A truncated mixture of two normals with parameters $(0, 0.005)$ and $(0.04, 0.005)$

Simulated observations. To construct the simulated time series of prices and interest rates on which we implement the recovery we fix a length of one year with daily observations so that $\tau = 1$ and $\Delta = \frac{1}{N} = \frac{1}{250}$. Next, we simulate the path of aggregate consumption according to the exact discretisation scheme

$$D_{t_n} = \mathbf{1}_{\{n=0\}} + \mathbf{1}_{\{n>0\}} D_{t_{n-1}} e^{-\frac{1}{2}v^2\Delta + v\sqrt{\Delta} z_n}$$

where $t_n = n/250$ denote the observation times and $\{z_n\}$ is a sequence of iid standard normal random variables under $\mathbf{P}^0$. The simulated time series for each test case is then obtained by applying the formulae of Propositions 1 and 4 along the simulated path of aggregate consumption.

Approximate problem. Let $\phi(x) := \|x\|^2$ denote the Euclidean norm. To perform the recovery for our test cases we need to numerically solve six instances of (34). This is a arduous task because (34) optimizes over infinite-dimensional objects and requires the computation of the triple integral (32) to evaluate its objective function. We circumvent both difficulties by considering the approximate problem

$$\inf_{\mathfrak{S} \in \mathcal{P}_d} \sum_{n=1}^{N} \left\| (r_{t_n}(\mathcal{S}_0^i) - r_{t_n}(\mathfrak{S}), P_{t_n}(\mathcal{S}_0^i) - P_{t_n}(\mathfrak{S}) \right\|^2 \Delta \tag{35}$$

where $\mathcal{S}_0^i$ denotes the initial consumption-weighted distribution of beliefs in the ith test case and the infimum is over the set $\mathcal{P}_d$ of discrete probabilities whose support is contained in the finite set

$$\mathcal{X}_d := \left\{ -0.02 + \frac{n}{16} |\mathcal{X}| : n = 0, 1, \ldots, 16 \right\} \in \mathbf{R}^{17}.$$

Such a parameterization makes the problem finite-dimensional and greatly facilitates the computation of the objective function. Indeed, Corollary 2 in Appendix B shows that the interest rate and asset price induced by a discrete $\mathfrak{S}$ are explicit functions of t and D_t/D_0 whenever γ is an integer. Furthermore, since these functions are computationally inexpensive, the error generated by solving instances of (35) instead of (34) can be made small without sacrificing the runtime of the implementation by increasing the dimension of $\mathcal{X}_d$.

Results. For each test case we numerically solve the approximate problem (35) using a basin-hopping algorithm to obtain an estimate of the initial consumption-weighted distribution of beliefs and then repeat the whole procedure a thousand times to gauge the distribution of the estimates.

Case	Mean	Std.	Min	25%	50%	75%	Max
				Loss			
1	$1.38e{-}29$	$1.26e{-}29$	$3.97e{-}30$	$6.83e{-}30$	$8.10e{-}30$	$1.91e{-}29$	$1.43e{-}28$
2	$4.22e{-}09$	$6.81e{-}09$	$6.61e{-}21$	$6.77e{-}11$	$6.28e{-}10$	$5.43e{-}09$	$4.61e{-}08$
3	$1.46e{-}10$	$1.84e{-}10$	$3.81e{-}13$	$1.31e{-}11$	$8.79e{-}11$	$1.98e{-}10$	$1.61e{-}09$
4	$5.45e{-}12$	$4.35e{-}11$	$1.44e{-}14$	$7.26e{-}13$	$1.65e{-}12$	$3.64e{-}12$	$1.27e{-}09$
5	$5.43e{-}12$	$1.34e{-}11$	$2.13e{-}16$	$4.23e{-}13$	$1.13e{-}12$	$3.97e{-}12$	$1.75e{-}10$
6	$3.60e - 10$	$4.77e - 10$	$1.10e - 14$	$1.84e - 11$	$1.93e - 10$	$5.37e - 10$	$5.07e - 09$
				Wasserstein distance			
1	$6.86e - 16$	$1.21e - 15$	$0.00e + 00$	$0.00e + 00$	$0.00e + 00$	$1.19e - 15$	$1.03e - 14$
2	$1.04e - 03$	$1.38e - 03$	$3.44e - 10$	$7.49e - 05$	$2.89e - 04$	$1.68e - 03$	$5.70e - 03$
3	$2.03e - 03$	$3.27e - 04$	$1.31e - 03$	$1.79e - 03$	$2.01e - 03$	$2.25e - 03$	$2.96e - 03$
4	$2.01e - 03$	$3.10e - 04$	$1.31e - 03$	$1.76e - 03$	$2.02e - 03$	$2.27e - 03$	$3.34e - 03$
5	$2.11e - 03$	$3.58e - 04$	$1.33e - 03$	$1.85e - 03$	$2.08e - 03$	$2.33e - 03$	$3.45e - 03$
6	$1.90e - 03$	$3.19e - 04$	$1.29e - 03$	$1.66e - 03$	$1.87e - 03$	$2.10e - 03$	$3.08e - 03$

Table 1: Summary statistics of the recovery procedure

Notes. This upper panel reports summary statistics across simulations for the value of the approximate problem (35) (upper panel) and for the Wasserstein distance

$$W(\mathfrak{S}^i, \mathcal{S}_0^i) := \inf_{\Gamma \in \mathcal{C}(\mathfrak{S}^i, \mathcal{S}_0^i)} \int_{\mathcal{X}^2} |x - y| \, \Gamma(dx, dy)$$

where $\mathcal{C}(\mathfrak{S}^i, \mathcal{S}_0^i)$ denotes the set of all couplings between between the true distribution $\mathcal{S}_0^i$ and the recovered distribution $\mathfrak{S}^i$ (lower panel).

The results of this procedure are reported in the top panel of Table 1 which provides summary statistics for the value of (35) across simulations and in Figure 2 which illustrates the differences between the true cumulative distribution function (CDF) and the average of the recovered CDFs for each test case. In the first two test cases the recovery is very effective. In particular, it perfectly identifies the point mass distribution of the first test cases and is able to discover that in the second test case most of the mass is concentrated on three points. This success is partly due to the fact that, since the supports of $\mathcal{S}_0^1$ and $\mathcal{S}_0^2$ are contained in the set $\mathcal{X}_d$, these two recoveries are only affected by the time discretisation used in the simulation but not by the coarse space discretisation that we applied to obtain a finite dimensional problem. The other four test cases are affected by both types of discretisations but our results show that, despite this, the recovery remains quite successful. In particular, and as can be seen from the figure, the algorithm effectively learns the slope of the cumulative distribution

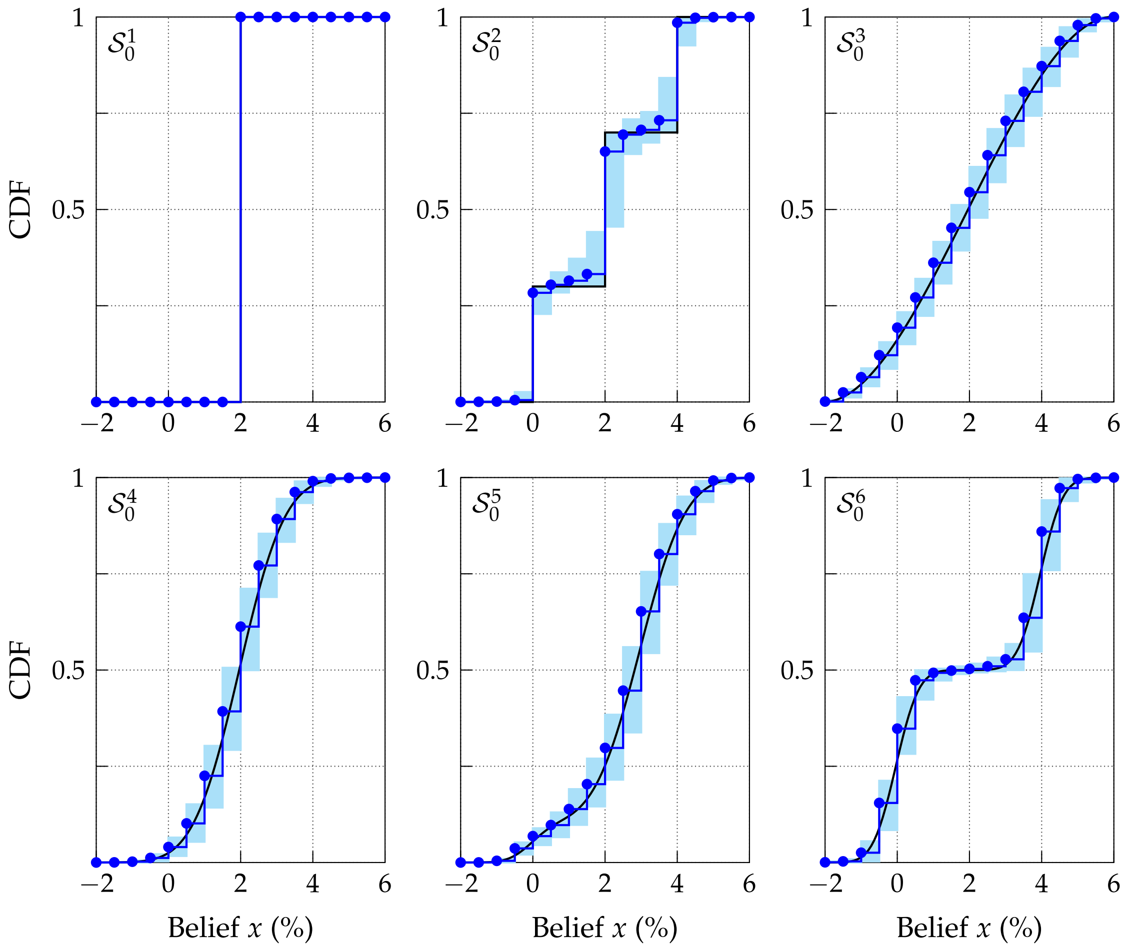

Figure 2: Recovered cumulative distributions

Notes. This figure shows the average of the cumulative distributions functions of the recovered distribution across 1,000 simulations (blue dotted) and the true cumulative distribution (thick black). In each panel, the shaded areas represent the intervals between the 5th and 95th percentile of recovered cumulative distribution values.

function and is thus able to correctly recover the regions of the type space where most of the mass is concentrated.

5 Extensions

5.1 Multidimensional beliefs

In this section we extend our main result to allow for *multiple sources of risk* and *multiple risky assets*. To this end we let $n \geq 1$ denote the number of risky assets and assume that the filtration is generated by a continuous *vector* process $D_t \in \mathbb{R}^n_{++}$ with quadratic variation given by

$$\langle D \rangle_t = \int_0^t \mathrm{diag}(D_u) V V^\top \mathrm{diag}(D_u) du$$

for some *invertible* volatility matrix $V \in \mathbb{R}^{n \times n}$. The ith component D_{it} of the vector D_t represents the dividend rate of the ith risky asset and the sum

$$\overline{D}_t := \mathbf{1}^\top D_t = \sum_{i=1}^n D_{it}$$

gives the aggregate supply of the consumption good.

The economy is populated by a continuum of investors with subjective discount rate ρ, constant relative risk aversion γ, and heterogenous beliefs about the growth rates of dividends. Specifically, we fix a bounded rectangle $\mathcal{X} \subset \mathbf{R}^n$ and assume that investors of type $x \in \mathcal{X}$ are endowed with a subjective probability measure $\mathbf{P}^x$ relative to which the process

$$B_t^x := \int_0^t V^{-1} \left(\mathrm{diag}(D_u)^{-1} dD_u - x du \right)$$

is a n-dimensional Brownian motion. In particular, investors of type x believe that the vector of dividends evolves according to

$$dD_t = \mathrm{diag}(D_t) \left(x dt + V dB_t^x \right).$$

As before, we denote by $\mu(dx)$ and $\mathcal{W}_0(dx)$ the distributions of types and the initial wealth-weighted distribution of beliefs. The only difference with the baseline model is that these distributions are now probability measures over a multidimensional set rather than over an interval.

The financial market operates in continuous-time and consists of $n+1$ assets: a riskless asset in zero net supply and n risky assets in positive supply of one unit each. The prices of these assets evolve according to (4) and

$$P_t = P_0 + \int_0^t \left(r_u P_u - D_u \right) du + \int_0^t \mathrm{diag}(P_u) \Sigma_u \left(dB_u^x + \theta_u^x du \right)$$

where the riskless interest rate r_t, the initial price *vector* P_0, the volatility *matrix* Σ_t, and the *vector* of type-specific market prices of risk

$$\theta_t^x = \theta_t^y + V^{-1}(x-y)\,t, \qquad (x,y) \in \mathcal{X}^2 \tag{36}$$

are to be determined endogenously in equilibrium.

Under the usual self-financing condition, the wealth of an investor of type x evolves according to

$$W_t^x = W_0^x + \int_0^t \left(r_u W_u^x - s_u \overline{D}_u\right) du + \int_0^t \pi_u^\top \operatorname{diag}(P_u)\Sigma_u \left(dB_u^x + \theta_u^x du\right) \tag{37}$$

where $s_t \in (0,1)$ represents her share of aggregate consumption and $\pi_t \in \mathbf{R}^n$ captures the number of shares that she holds in each of the risky assets. The optimization of such an investor is accordingly given by (7) with the budget constraint (6) replaced by (37) and, as before, we denote its solution by (s_t^x, π_t^x) whenever it exists.

Definition 2. *An equilibrium is a list*

$$\mathcal{E} = \langle r_t, \Sigma_t, \theta_t^x, P_0 \rangle$$

such that (36) *holds;* $\det \Sigma_t > 0$ *almost everywhere; the optimal strategy* (s_t^x, π_t^x) *in* (7) *exists for all types; and the market clearing conditions*

$$\int_{\mathcal{X}} s_t^x \mu(dx) = 1, \qquad \int_{\mathcal{X}} \pi_t^x \mu(dx) = \mathbf{1}, \qquad \int_{\mathcal{X}} W_t^x \mu(dx) = \mathbf{1}^\top P_t$$

hold almost everywhere.

Since the asset volatility matrix Σ_t is invertible at all times, we have that markets are complete in equilibrium. Therefore, the same arguments as in the unidimensional case show that the consumption share s_t^x and pricing kernel H_t^x of investors of type x are given by (8) and (9) suitably modified to account for the fact that the market price of risk θ_t^x is now a vector process. The consumption-weighted distribution of beliefs $\mathcal{S}_t$ that prevails in equilibrium is then given by (16) and we have the following multidimensional counterpart to Proposition 1

Proposition 6. *In equilibrium, the type-specific market prices of risk and the riskless rate of interest are given by*

$$\theta_t^x = \gamma V^\top \omega_t + V^{-1}(x - m_t)$$

and

$$r_t = \rho + \gamma \omega_t^\top m_t - \frac{\gamma}{2}(1+\gamma)\|V^\top \omega_t\|^2 + \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\mathrm{Tr}(V^{-1}C_t V^{-1\top})$$

where $\omega_t := D_t/\overline{D}_t \in \mathbf{R}^n$ *measures the share of aggregate consumption paid out by each of the traded risky asset and*

$$m_t := \int_{\mathcal{X}} x\mathcal{S}_t(dx) \in \mathbf{R}^n$$
$$C_t := \int_{\mathcal{X}} xx^\top \mathcal{S}_t(dx) - m_t m_t^\top \in \mathbf{R}^{n\times n}$$

denote, respectively, the mean and the variance-covariance matrix of $\mathcal{S}_t$*.*

Proof. See Appendix D.1. ■

The continuous observation of (r_t, P_t, D_t) implies the continuous observation of the market price of risk $\theta_t^{\mathbf{0}}$ which in turn implies the continuous observation of m_t and we prove that recovery is possible by showing that the observation of this process allows to reconstruct *all the moments*

$$M_{kt} := \int_{\mathcal{X}} \left(\prod_{i=1}^n x_i^{k_i}\right)\mathcal{S}_t(dx), \qquad k \in \mathbf{N}^n,$$

of the consumption-weighted distribution of beliefs. To state the result, let $\mathbf{e}_i \in \mathbf{R}^n$ be the ith basis vector and denote by

$$Q_t(X) := \langle X, \log D\rangle_t = \begin{pmatrix} \langle X, \log D_1\rangle_t \\ \langle X, \log D_2\rangle_t \\ \vdots \\ \langle X, \log D_n\rangle_t \end{pmatrix} \in \mathbf{R}^n$$

the quadratic covariation between an adapted process $X_t \in \mathbf{R}$ and the componentwise logarithm of the dividend process. The following multidimensional counterpart to Theorem 1 constitutes the main result of this section.

Theorem 3. *In equilibrium*

$$\begin{pmatrix} M_{k+\mathbf{e}_1,t} \\ M_{k+\mathbf{e}_2,t} \\ \vdots \\ M_{k+\mathbf{e}_n,t} \end{pmatrix} = m_t M_{kt} + \gamma \dot{Q}_t(M_k), \qquad \forall k \in \mathbf{N}^n.$$

In particular, if $\tau > 0$ then the consumption-weighted distributions of beliefs $\{\mathcal{S}_t\}_{t\leq\tau}$ can be recovered from the continuous observation of $\{(r_t, P_t, D_t)\}_{t\leq\tau}$.

Proof. See Appendix D.1. ■

Building on Theorem 3 one can recover the wealth-weighted distribution of beliefs by exploiting the fact that, as in the baseline model,

$$\mathcal{W}_t(dx) = \left(\frac{D_t^\top \mathbf{1}}{P_t^\top \mathbf{1}}\right) \Lambda_t^x \mathcal{S}_t(dx)$$

for some process Λ_t^x that tracks the wealth-consumption ratio of investors of type x. See (49) in Appendix D.1 for the expression of this process in the context of this section. One can also build on Theorem 3 to establish a multidimensional analog of Theorem 2 and use that result to devise an optimization-based approach to the multidimensional recovery problem along the lines of Proposition 5. We leave the details of this extension to the interested reader.

5.2 Stochastic beliefs

For our second extension, we consider a more general structure of beliefs. To simplify the exposition, we present this extension in the context of a one dimensional model but similar results can also be established in the multidimensional case.

Our point of departure from the baseline model of Section 2 is that the evolution of the aggregate consumption process $D_t > 0$ perceived by investors of type $x \in \mathcal{X}$ is given by (compare with (2))

$$dD_t = D_t \left(\alpha_t^x dt + v dB_t^x\right)$$

where B_t^x is an observed Brownian motion under $\mathbf{P}^x$ and α_t^x is an $\mathbf{F}$-adapted process that models the expected growth rate of aggregate consumption of investors of type

x. Importantly, and as illustrated by the following examples, the fact that the expected growth rate now varies across time and states of nature allows to consider models in which investors learn from their observation of aggregate consumption rather than maintain constant beliefs.

Example 1. Assume that

$$dD_t = D_t\left(\alpha dt + v dB_t\right)$$

where α is unknown to investors and B_t is an *unobserved* Brownian motion. Investors gradually learn about the value of α from their observations of aggregate consumption starting and have heterogenous priors. Specifically, let $\mathcal{X}$ be compact interval and assume that investors of type $x \in \mathcal{X}$ initially believe that α is normally distributed with mean x and standard deviation $z > 0$. Classical results in filtering theory (see Liptser and Shiryaev (2001, Chapter 8)) show that in this case

$$\alpha_t^x := \mathbf{E}_t^x[\alpha] = \left(\frac{v^2}{z^2 t + v^2}\right) x + \left(1 - \frac{v^2}{z^2 t + v^2}\right)\left(\frac{v^2}{2} + \frac{1}{t}\log\frac{D_t}{D_0}\right) \tag{38}$$

is a time-varying convex combination of the prior mean x and maximum likelihood estimate of the growth rate over $[0, t]$.

Example 2. Assume that

$$dD_t = D_t\left(\alpha_t dt + v dB_t\right)$$

where B_t is an *unobserved* Brownian motion and α_t is a continuous-time Markov chain with two states $\alpha_L < \alpha_H$ and *known* transition intensities $\lambda_{LH}, \lambda_{HL} > 0$. In this example, investors of type $x \in \mathcal{X} = [0, 1]$ start from the prior that

$$\alpha_0^x := \mathbf{E}^x[\alpha_0] = \alpha_L + x\left(\alpha_H - \alpha_L\right)$$

and learn about the evolution of α_t by observing the path of aggregate consumption. Classical results in filtering theory (see Liptser and Shiryaev (2001, Chapter 9)) show that in this case

$$d\alpha_t^x = \left(\lambda_{HL} + \lambda_{LH}\right)\left(\alpha^* - \alpha_t^x\right) dt + \left(\alpha_t^x - \alpha_L\right)\left(\alpha_H - \alpha_t^x\right)\frac{dB_t^x}{v} \tag{39}$$

where the constant

$$\alpha^* := \alpha_L + \frac{\lambda_{LH}}{\lambda_{HL} + \lambda_{LH}} (\alpha_H - \alpha_L)$$

gives the unconditional mean of the underlying Markov chain.

The consumption-weighted distribution of beliefs $\mathcal{S}_t$ that prevails in equilibrium is then given by (12) and (16) albeit with

$$Z_t^x := \exp\left(-\frac{1}{2v^2}\int_0^t \left|\alpha_u^x - \alpha_u^0\right|^2 du + \frac{1}{v}\int_0^t \left(\alpha_u^x - \alpha_u^0\right) dB_u^0\right), \tag{40}$$

and proceeding as the in baseline case delivers the following characterization of the equilibrium interest rate and market price of risk.

Proposition 7. *In equilibrium*

$$\begin{aligned}\theta_t^x &= \gamma v + \frac{1}{v}(\alpha_t^x - A_{1t})\\ r_t &= \rho + \gamma A_{1t} - \frac{\gamma}{2}(1+\gamma)v^2 + \frac{1}{2v^2}\left(1 - \frac{1}{\gamma}\right)\left(A_{2t} - A_{1t}^2\right)\end{aligned}$$

where

$$A_{nt} := \int_{\mathcal{X}} (\alpha_t^x)^n \mathcal{S}_t(dx)$$

denote the cross-sectional moments of perceived growth rates under $\mathcal{S}_t$.

Proof. The proof is similar to that of Proposition 1. We omit the details. ■

The equilibrium is the same as in the baseline model except that x is replaced by the perceived growth rate α_t^x. The recovery procedure is however more complicated for two reasons. First, the process A_{1t} that can be recovered from the observation of the market price of risk is a cross-sectional moment of the process α_t^x under $\mathcal{S}_t$ rather than a moment of $\mathcal{S}_t$ itself. Second, since the perceived growth rates are stochastic they may have non zero quadratic covariation with aggregate consumption which prevent us from obtaining a meaningful recursion by successive applications of Q_t. To circumvent these difficulties we construct a sequence of *pseudo*-moments that can be recovered from the observation of $\{(A_{1t}, D_t)\}_{t\leq\tau}$ through a recursion similar to (30) and provide conditions under which this sequence is sufficiently rich to allow for the recovery of the consumption-weighted distribution of beliefs.

To state the results, let $\mathcal{C}_n \subseteq \mathbf{N}^n$ denote the set of vectors such that $\sum_{i=1}^n ik_i = n$ and fix a sequence $\mathbf{c} = \{c_{nk}\}_{k\in\mathcal{C}_n, n\in\mathbf{N}}$ such that $c_{11} = 1$. Next, consider the sequences of processes given by

$$\alpha_{nt}^x := \mathbf{1}_{\{n=1\}}\alpha_t^x + \mathbf{1}_{\{n>1\}}\dot{Q}_t(\alpha_{n-1}^x) \tag{41}$$

and

$$b_{nt}(x;\mathbf{c}) := \sum_{k\in\mathcal{C}_n} c_{nk}\left[\prod_{i=1}^n (\alpha_{it}^x)^{k_i}\right];$$

and define a sequence of pseudo-moments by setting

$$\beta_{nt}(\mathbf{c}) := \int_{\mathcal{X}} b_{nt}(x;\mathbf{c})\,\mathcal{S}_t(dx).$$

Importantly, the facts that $c_{11} = 1$ and $\mathcal{C}_1 = \{\mathbf{e}_1\}$ imply that $\beta_{1t}(\mathbf{c}) = A_{1t}$ can be recovered from the observation of the market price of risk. The following theorem provides sufficient conditions for recovery in this extended setting and constitutes the main result of this section.

Theorem 4. *There exists $\hat{\mathbf{c}}$ that is independent from $\mathcal{S}_0$ and such that*

$$\beta_{n+1,t}(\hat{\mathbf{c}}) = \frac{1}{\gamma}\beta_{1t}(\hat{\mathbf{c}})\beta_{nt}(\hat{\mathbf{c}}) + \dot{Q}_t(\beta_n(\hat{\mathbf{c}})), \qquad \forall n \geq 1. \tag{42}$$

In particular, if $\tau > 0$ and $\operatorname{span}\left(\{b_{n0}(\cdot;\hat{\mathbf{c}})\}_{n\in\mathbf{N}}\right)$ is dense in the space of continuous functions on $\mathcal{X}$ then the consumption-weighted distributions of beliefs $\{\mathcal{S}_t\}_{t\leq\tau}$ can be recovered from the the continuous observation of $\{(r_t, P_t, D_t)\}_{t\leq\tau}$.

Proof. See Appendix D.2 ■

The above spanning condition is clearly satisfied in the baseline model where we have $\alpha_t^x \equiv x$ because in that case $b_{n0}(x;\hat{\mathbf{c}}) = x^n/\gamma^{n-1}$ and $\operatorname{span}\left(\{b_{n0}(\cdot,\hat{\mathbf{c}})\}_{n\in\mathbf{N}}\right)$ is the set of all polynomials. The following proposition shows that it also holds in Examples 1 & 2 and concludes this section.

Proposition 8. *The spanning condition of Theorem 4 holds*

1. *in Example 1,*
2. *in Example 2 provided that $1/\gamma \notin \mathbf{N}$.*

Proof. See Appendix D.2. ■

6 Conclusion

We show that, in a continuous-time economy populated by a continuum of investors with homogeneous CRRA utility and heterogenous beliefs about the growth rate of aggregate consumption, the support of the distribution of beliefs and the distributions of consumptions or wealth shares across beliefs can be recovered from the observation of equilibrium prices. The theoretical result relies on continuous observations that are not available in practice. We sidestep this difficulty by showing that the recovery problem can be formulated as a pathwise optimization problem whose solution may be reliably approximated using only discrete observations and confirm the validity of this approach on simulated data.

Although our baseline model may seem limited, we demonstrate that the results extend to more sophisticated environments with multiple assets and stochastic beliefs. A natural question is whether our pathwise methodology can be generalized to allow for additional dimensions of heterogeneity among investors such as heterogenous risk aversions or heterogenous subjective discount rates. We leave this challenging extension for future research.

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A Transversality

In Section 3.3 we claimed that to ensure that the transversality condition (25) holds in the economy with heterogenous types it is sufficient to assume that it holds in all the underlying homogeneous economies. This validity of this claim is a direct consequence of the following lemma:

Lemma 1. *Fix $t \geq 0$ and assume that condition (26) holds. Then*

$$G_{tu}(\mathfrak{S}) := e^{-\rho(u-t)} \left(\frac{D_u}{D_t} \right)^{1-\gamma} \frac{\Phi_u(\mathfrak{S})}{\Phi_t(\mathfrak{S})} \leq \Gamma_u, \qquad u \geq t, \tag{43}$$

for some adapted process that is integrable relative to $\mathrm{Leb}[t, \infty) \otimes \mathbf{P}_t^0$*. In particular, the family of random variables*

$$\{G_{tu}(\mathfrak{S}) : \mathfrak{S} \in \mathcal{P}\}$$

is uniformly integrable relative to $\mathrm{Leb}[t, \infty) \otimes \mathbf{P}_t^0$.

Proof. Fix $t \geq 0$ and let $\mathfrak{S}_t \in \mathcal{P}$ be defined as in (12) but with $\mathfrak{S}$ in place of $\mathcal{S}_0$. The definition of the density process and the fact that the type space is a bounded interval jointly imply that

$$\begin{aligned}
\Phi_{tu}(\mathfrak{S}) &\leq \left(\int_{\mathcal{X}} (Z_{tu}^x)^{\frac{1}{\gamma}} \mathfrak{S}_t(dx) \right)^{\gamma} \\
&\leq \max_{x \in \mathcal{X}} Z_{tu}^x = \max_{x \in \mathcal{X}} \left\{ e^{\frac{1}{2}\left(\frac{B_u^0 - B_t^0}{\sqrt{u-t}} \right)^2 - \frac{1}{2}\left(\frac{x}{v}\sqrt{u-t} - \frac{B_u^0 - B_t^0}{\sqrt{u-t}} \right)^2} \right\} \\
&= \begin{cases} Z_{tu}^{\overline{x}}, & \text{if } B_u^0 - B_t^0 > \frac{\overline{x}}{v}(u-t), \\ e^{\frac{1}{2}\left(\frac{B_u^0 - B_t^0}{\sqrt{u-t}} \right)^2}, & \text{if } B_u^0 - B_t^0 \in \frac{\mathcal{X}}{v}(u-t), \\ Z_{tu}^{\underline{x}}, & \text{if } B_u^0 - B_t^0 < \frac{\underline{x}}{v}(u-t). \end{cases}
\end{aligned}$$

where the subscript ut indicates the ratio of the value of a process at time u to its value at time $t \leq u$. On the other hand, the definition of λ implies that on the set where the

Brownian increment lies in $(\mathcal{X}/v)(u-t)$ we have

$$-\frac{1}{u-t}\log\left(e^{-\rho(u-t)}D_{tu}^{1-\gamma}\right) = \lambda^0 - v(1-\gamma)\left(\frac{B_u^0 - B_t^0}{u-t}\right)$$
$$= \lambda + \max_{x\in\mathcal{X}}\left\{v(1-\gamma)\left(\frac{x}{v} - \frac{B_u^0 - B_t^0}{u-t}\right)\right\} \geq \lambda.$$

Combining these estimates shows that (43) holds with

$$0 \leq \Gamma_u := \begin{cases} e^{-\rho(u-t)}D_{tu}^{1-\gamma}Z_{tu}^{\overline{x}}, & \text{if } B_u^0 - B_t^0 > \frac{\overline{x}}{v}(u-t), \\ e^{-\lambda(u-t)}e^{\frac{1}{2}\left(\frac{B_u^0-B_t^0}{\sqrt{u-t}}\right)^2}, & \text{if } B_u^0 - B_t^0 \in \frac{\mathcal{X}}{v}(u-t), \\ e^{-\rho(u-t)}D_{tu}^{1-\gamma}Z_{tu}^{\underline{x}}, & \text{if } B_u^0 - B_t^0 < \frac{\underline{x}}{v}(u-t), \end{cases}$$

and the first part of the statement now follows by observing that

$$\mathbf{E}_t^0\left[\int_t^\infty \Gamma_u du\right] \leq \sum_{x\in\partial\mathcal{X}}\mathbf{E}_t^0\left[\int_t^\infty e^{-\rho(u-t)}D_{tu}^{1-\gamma}Z_{tu}^x du\right]$$
$$+\int_t^\infty e^{-\lambda(u-t)}\mathbf{E}_t^0\left[e^{\frac{1}{2}\left(\frac{B_u^0-B_t^0}{\sqrt{u-t}}\right)^2}\mathbf{1}_{\{B_u^0-B_t^0\in\frac{\mathcal{X}}{v}(u-t)\}}\right]du$$
$$= \sum_{x\in\partial\mathcal{X}}\frac{1}{\lambda^x} + \int_t^\infty e^{-\lambda(u-t)}\frac{|\mathcal{X}|}{v}\left(\frac{u-t}{2\pi}\right)^{\frac{1}{2}}du \leq \frac{2}{\lambda} + \frac{|\mathcal{X}|}{v\sqrt{8\lambda^3}}$$

where the equality is due to the fact that the reduced increment of the Brownian motion is a standard normal under $\mathbf{P}_t^0$ and the second inequality follows from the computation of the indefinite time integral and the definition of λ. The second part then follows from the dominated convergence theorem. ■

B Price representations

Proof of Proposition 4 (1st part). The definition of D, Z^x, and B^0 imply that

$$(Z_t^x)^{1/\gamma} = \psi\left(t, x, \frac{D_t}{D_0}\right)$$

and

$$e^{-\rho\tau}\left(\frac{D_{t+\tau}}{D_t}\right)^{1/\gamma-1}\left(\frac{Z^x_{t+\tau}}{Z^x_t}\right)^{\frac{1}{\gamma}} = \xi\left(\tau, x, \frac{B^0_{t+\tau}-B^0_t}{\sqrt{\tau}}\right).$$

Combining these identities with (31a) gives

$$e^{-\rho\tau}\left(\frac{D_{t+\tau}}{D_t}\right)^{1-\gamma}\frac{\Phi_{t+\tau}(\mathfrak{S})}{\Phi_t(\mathfrak{S})} = \left\{\frac{\int_{\mathcal{X}}\xi\left(\tau, x, \frac{B^0_{t+\tau}-B^0_t}{\sqrt{\tau}}\right)\psi\left(t, x, \frac{D_t}{D_0}\right)\mathfrak{S}(dx)}{\int_{\mathcal{X}}\psi\left(t, y, \frac{D_t}{D_0}\right)\mathfrak{S}(dy)}\right\}^{\gamma}$$

and the desired representation of the price-dividend ratio now follows from (31c) by letting $u = t+\tau$ and using the fact that $(B^0_{t+\tau} - B^0_t)/\sqrt{\tau}$ is a standard normal random variable conditional on B^0_t. ■

Proof of Proposition 4 (Equilibrium existence). Fix an initial $\mathfrak{S} = \mathcal{S}_0$ and denote by $\mathcal{W}_t$ the induced distribution of wealth shares across beliefs. To establish the existence of an equilibrium we need to show that

$$\mathbf{E}^0\int_0^\infty \mathbf{1}_{\{\text{diff}_t(P)=0\}}dt \equiv \mathbf{E}^0\int_0^\infty \mathbf{1}_{\left\{\text{diff}_t\left(\frac{P}{D}\right)D_t+vP_t=0\right\}}dt = 0$$

where the first equality follows from Itô's lemma and Fubini's theorem implies that it is enough to show that

$$\mathbf{P}^0\left[\left\{\text{diff}_t\left(\frac{P}{D}\right)D_t - vP_t = 0\right\}\right] = 0 \tag{44}$$

for any fixed $t \geq 0$. The definition of the Brownian motion B^0_t and the first part of the proof imply that

$$\frac{P_t}{D_t} = \Pi\left(t, B^0_t\right) := \int_0^\infty d\tau\int_{\mathbf{R}}\mathbf{n}(dz)\left(\frac{I_1\left(t,\tau,z,B^0_t\right)}{I_0\left(t,\tau,z,B^0_t\right)}\right)^{\gamma}$$

where

$$I_n(t,\tau,z,b) := \int_{\mathcal{X}}\left(\xi(\tau,x,z)\right)^n e^{\frac{x}{\gamma v}b-\frac{1}{2\gamma}\left(\frac{x}{v}\right)^2 t}\mathcal{S}_0(dx).$$

For fixed (t,τ,z) the integrand in the definition of I_n is analytic in $(x,b) \in \mathcal{X}\times\mathbf{R}$ and nonnegative. Therefore, it follows from the combination of Tonelli's theorem and

Morera's theorem that I_n is analytic in $b \in \mathbf{R}$ for any fixed (t, τ, z). Since

$$I_0(t, \tau, z, b) \geq \min \left\{ e^{\frac{x}{\gamma v} b - \frac{1}{2\gamma} \left(\frac{x}{v}\right)^2 t}, e^{\frac{\overline{x}}{\gamma v} b - \frac{1}{2\gamma} \left(\frac{\overline{x}}{v}\right)^2 t} \right\} > 0$$

due to the concavity $x \mapsto \frac{x}{\gamma v} b - \frac{1}{2\gamma} \left(\frac{x}{v}\right)^2 t$ we deduce that $(I_n / I_0)^\gamma$ is analytic in $b \in \mathbf{R}$ for any (t, τ, z). Another combined application of Tonelli's theorem and Morera's theorem then shows that Π is analytic in $b \in \mathbf{R}$ for any $t \geq 0$ and it follows that

$$\text{diff}_t \left(\frac{P}{D} \right) - v \left(\frac{P_t}{D_t} \right) = \Pi_b(t, B_t^0) - v\Pi(t, B_t^0) =: \varphi(t, B_t^0)$$

is an analytic function of $B_t^0 = b \in \mathbf{R}$ for any $t \geq 0$. Therefore, if we can show that for any fixed $t \geq 0$ there exists at least one point b such that $\varphi(t, b) \neq 0$ then Theorem B.3 of Anderson and Raimondo (2008) will imply that

$$\mathcal{N}_t := \{ b \in \mathbf{R} : \varphi(t, b) = 0 \}$$

has Lebesgue measure zero for any fixed $t \geq 0$ and, thereby, ensure that condition (44) is satisfied for all $t \geq 0$.

To complete the proof we now need to show that for any fixed $t \geq 0$ there exist a point b such that $\varphi(t, b) \neq 0$. The first part of the proof and the definition of B_t^0 imply that the moments of the distributions of consumption and wealth shares across beliefs can be written as

$$M_{nt} = M_n(t, B_t^0) := \frac{A_n(t, B_t^0)}{A_0(t, B_t^0)}$$

and

$$\mathcal{M}_{nt} = \mathcal{M}_n(t, B_t^0) := \frac{\int_0^\infty d\tau \int_{\mathbf{R}} \mathbf{n}(dz) \mathcal{A}_0 \left(\tau, t, z, B_t^0\right)^{\gamma - 1} \mathcal{A}_n \left(\tau, t, z, B_t^0\right)}{\int_0^\infty d\tau \int_{\mathbf{R}} \mathbf{n}(dz) \mathcal{A}_0 \left(\tau, t, z, B_t^0\right)^\gamma}$$

with

$$\begin{aligned} A_n(t, b) &:= \int_{\mathcal{X}} e^{\frac{x}{\gamma v} b - \frac{1}{2\gamma} \left(\frac{x}{v}\right)^2 t} x^n \mathcal{S}_0(dx) \\ \mathcal{A}_n(\tau, t, z, b) &:= \int_{\mathcal{X}} \xi(\tau, x, z) e^{\frac{x}{\gamma v} b - \frac{1}{2\gamma} \left(\frac{x}{v}\right)^2 t} x^n \mathcal{S}_0(dx). \end{aligned}$$

As a result, it follows from Proposition 3 that

$$\varphi(t,b) = \Pi(t,b)\sigma(t,b) \quad \text{where} \quad \sigma(t,b) := v + \frac{1}{v}(\mathcal{M}_1(t,b) - M_1(t,b))$$

and, since $0 < \Pi(t,b) < \infty$, it now suffices to show that for any $t \geq 0$ there exists at least one value b of the space variable such that $\sigma(t,b) \neq 0$. Assume towards a contradiction that $\sigma(t,b) = 0$ for *some* $t \geq 0$ and *all* b. Combining this assumption with Lemma 2 below shows that

$$0 = \frac{\partial^n \sigma}{\partial b^n}(t,b) = \frac{1}{v^{n+1}}\left[\mathcal{K}_{n+1}(t,b) - \frac{1}{\gamma^n}K_{n+1}(t,b)\right]$$

where $\mathcal{K}_{n+1}(t,b)$ and $K_{n+1}(t,b)$ denote the cumulants of the distributions of wealth and consumption shares across beliefs seen as functions of time and the Brownian motion. In particular, we have that

$$\begin{aligned} K_1(t,b) &= v^2 + \mathcal{K}_1(t,b) \\ K_{n+1}(t,b) &= \gamma^n \mathcal{K}_{n+1}(t,b) \end{aligned}$$

for all $(b,n) \in \mathbf{R} \times \mathbf{N}$ and it follows that the moment generating functions $(G_t, \mathcal{G}_t)$ of the two distributions at time t are related by

$$\begin{aligned} \log G_t(y) = \log \int_{\mathcal{X}} e^{xy} \mathcal{S}_t(dx) &= \sum_{n=1}^{\infty} K_n(t, B_t^0) \frac{y^n}{n!} \\ &= v^2 y + \frac{1}{\gamma} \sum_{n=1}^{\infty} \mathcal{K}_n(t, B_t^0) \frac{(\gamma y)^n}{n!} \\ &= v^2 y + \log \int_{\mathcal{X}} e^{x\gamma y} \mathcal{W}_t(dx) = v^2 y + \log \left[\mathcal{G}_t(\gamma y)^{\frac{1}{\gamma}}\right]. \end{aligned} \tag{45}$$

If $\gamma \geq 1$ then Jensen's inequality shows that we have $\mathcal{G}_t(y) \leq \mathcal{G}_t(\gamma y)^{\frac{1}{\gamma}}$. Combining this inequality with (45) and Lemma 3 below then shows that

$$\lim_{y \to \infty} e^{-yz} G_t(y) \geq \lim_{y \to \infty} e^{-y(z - v^2)} \mathcal{G}_t(y) = \infty$$

for all $z \in (\bar{z}, \bar{z} + v^2)$ where $\bar{z}$ denotes the largest point in $\operatorname{supp}(\mathcal{W}_t)$. By Lemma 3 again this implies that any such z belongs to $\operatorname{supp}(\mathcal{S}_t)$ which in turn implies that $\operatorname{supp}(\mathcal{W}_t) \neq \operatorname{supp}(\mathcal{S}_t)$ and contradicts the fact that the probability measures $\mathcal{S}_t$ and

$\mathcal{W}_t$ are equivalent. If instead $\gamma < 1$ then Jensen's inequality gives

$$\mathcal{G}_t(\gamma y)^{\frac{1}{\gamma}} \leq \mathcal{G}_t(y)$$

and a similar argument provides the required contradiction by showing that in this case also $\text{supp}(\mathcal{S}_t) \neq \text{supp}(\mathcal{W}_t)$. ■

Lemma 2. *The moments and cumulants of the distribution of wealth shares across beliefs satisfy* $\dot{Q}_t(\mathcal{M}_n) = \mathcal{M}_{n+1t} - \mathcal{M}_{nt}\mathcal{M}_{1t}$ *and* $\dot{Q}_t(\mathcal{K}_n) = \mathcal{K}_{n+1t}$.

Proof. The result can be established by following the same steps as in the proofs of Theorem 1 and Proposition 3. We omit the details. ■

Lemma 3. *Assume that ϕ is a probability distribution with bounded support and denote its moment generating function by $\Gamma(y)$. Then*

$$\lim_{y\to-\infty} \frac{\Gamma(y)}{e^{zy}} = \begin{cases} 0 & \textit{if } z < \underline{z} \\ \infty & \textit{if } z > \underline{z} \end{cases} \quad \textit{and} \quad \lim_{y\to\infty} \frac{\Gamma(y)}{e^{zy}} = \begin{cases} 0 & \textit{if } z > \overline{z} \\ \infty & \textit{if } z < \overline{z} \end{cases}$$

where $\underline{z}$ and $\overline{z}$ denote the lowest and largest points in the support of ϕ.

Proof. Assume that $\underline{z} < \overline{z}$ for otherwise there is nothing to prove. For any $z > \overline{z}$ we have that

$$0 \leq \lim_{y\to\infty} \frac{\Gamma(y)}{e^{zy}} = \lim_{y\to\infty} \int_{\underline{z}}^{\overline{z}} e^{(x-z)y}\phi(dx) \leq \lim_{y\to\infty} e^{(\overline{z}-z)y} \int_{\underline{z}}^{\overline{z}} \phi(dx) = 0$$

while for any $z < \overline{z}$ we have that

$$\begin{aligned} \lim_{y\to\infty} \frac{\Gamma(y)}{e^{zy}} &\geq \lim_{y\to\infty} \frac{\Gamma(y)}{e^{y(z\vee\underline{z})}} \\ &= \lim_{y\to\infty} \left\{ \int_{\underline{z}}^{z\vee\underline{z}} e^{y(x-z\vee\underline{z})}\phi(dx) + \int_{z\vee\underline{z}}^{\overline{z}} e^{y(x-z\vee\underline{z})}\phi(dx) \right\} \\ &\geq \lim_{y\to\infty} \int_{z\vee\underline{z}}^{\overline{z}} e^{y(x-z\vee\underline{z})}\phi(dx) \geq \lim_{y\to\infty} \int_{z\vee\underline{z}}^{\overline{z}} e^{y(x-z\vee\underline{z})}\mathbf{1}_{\{x\neq z\vee\underline{z}\}}\phi(dx) = \infty \end{aligned}$$

where the last equality follows from the monotone convergence theorem. The proof of the limits as $y \to -\infty$ are similar. ■

Corollary 2. *Assume that condition (26) holds, that γ is an integer, and that $\mathfrak{S}$ is a discrete probability measure with support $\{\mathbf{x}_i\}$ and probability vector $\{\mathbf{p}_i\}$. Then*

$$M_{nt}(\mathfrak{S}) = \sum_i \mathbf{x}_i^n \left(\frac{\psi\left(t, \mathbf{x}_i, \frac{D_t}{D_0}\right) \mathbf{p}_i}{\sum_{k=1}^n \psi\left(t, \mathbf{x}_k, \frac{D_t}{D_0}\right) \mathbf{p}_k} \right)$$

$$\frac{P_t(\mathfrak{S})}{D_t} = \sum_{|\ell|=\gamma} \binom{\gamma}{\ell} \frac{1}{\Theta(\ell)} \prod_{i=1}^n \left(\frac{\psi\left(t, \mathbf{x}_i, \frac{D_t}{D_0}\right) \mathbf{p}_i}{\sum_{k=1}^n \psi\left(t, \mathbf{x}_k, \frac{D_t}{D_0}\right) \mathbf{p}_k} \right)^{\ell_i}$$

where ψ is defined as in (33),

$$\Theta(\ell) := \sum_{i=1}^n \lambda^{\mathbf{x}_i} \left(\frac{\ell_i}{\gamma}\right) + \frac{1}{2v^2} \sum_{i=1}^n \mathbf{x}_i^2 \left(\frac{\ell_i}{\gamma}\right) - \frac{1}{2v^2} \left[\sum_{i=1}^n \mathbf{x}_i \left(\frac{\ell_i}{\gamma}\right) \right]^2 > 0,$$

and the sum extends over all vectors $\ell \in \mathbf{N}^n$ whose components add up to γ.

Proof. If γ is an integer and $\mathfrak{S}$ is discrete then the multinomial theorem implies that the integrand in (32) can be expressed as

$$\left\{ \sum_i \xi\left(\tau, \mathbf{x}_i, z\right) \mathbf{p}_{it}(\mathfrak{S}) \right\}^\gamma = \sum_{|\ell|=\gamma} \binom{\gamma}{\ell} e^{\sum_i \ell_i \log \xi(\tau, \mathbf{x}_i, z)} \prod_i \mathbf{p}_{it}(\mathfrak{S})^{\ell_i}$$

where

$$\mathbf{p}_{it}(\mathfrak{S}) := \frac{\psi\left(t, \mathbf{x}_i, \frac{D_t}{D_0}\right) \mathbf{p}_i(\mathfrak{S})}{\sum_k \psi\left(t, \mathbf{x}_k, \frac{D_t}{D_0}\right) \mathbf{p}_k(\mathfrak{S})}$$

denotes the consumption-weighted mass of investors of type $\mathbf{x}_i$ at time $t \geq 0$. The conclusion follows by substituting this expression into (32) and integrating first with respect to $\mathbf{n}(dz)$ then with respect to $d\tau$. The reported expression of the moments follows directly from the definition of $\mathbf{p}_t$. We omit the details. ∎

C Convergence of the discrete problem

Proof of Proposition 5. Since

$$\mathcal{L}(\mathcal{S}; N) = \mathcal{L}(\mathcal{S}, \mathcal{S}; N) = 0$$

we have that

$$
\begin{aligned}
A_N^* &:= \operatorname*{argmin}_{\mathfrak{S}\in\mathcal{P}} \mathcal{L}(\mathcal{S},\mathfrak{S};N) \\
&= \{\mathfrak{S}\in\mathcal{P} : (r_t(\mathfrak{S}),P_t(\mathfrak{S})) = (r_t(\mathcal{S}),P_t(\mathcal{S}))\ \forall t\in\Pi_N\} \neq \emptyset
\end{aligned}
$$

and the fact that $\Pi_N \subset \Pi_{N+1}$ implies that the map $N \mapsto A_N^*$ is decreasing in the sense of set inclusion. Now let the sequence $\{\mathfrak{S}_N^*\}$ be as in the statement and consider the sequence of nonnegative numbers defined by

$$
0 \le W(\mathfrak{S}_N^*,\mathcal{S}) \le T_N := \sup_{\mathfrak{S}\in A_N^*} W(\mathfrak{S},\mathcal{S})
$$

where $W(p,q)$ denotes the Wasserstein distance. Since the map $N \mapsto A_N^*$ is decreasing we have that $\{T_N\}$ is bounded from below and decreasing. This in turn implies that $\{T_N\}$ converges to some limit T^* and it now suffices to show that this limit is zero.

If there exists an index N^* such that $T_{N^*} = 0$ then $T_N = 0$ for all $N \ge N^*$ and the desired result follows immediately. If not then we pick a sequence of strictly positive real numbers $\{\varepsilon_N\}$ such that

$$
T_N > \varepsilon_N > 0 = \lim_{k\to\infty} \varepsilon_k
$$

and a sequence $\{\mathfrak{S}_N\} \subseteq \mathcal{P}$ such that $\mathfrak{S}_N \in A_N^*$ and

$$
W(\mathfrak{S}_N,\mathcal{S}) \ge T_N - \varepsilon_N.
$$

for every N. Because $\mathcal{X}$ is compact we have that $(\mathcal{P}, W(\cdot))$ is compact by application of Prohorov's theorem. In particular, there exists a probability $\mathfrak{S}^* \in \mathcal{P}$ and a subsequence $\{\mathfrak{S}_{N_k}\}$ such that $\mathfrak{S}_{N_k} \in A_{N_k}^*$ for each k and

$$
\lim_{k\to\infty} W\left(\mathfrak{S}_{N_k},\mathfrak{S}^*\right) = 0.
$$

Since

$$
\mathcal{L}\left(\mathcal{S},\mathfrak{S}_{N_k};M\right) = 0, \qquad \forall N_k > M,
$$

we have that

$$\lim_{M\to\infty}\lim_{k\to\infty}\mathcal{L}\left(\mathcal{S},\mathfrak{S}_{N_k};M\right)=0.$$

On the other hand, Lemma 6 below shows that the map $\mathfrak{S}\mapsto\mathcal{L}(\mathcal{S},\mathfrak{S})$ is weakly continuous and that

$$\lim_{M\to\infty}\sup_{\mathfrak{S}\in\mathcal{P}}\left|\mathcal{L}\left(\mathcal{S},\mathfrak{S};M\right)-\mathcal{L}\left(\mathcal{S},\mathfrak{S}\right)\right|=0.$$

Combining these properties with the Moore-Osgood theorem then shows that

$$\begin{aligned}0=\lim_{M\to\infty}\lim_{k\to\infty}\mathcal{L}\left(\mathcal{S},\mathfrak{S}_{N_k};M\right)&=\lim_{k\to\infty}\lim_{M\to\infty}\mathcal{L}\left(\mathcal{S},\mathfrak{S}_{N_k};M\right)\\&=\lim_{k\to\infty}\mathcal{L}\left(\mathcal{S},\mathfrak{S}_{N_k}\right)=\mathcal{L}\left(\mathcal{S},\mathfrak{S}^*\right)\end{aligned}$$

which implies $\mathfrak{S}^*=\mathcal{S}$ by Theorem 2 and the result follows since

$$0\leq T^*=\lim_{k\to\infty}T_{N_k}\leq\lim_{k\to\infty}W(\mathfrak{S}_{N_k},\mathcal{S})+\varepsilon_k=W(\mathfrak{S}^*,\mathcal{S})+\lim_{k\to\infty}\varepsilon_k=0$$

by definition of the sequences $\{\varepsilon_N\}$ and $\{\mathfrak{S}_N\}$. ■

Lemma 4. *The map*

$$[0,\tau]\times\mathcal{P}\ni(t,\mathfrak{T})\mapsto\mathfrak{T}_t(dx):=\left(\frac{Z_t^x}{\Phi_t(\mathfrak{T})}\right)^{\frac{1}{\gamma}}\mathfrak{T}(dx)\in\mathcal{P}$$

is weakly continuous for any $\tau>0$.

Proof. The definition of $\Phi_t(\mathfrak{T})$ and the Portmanteau theorem imply that proving the claim is equivalent to showing that the map

$$[0,\tau]\times\mathcal{P}\ni(t,\mathfrak{T})\mapsto\int_{\mathcal{X}}g(x)\mathfrak{T}_t(dx)=\frac{\int_{\mathcal{X}}g(x)(Z_t^x)^{1/\gamma}\mathfrak{T}(dx)}{\int_{\mathcal{X}}(Z_t^x)^{1/\gamma}\mathfrak{T}(dx)}\in\mathbf{R}\tag{46}$$

is weakly continuous for any continuous g. Since

$$\int_{\mathcal{X}}(Z_t^x)^{1/\gamma}\mathfrak{T}(dx)\geq\min_{x\in\mathcal{X}}(Z_t^x)^{1/\gamma}>0$$

the latter property is in turn equivalent to the weak continuity of the numerator and the denominator on the right of (46) and the conclusion now follows from Lemma

5 below by observing that for any fixed ω the functions $f(t,x) = (Z_t^x(\omega))^{1/\gamma}$ and $h(t,x) = g(x)f(t,x)$ are both jointly continuous in (t,x). ■

Lemma 5. *The map*

$$[0,\tau] \times \mathcal{P} \ni (t,\mathfrak{T}) \mapsto H(t,\mathfrak{T}) := \int_{\mathcal{X}} h(t,x)\mathfrak{T}(dx)$$

is weakly continuous for any $\tau > 0$ and any continuous $h : \mathbf{R}_+ \times \mathcal{X} \to \mathbf{R}$.

Proof. Assume that $\{(t_n,\mathfrak{T}_n)\} \subseteq [0,\tau] \times \mathcal{P}$ converges weakly to $(t,\mathfrak{T})$. Since h is continuous and $[0,\tau] \times \mathcal{X}$ is compact we have that h is *uniformly* continuous on that set. In particular, for any given $\varepsilon > 0$ there exists n_ε such that

$$n \geq n_\varepsilon \Rightarrow \sup_{x \in \mathcal{X}} |h(t_n,x) - h(t,x)| < \varepsilon.$$

On the other hand, since $h(t,\cdot)$ is continuous and $\{\mathfrak{T}_n\}$ converges weakly to $\mathfrak{T}$ we know from the Portmanteau theorem that

$$\lim_{n\to\infty} \int_{\mathcal{X}} h(t,x)\mathfrak{T}_n(dx) = \int_{\mathcal{X}} h(t,x)\mathfrak{T}(dx)$$

and it follows that for any $\varepsilon > 0$ there exists n'_ε such that

$$n \geq n'_\varepsilon \Rightarrow \left| \int_{\mathcal{X}} h(t,x)(\mathfrak{T}_n(dx) - \mathfrak{T}(dx)) \right| < \varepsilon.$$

Combining these properties shows that

$$\begin{aligned}|H(t_n,\mathfrak{T}_n) - H(t,\mathfrak{T})| &\leq \int_{\mathcal{X}} |h(t_n,x) - h(t,x)|\,\mathfrak{T}_n(dx) \\ &+ \left| \int_{\mathcal{X}} h(t,x)(\mathfrak{T}_n(dx) - \mathfrak{T}(dx)) \right| < \int_{\mathcal{X}} \varepsilon\,\mathfrak{T}_n(dx) + \varepsilon = 2\varepsilon\end{aligned}$$

for all $n \geq \max\{n_\varepsilon, n'_\varepsilon\}$ and the proof is complete. ■

Lemma 6. *Fix and arbitrary $\mathcal{S} \in \mathcal{P}$ and assume that condition (26) holds.*

1. *The map*

$$[0,\tau] \times \mathcal{P} \ni (t,\mathfrak{T}) \mapsto (r_t(\mathfrak{T}), P_t(\mathfrak{T})) \in \mathbf{R}^2$$

is weakly continuous for any $\tau > 0$.

2. *The sequence* $\{\mathcal{L}(\mathfrak{T},\mathcal{S};N)\}$ *converges to* $\mathcal{L}(\mathfrak{T},\mathcal{S})$ *uniformly in* $\mathfrak{T}\in\mathcal{P}$.

3. *The map* $\mathcal{P}\ni\mathfrak{T}\mapsto\mathcal{L}(\mathfrak{T},\mathcal{S})\in\mathbf{R}$ *is weakly continuous.*

Proof of Claim 1. Since the interest rate $r_t(\mathfrak{T})$ is a function of $(M_{1t}(\mathfrak{T}),M_{2t}(\mathfrak{T}))$ we have that the required continuity follows from Lemma 4 and the fact that

$$M_{nt}(\mathfrak{T})=\int_{\mathcal{X}}x^n\mathfrak{T}_t(dx).$$

Turning to the stock price, we have that

$$\frac{P_t(\mathfrak{T})}{D_t}=U_t(\mathfrak{T}):=\mathbf{E}_t^0\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{D_u}{D_t}\right)^{1-\gamma}\frac{\Phi_u(\mathfrak{T})}{\Phi_t(\mathfrak{T})}du\right]\tag{47}$$

and it thus suffices to show that the price-dividend ratio $P_t(\mathfrak{T})$ is weakly continuous on $[0,\tau]\times\mathcal{P}$. Combining (24) with the fact that

$$\left(\frac{Z_u^x}{Z_t^x}\right)^{\frac{1}{\gamma}}=\psi\left(u-t,x,\frac{D_u}{D_t}\right):=e^{\frac{x(x-v^2)}{2\gamma v^2}(u-t)}\left(\frac{D_u}{D_t}\right)^{\frac{x}{\gamma v^2}}$$

and effecting the change of variable $s=u-t$ in (47) shows that the price dividend ratio can be written as

$$U_t(\mathfrak{T})=\mathbf{E}_t^0\left[\int_t^\infty e^{-\rho s}\left\{\int_{\mathcal{X}}\psi\left(s,x,\frac{D_{t+s}}{D_t}\right)\mathfrak{T}_t(dx)\right\}^\gamma ds\right].$$

Since

$$\mathbf{P}_t^0\left[\left(\frac{D_{t+s}}{D_t}\right)\in dy\right]=\mathbf{P}^0\left[\left(\frac{D_s}{D_0}\right)\in dy\right]$$

this alternative representation implies that $U_t(\mathfrak{T})=U_0(\mathfrak{T}_t)$ and it now suffices to prove that the map

$$\mathfrak{S}\mapsto U_0(\mathfrak{S})=\mathbf{E}^0\left[\int_0^\infty e^{-\rho s}\left(\frac{D_s}{D_0}\right)^{1-\gamma}\Phi_s(\mathfrak{S})ds\right]$$

is weakly continuous. To this end assume that the sequence $\{\mathfrak{S}_n\}\subseteq\mathcal{P}$ converges weakly to some $\mathfrak{S}$. Because

$$\Phi_s(\mathfrak{S}_n)=\int_{\mathcal{X}}(Z_s^x)^{\frac{1}{\gamma}}\mathfrak{S}_n(dx),\qquad s\geq 0,$$

and $x \mapsto Z_s^x$ is continuous for any fixed (s, ω) we have that

$$G_{0s}(\mathfrak{S}_n) = e^{-\rho s} \left(\frac{D_s}{D_0} \right)^{1-\gamma} \Phi_s(\mathfrak{S}_n)$$

converges almost everywhere to $G_{0s}(\mathfrak{S})$ and the desired result now follows from Lemma 1 and the dominated convergence theorem. ■

Proof of Claim 2. Combining Claim 1 and the fact that any norm on $\mathbf{R}^2$ is continuous shows that the map

$$[0, \tau] \times \mathcal{P} \ni (t, \mathfrak{T}) \mapsto f_t(\mathfrak{T}) = \phi\left((r_t(\mathfrak{T}), P_t(\mathfrak{T})) - (r_t(\mathcal{S}), P_t(\mathcal{S}))\right) \in \mathbf{R}_+ \tag{48}$$

is weakly continuous. Since the set $[0, \tau] \times \mathcal{P}$ is compact this implies that for any $\varepsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$|t - s| < \delta_\varepsilon \Rightarrow \sup_{\mathfrak{T} \in \mathcal{P}} |f_t(\mathfrak{T}) - f_s(\mathfrak{T})| < \varepsilon.$$

Using this property we deduce that

$$\begin{aligned} \sup_{\mathfrak{T} \in \mathcal{P}} |\mathcal{L}(\mathfrak{T}, \mathcal{S}; N) - \mathcal{L}(\mathfrak{T}, \mathcal{S})| &= \sup_{\mathfrak{T} \in \mathcal{P}} \left| \sum_{n=1}^{N} \int_{t_{n-1,N}}^{t_{n,N}} \left(f_s(\mathfrak{T}) - f_{t_{n,N}}(\mathfrak{T})\right) ds \right| \\ &\leq \sum_{n=1}^{N} \int_{t_{n-1,N}}^{t_{n,N}} \sup_{\mathfrak{T} \in \mathcal{P}} \left|f_s(\mathfrak{T}) - f_{t_{n,N}}(\mathfrak{T})\right| ds < \varepsilon\tau \end{aligned}$$

for all N such that $\max_n |\Delta_{n,N}| < \delta_\epsilon$ and the proof is complete. ■

Proof of Claim 3. Assume that $\{\mathfrak{T}_n\} \subseteq \mathcal{P}$ converges weakly to $\mathfrak{T}$. Since the map in (48) is weakly continuous on the compact set $[0, \tau] \times \mathcal{P}$ be have that for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$W(\mathfrak{T}, \mathfrak{T}_n) < \delta_\varepsilon \Rightarrow \sup_{t \in [0, \tau]} |f_t(\mathfrak{T}) - f_t(\mathfrak{T}_n)| < \frac{\varepsilon}{\tau}$$

where W denotes the Wasserstein distance on $\mathcal{P}$. Using this property together with the definition of $\mathcal{L}(\mathfrak{T}, \mathcal{S})$ we deduce that

$$W(\mathfrak{T}, \mathfrak{T}_n) < \delta_\varepsilon \Rightarrow |\mathcal{L}(\mathfrak{T}_n, \mathcal{S}) - \mathcal{L}(\mathfrak{T}, \mathcal{S})| \leq \sup_{t \in [0, \tau]} \tau |f_t(\mathfrak{T}) - f_t(\mathfrak{T}_n)| < \varepsilon$$

and the proof is complete. ■

D Proofs of the extensions

D.1 Multidimensional beliefs

Proof of Proposition 6. In equilibrium markets are complete. Therefore, the optimal consumption share of an investor of type x is given by

$$s_t^x = s_0^x \left(e^{\rho t} H_t^0 / Z_t^x\right)^{-\frac{1}{\gamma}} \left(\overline{D}_0 / \overline{D}_t\right)$$

where

$$H_t^{\mathbf{0}} = \exp\left(-\int_0^t r_u du - \frac{1}{2}\int_0^t \|\theta_u^{\mathbf{0}}\|^2 du - \int_0^t (\theta_u^{\mathbf{0}})^\top dB_u^{\mathbf{0}}\right)$$

is the unique state price density relative to $\mathbf{P}^0$ and

$$Z_t^x := \left.\frac{d\mathbf{P}^x}{d\mathbf{P}^{\mathbf{0}}}\right|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\|V^{-1}x\|^2 t + \left(V^{-1}x\right)^\top B_t^0\right).$$

is the density of $\mathbf{P}^x$ relative to the probability $\mathbf{P}^{\mathbf{0}}$ under which the drift of aggregate consumption is equal to $\mathbf{0}$. Using the consumption market clearing condition we then get that the state price density

$$H_t^{\mathbf{0}} = e^{-\rho t}\Phi_t \left(\overline{D}_t / \overline{D}_0\right)^{-\gamma}$$

where the auxiliary Φ_t is defined as in (14) albeit with the above Z_t^x and the desired results now follow—as in the baseline model—by using Itô's lemma to compute the dynamics of the state price density process. ∎

Proof of Theorem 3. The absolute continuity relation relation between $\mathcal{S}_t$ and $\mathcal{S}_0$ shows that we have

$$M_{kt} = \int_{\mathcal{X}} \left(\prod_{i=1}^N x_i^{k_i}\right) \left(\frac{Z^x}{\Phi}\right)^{\frac{1}{\gamma}} \mathcal{S}_0(dx).$$

Applying Itô's lemma to this expression gives

$$\begin{aligned}
\text{diff}_t(M_k) &= \int_{\mathcal{X}} \left(\prod_{i=1}^{N} x_i^{k_i}\right) \text{diff}_t\left(\frac{Z^x}{\Phi}\right)^{\frac{1}{\gamma}} \mathcal{S}_0(dx) \\
&= \frac{1}{\gamma}\int_{\mathcal{X}} \left(\prod_{i=1}^{N} x_i^{k_i}\right) \left[V^{-1}(x - m_t)\right] \left(\frac{Z_t^x}{\Phi_t}\right)^{\frac{1}{\gamma}} \mathcal{S}_0(dx) \\
&= \frac{1}{\gamma}\int_{\mathcal{X}} \left(\prod_{i=1}^{N} x_i^{k_i}\right) \left[V^{-1}(x - m_t)\right] \mathcal{S}_t(dx) = (\gamma V)^{-1}(N_{kt} - M_{kt} m_t)
\end{aligned}$$

with the vector defined by

$$N_{kt}^{\top} := \begin{pmatrix} M_{k+\mathbf{e}_1,t} & M_{k+\mathbf{e}_2,t} & \cdots & M_{k+\mathbf{e}_n,t} \end{pmatrix}^{\top}$$

and the moment recursion in the statement follows by noting that

$$\begin{aligned}
\gamma \dot{Q}_t(M_k) &= \gamma \text{diff}_t(\log D)^{\top} \text{diff}_t(M_k) \\
&= V V^{-1}(N_{kt} - M_{kt} m_t) = N_{kt} - M_{kt} m_t
\end{aligned}$$

The rest of the proof follows from arguments similar to those of Theorem 1. We omit the details. To recover the wealth-weighted distribution of beliefs distribution $\mathcal{W}_t$ from the consumption-weighted distribution $\mathcal{S}_t$ it suffices to note that

$$\mathcal{W}_t(dx) = \left(\overline{D_t}/P_t^{\top}\mathbf{1}\right) \Lambda_t^x \mathcal{S}_t(dx)$$

where the process

$$\Lambda_t^x := \mathbf{E}_t^0\left[\int_t^{\infty} e^{-\rho(u-t)} \left(\frac{\overline{D}_u}{\overline{D}_t}\right)^{1-\gamma} \left(\frac{\Phi_u}{\Phi_t}\right)^{1-\frac{1}{\gamma}} \left(\frac{Z_u^x}{Z_t^x}\right)^{\frac{1}{\gamma}} du\right] \tag{49}$$

captures the wealth-consumption ratio of an investor of type x. ■

D.2 Stochastic beliefs

Proof of Theorem 4. Equation (41) implies that

$$\text{diff}_t(\alpha_n^x) = \frac{\alpha_{n+1t}^x}{v}, \qquad n \geq 1.$$

Combining this identity with Itô's lemma, the definition of $\beta_{nt}(\mathbf{c})$, and the relation between $\mathcal{S}_0$ and $\mathcal{S}_t$ shows that we have

$$\begin{aligned}
\dot{Q}_t(\beta_n(\mathbf{c})) &= v\,\mathrm{diff}_t(\beta_n(\mathbf{c})) \\
&= \int_{\mathcal{X}} \sum_{k\in\mathcal{C}_n}^{n} v\,c_{nk}\mathrm{diff}_t\left[\left(\prod_{j=1}^{n}(\alpha_j^x)^{k_j}\right)\left(\frac{Z^x}{\Phi}\right)^{\frac{1}{\gamma}}\right]\left(\frac{Z_t^x}{\Phi_t}\right)^{-\frac{1}{\gamma}}\mathcal{S}_t(dx) \\
&= \int_{\mathcal{X}} \sum_{k\in\mathcal{C}_n} c_{nk}\left(\prod_{j=1}^{n}(\alpha_{jt}^x)^{k_j}\right)\left[\frac{1}{\gamma}\left(\alpha_{1t}^x - A_{1t}\right) + \sum_{i=1}^{n} k_i\left(\frac{\alpha_{i+1,t}^x}{\alpha_{i,t}^x}\right)\right]\mathcal{S}_t(dx) \\
&= \beta_{n+1t}(\mathbf{c}) - \frac{1}{\gamma}\beta_{1t}(\mathbf{c})\beta_{nt}(\mathbf{c}) \\
&+ \int_{\mathcal{X}} \sum_{k\in\mathcal{C}_n} c_{nk}\left(\prod_{j=1}^{n}(\alpha_{jt}^x)^{k_j}\right)\left[\frac{\alpha_{1t}^x}{\gamma} + \sum_{i=1}^{n} k_i\left(\frac{\alpha_{i+1,t}^x}{\alpha_{i,t}^x}\right)\right]\mathcal{S}_t(dx) \\
&- \int_{\mathcal{X}} \sum_{\ell\in\mathcal{C}_{n+1}} c_{n+1\ell}\left(\prod_{j=1}^{n+1}(\alpha_{jt}^x)^{\ell_j}\right)\mathcal{S}_t(dx).
\end{aligned} \tag{50}$$

To proceed further note that we have

$$\left\{(k,0) + \mathbf{e}_1^{n+1} : k \in \mathcal{C}_n\right\} = \{\ell : \ell \in \mathcal{C}_{n+1} \text{ and } \ell_1 > 0\}$$

and

$$\begin{aligned}
&\left\{(k,0) + \mathbf{e}_{i+1}^{n+1} - \mathbf{e}_i^{n+1} : k \in \mathcal{C}_n \text{ and } i \in \{1,\ldots,n\}\right\} \\
&\qquad = A \cup \left\{\mathbf{e}_{n+1}^{n+1}\right\} \bigcup_{i=1}^{n-1} \{\ell : \ell \in \mathcal{C}_{n+1} \text{ and } \ell_{i+1} > 0\}
\end{aligned}$$

for some set of $A \subseteq \mathbf{Z}^{n+1}$ such that $(a_1,\ldots,a_{n+1}) \in \mathbf{Z}^{n+1}\backslash\mathcal{C}^{n+1}$ for all $a \in A$. As a result, the sums in the second to last line of (50) can be expressed as

$$\sum_{k\in\mathcal{C}_n}(c_{nk}/\gamma)\left(\prod_{j=1}^{n}(\alpha_{jt}^x)^{k_j}\right)\alpha_{1t}^x = \sum_{\ell\in\mathcal{C}_{n+1}\backslash\mathbf{e}_{n+1}^{n+1}}\left(c_{n,(\ell_1-1,\ell_2,\ldots,\ell_n)}/\gamma\right)\left(\prod_{j=1}^{n+1}(\alpha_{jt}^x)^{\ell_j}\right)$$

and

$$\sum_{k\in\mathcal{C}_n}\sum_{i=1}^{n} c_{nk}\left(\prod_{j=1}^{n}(\alpha_{jt}^x)^{k_j}\right)k_i\left(\frac{\alpha_{i+1,t}^x}{\alpha_{i,t}^x}\right)$$
$$= c_{n,\mathbf{e}_n^n}\alpha_{n+1t}^x + \sum_{\ell\in\mathcal{C}_{n+1}\setminus\mathbf{e}_{n+1}^{n+1}}\left(\prod_{j=1}^{n}(\alpha_{jt}^x)^{\ell_j}\right)\sum_{i=1}^{n-1} c_{n,(\ell_1,\ldots,\ell_i+1,\ell_{i+1}-1,\ldots,\ell_n)}(1+\ell_i).$$

with the convention that $c_{nk} = 0$ for all $(n,k) \in \mathbf{N}\times\mathbf{Z}^n\backslash\mathcal{C}^n$ and substituting these expressions into (50) we conclude that (42) holds provided that

$$\hat{c}_{n+1,\ell} = \mathbf{1}_{\{\ell\in\mathcal{C}_{n+1}\}}\begin{cases}1, & \text{if } \ell = \mathbf{e}_{n+1}^{n+1}\\ \hat{c}_{n(\ell_1-1,\ell_2,\ldots,\ell_n)}/\gamma & \\ +\sum_{i=1}^{n-1}(1+\ell_i)\,\hat{c}_{n,(\ell_1,\ldots,\ell_i+1,\ell_{i+1}-1,\ldots,\ell_n)}, & \text{if } \ell \neq \mathbf{e}_{n+1}^{n+1}\end{cases} \tag{51}$$

for all $n \geq 1$ subject to $\hat{c}_{11} = 1$. Now assume that $\mathbf{c}$ satisfies the above recursion and suppress the dependence on $\hat{\mathbf{c}}$ to simplify the notation. Using the recursion (42) and proceeding as in the baseline case shows that $\{\beta_{n0}\}_{n\in\mathbf{N}}$ can be recovered from the continuous observation of (r_t, D_t, P_t) over a non empty interval $[0,\tau)$. To show that the initial consumption-weighted distribution of beliefs can then be recovered from this sequence it suffices to show that if $\operatorname{span}(\{b_{n0}\}_{n\in\mathbf{N}})$ is dense in the space of continuous functions then

$$(\mathcal{R},\mathcal{S}) \in \mathcal{P}^2 \text{ and } \mathcal{R}\neq\mathcal{S} \Longrightarrow \{\beta_{n0}(\mathcal{R})\}_{n\in\mathbf{N}} \neq \{\beta_{n0}(\mathcal{S})\}_{n\in\mathbf{N}}. \tag{52}$$

This is established as follows: Since $\mathcal{R}\neq\mathcal{S}$, the determinacy of the moment problem on the compact set $\mathcal{X}$ implies that there exists $\nu\in\mathbf{N}$ and $\varepsilon > 0$ such that

$$|M_{\nu 0}(\mathcal{R}) - M_{\nu 0}(\mathcal{S})| > \varepsilon.$$

On the other hand, if $\operatorname{span}(\{b_{n0}\}_{n\in\mathbf{N}})$ is dense in the space of continuous functions then there exists $a\in\mathbf{R}^N$ such that

$$\sup_{x\in\mathcal{X}}\left|x^\nu - \sum_{n=1}^{N} a_n b_{n0}(x)\right| < \frac{\varepsilon}{2}.$$

In particular, we have that

$$\left| M_{\nu 0}(\mathfrak{S}) - \sum_{n=1}^{N} a_n \beta_{n0}(\mathfrak{S}) \right| \leq \int_{\mathcal{X}} \left| x^{\nu} - \sum_{n=1}^{N} a_n b_{n0}(x) \right| \mathfrak{S}(dx) < \frac{\varepsilon}{2}$$

for any $\mathfrak{S} \in \mathcal{P}$ and, therefore,

$$\begin{aligned} &\left| \sum_{n=1}^{N} a_n (\beta_{n0}(\mathcal{R}) - \beta_{n0}(\mathcal{S})) \right| \\ &\qquad = \left| M_{\nu 0}(\mathcal{R}) - M_{\nu 0}(\mathcal{S}) + \sum_{\mathfrak{S}=\mathcal{R},\mathcal{S}} \left(\sum_{n=1}^{N} a_n \beta_{n0}(\mathfrak{S}) - M_{\nu 0}(\mathfrak{S}) \right) \right| \\ &\qquad \geq |M_{\nu 0}(\mathcal{R}) - M_{\nu 0}(\mathcal{S})| - \sum_{\mathfrak{S}=\mathcal{R},\mathcal{S}} \left| \sum_{n=1}^{N} a_n \beta_{n0}(\mathfrak{S}) - M_{\nu 0}(\mathfrak{S}) \right| > 0 \end{aligned}$$

which shows that (52) is satisfied. To recover the consumption-weighted distribution of beliefs at time $t \in (0, \tau]$ one uses (16) and the wealth-weighted distribution of beliefs can then be recovered from (21) with Z_t^x defined as in (40). ■

Proof of Proposition 8.1). A direct calculation using (38) shows that in the gaussian setting of Example 1 we have

$$\begin{aligned} \alpha_{10}^x &= \alpha_0^x = x, \\ \alpha_{20}^x &= z_0, \\ \alpha_{n0}^x &= 0, \qquad \forall n \geq 3. \end{aligned}$$

Combining this with the fact that $\hat{c}_{n,n\mathbf{e}_1^n} = \gamma^{1-n}$ from (51) shows that

$$b_{n0}(x) = \gamma^{1-n} x^n + \sum_{k \in \mathcal{C}_n \setminus n\mathbf{e}_1^n} \hat{c}_{nk} x^{k_1} z_0^{k_2}$$

is a polynomial of degree n and the desired result now follows by application of the Stone-Weierstrass theorem. ■

Proof of Proposition 8.2). A direct calculation using (39) shows that in the Markov chain setting of Example 2 we have

$$\alpha_{n0}^x = \alpha_L \mathbf{1}_{\{n=1\}} + (\alpha_H - \alpha_L)^n \sum_{k=1}^{n} m_{kn} x^k$$

where the coefficients m_{kn} are recursively defined by

$$m_{k;n+1} = (-1)^n n! \mathbf{1}_{\{k=n+1\}} + \mathbf{1}_{\{k=1\}} + [k m_{kn} - (k-1)\, m_{k-1;n}]\, \mathbf{1}_{\{1<k<n+1\}}.$$

A tedious calculation along the lines of 50 then shows that

$$b_{n0}(x) = \left[(\alpha_H - \alpha_L)^n \prod_{k=1}^{n-1} \left(\frac{1}{\gamma} - k \right) \right] x^n + P_{n-1}(x)$$

where $P_{n-1}(x)$ is a polynomial of degree $n-1$ and the result now follows from the Stone-Weierstrass theorem provided that the risk tolerance $1/\gamma \notin \mathbf{N}$. ■