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Heterogeneity in decentralized asset markets

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We study a canonical model of decentralized exchange for a durable good or asset, where agents are assumed to have time-varying, heterogeneous utility types. Whereas the existing literature has focused on the special case of two types, we allow agents' utility to be drawn from an arbitrary distribution. Our main contribution is methodological: we provide a solution technique that delivers a complete characterization of the equilibrium, in closed form, both in and out of the steady state. This characterization offers a richer framework for confronting data from real-world markets and reveals a number of new economic insights. In particular, we show that heterogeneity magnifies the impact of frictions on equilibrium outcomes and that this impact is more pronounced on price levels than on price dispersion and welfare.

Keywords. Over-the-counter markets, search frictions, bargaining, heterogeneity, price dispersion.

JEL classification. G11, G12, G21.

1. Introduction

We consider a canonical model of decentralized exchange for an indivisible durable good or asset, such as a house, a bond, a credit-default swap, a commercial aircraft, a painting, or even an idea for a company. A fixed measure of agents are periodically and randomly matched in pairs, and bargain over the price if there are gains from trade. Agents can hold either 0 or 1 unit of the asset, and have time-varying utility types that generate heterogeneous valuations. Importantly, whereas the existing literature (e.g.,

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Duffie, Gârleanu, and Pedersen (henceforth DGP) (2005, 2007)) has focused on the special case of two utility types, we allow agents' utility types to be drawn from an arbitrary distribution. Our main contribution is methodological: we provide a solution technique that delivers a complete characterization of the equilibrium, in closed form, both in and out of the steady state.

This characterization is valuable for several reasons. First, by establishing that the model with arbitrary heterogeneity is much richer than the special case with only two types, yet equally tractable, our characterization can be used to confront a broader set of empirical observations in real-world markets. For example, unlike the special case with only two types, our model is capable of generating dispersion in both the terms of trade and the time it takes different investors to buy or sell—both important features of markets for durable goods and assets, especially assets traded in over-the-counter markets. Indeed, in a companion paper (Hugonnier, Lester, and Weill (2020)), we apply some of the techniques developed here to assess the role of search and bargaining frictions in dealer-intermediated over-the-counter markets and show that our framework captures some of the key features of the municipal bond market.

Second, solving the model without imposing arbitrary restrictions on the distribution of utility types reveals a common, underlying structure that unifies a broad class of search-theoretic models and provides an important bridge to the literature on asset pricing in frictionless environments. In particular, we show that, as in standard asset pricing models, an agent's private valuation for the asset in our setting can be represented as the present value of future dividend flows to a hypothetical investor whose stochastic discount factor reflects the relevant search and bargaining frictions. Last, studying the relationship between search and bargaining frictions and heterogeneity in valuations reveals new economic insights. We highlight two. First, in an environment with decentralized trade and heterogeneous valuations, we show that trade is mostly concentrated among agents with utility types near the marginal type, as defined in a frictionless benchmark. Hence, even absent heterogeneity in trading speed or inventory capacity, our results suggest an underlying gravitational pull toward a market structure in which a small "core" of agents emerge as natural intermediaries. Second, we show that heterogeneity magnifies the impact of frictions on equilibrium outcomes, and that this impact is more pronounced on price levels than on price dispersion and welfare. As a result, using observed price dispersion to quantify the effect of search frictions on price discounts or premia can be misleading, as price dispersion can essentially vanish while price levels are still far from their frictionless counterpart. A practical implication of this finding is that frictions can have relatively large effects on yield spreads in over-the-counter (OTC) credit markets even when markets appear highly liquid according to traditional measures like volume or price dispersion.

The paper proceeds as follows. After briefly reviewing the literature, we lay out the environment in Section 2. In Section 3, we develop the methodology that allows us to characterize the equilibrium in closed form, both in and out of steady state, for an arbitrary *initial* distribution of utility types. Importantly, in our analysis of the individual optimization problems, we establish several key properties of reservation values that hold irrespective of the cross-sectional distributions that agents take as given. This allows us to eschew the usual guess-and-verify approach and ensures the uniqueness of our equilibrium. Finally, in Section 4, we exploit our characterization to study the relationship between heterogeneity in valuations, asset prices, trading volume, and welfare as trading frictions vanish.

1.1 *Related literature*

This paper belongs to the literature that applies search-and-matching theory to study decentralized markets for durable goods or assets. For an extensive review of this literature, we refer the reader to Hugonnier, Lester, and Weill (2020), and provide here a more narrow discussion of papers that study unintermediated or “pure” decentralized asset markets.

The present paper merges and replaces two working papers, Hugonnier (2012) and Lester and Weill (2013), in which we independently developed the methods to characterize equilibria with arbitrary heterogeneity in valuations.[1] Among other early attempts to study pure decentralized trade with more than two types, Gavazza (2011) and Afonso and Lagos (2015) are most closely related to our work.[2] In particular, in an online appendix, Gavazza (2011) proposes a model of pure decentralized trade with a continuum of types, but focuses on the case in which investors trade only once between preference shocks. In contrast, many of the insights that arise in our environment derive from the many trading opportunities that arise between preference shocks. In Afonso and Lagos (2015), the heterogeneity in valuations derives from allowing investors to take on arbitrary (discrete) asset positions. Though several insights from Afonso and Lagos (2015) also arise in our environment, the two papers differ in both methodology and focus: while they establish many results via numerical methods in an attempt to confront trading patterns in the federal funds market, we derive a variety of analytical results that allow us to study implications for volume and prices across a broad range of OTC markets.

A number of subsequent papers have explored applications of our results, as well as alternative dimensions of heterogeneity that are relevant in OTC markets, including Shen, Wei, and Yan (2020), Üslü (2019), Sagi (2015), Farboodi, Jarosch, and Shimer (2018), Farboodi, Jarosch, Menzio, and Wiriadinata (2018), Bethune, Sultanum, and Trachter (2018), Zhang (2017), Liu (2018), Tse and Xu (2020), and Yang and Zeng (2019). However, the most closely related work is our companion paper, Hugonnier, Lester, and Weill (2020). In that paper, we study a market with two distinct types of agents, customers and dealers, where the dealers themselves trade in a decentralized market. To study characteristics of the intermediation process that have been documented using new, transaction-level data sets, including so-called intermediation chains that are common in dealer-intermediated OTC markets, we assume that dealers have heterogenous and continuously distributed private flow valuations for the asset (or inventory costs).

[1] In contrast to the current paper and Lester and Weill (2013), where heterogeneous valuations are “hardwired” into investors’ preferences, Hugonnier (2012) considers an environment where investors’ valuations differ because of heterogeneity in beliefs about the growth rate of the dividend process and studies conditions under which the speculative behavior highlighted by Harrison and Kreps (1978) in frictionless markets also arises in a decentralized market setting.

[2] See also Cujean and Praz (2013) and Neklyudov (2019).

Then, exploiting the techniques developed here to characterize the steady-state equilibrium of the decentralized interdealer market, we derive a number of testable implications, calibrate the structural parameters to key moments from the municipal bond market, and explore the model's quantitative predictions regarding the relationship between the frictions that we estimate in the market, observable outcomes like bid–ask spreads, and unobservable outcomes like welfare.

In contrast with Hugonnier, Lester, and Weill (2020), the current paper has a sharper methodological focus: to provide a technical toolkit for analyzing decentralized markets for assets or durable goods. For one, the model we study here is different, with no ex ante distinction between customers and dealers, which allows us to characterize all equilibrium objects in closed form and to establish uniqueness of equilibrium. Second, with no ex ante heterogeneity across agents, our solution techniques apply equally well to continuous distributions and those with mass points, so that our framework nests earlier, discrete-type models as special cases (including, e.g., Duffie, Gârleanu, and Pedersen (2005, 2007)). As we will argue below, this level of generality ultimately reveals deeper properties of a broad class of search-and-matching models of pure decentralized asset markets. Third, in the current environment, we are able to characterize the dynamics of equilibria outside of the steady state, starting from any initial distribution of utility types, making it straightforward to analyze the market's response to a variety of aggregate shocks, including a change in the quantity of the asset available (e.g., an issuance shock) or a change in the distribution of valuations (e.g., an aggregate liquidity shock). Finally, leveraging our explicit solutions, we are able to study the properties of equilibria as trading frictions vanish, which reveals that heterogeneity magnifies the impact of search frictions on allocations, prices, and welfare.

2. The model

2.1 *Environment*

We consider a continuous-time, infinite-horizon model with time indexed by $t \geq 0$. The economy is populated by a unit measure of infinitely lived, risk-neutral investors who discount the future at rate $r > 0$. There is one indivisible, durable asset in fixed supply, $s \in (0, 1)$, and one perishable good that we treat as the numéraire. Investors can hold either zero or one unit of the asset.

Preferences The instantaneous utility function of an investor at time t is $c_t + q_t \delta_t$, where c_t denotes the investor's net consumption of the numéraire good ($c_t < 0$ if the investor produces more than he consumes), $q_t \in \{0, 1\}$ denotes the investor's asset holdings, and δ_t denotes the utility flow the investor receives from holding a unit of the asset. We assume that δ_t differs across investors and, for each investor, changes over time. In particular, let $F_0(\delta)$ denote the cumulative distribution of utility types at $t = 0$. Moreover, we suppose that each investor receives independent and identically distributed (i.i.d.) preference shocks that arrive according to a Poisson process with intensity γ, whereupon the investor draws a new utility flow δ' from a (potentially different) cumulative distribution function $F(\delta')$.[3]

Given these assumptions, the cumulative distribution of utility types across the population evolves according to

$$\dot{F}_t(\delta) = \gamma\big(F(\delta) - F_t(\delta)\big),$$

where $\dot{F}_t$ denotes the time derivative. One can easily derive the explicit solution to this ordinary differential equation,

$$F_t(\delta) = F(\delta) + e^{-\gamma t}\big(F_0(\delta) - F(\delta)\big),$$

and see that it converges to the long-run distribution $F(\delta)$ as $t \to \infty$. Note that, at this point, we place very few restrictions on the exogenous distributions $F_0(\delta)$ and $F(\delta)$: our solution method applies equally well to discrete distributions such as the two-point distribution of DGP, continuous distributions, and a mixture of the two, and does not require that $F_0(\delta)$ be absolutely continuous with respect to $F(\delta)$. As a result, our framework allows transient initial conditions that can be used to model the recovery of the market following a liquidity shock. We only require that $\text{supp}(F_0) \cup \text{supp}(F)$ is included in a compact interval and make it sufficiently large so that there are no mass points at the boundaries. For simplicity, we normalize this interval to $[0, 1]$.

Matching and trade Investors trade in a purely decentralized market in which each investor initiates contact with another randomly selected investor according to a Poisson process with intensity $\lambda/2$. If two investors are matched and there are gains from trade, they bargain over the price of the asset. The outcome is taken to be the Nash bargaining solution, in which the investor with asset holdings $q \in \{0, 1\}$ has bargaining power $\theta_q \in (0, 1)$, with $\theta_0 + \theta_1 = 1$.

The state variable An important object of interest throughout our analysis will be the joint distribution of utility types and asset holdings. The standard approach in the literature, following DGP, is to characterize this distribution by analyzing the density or *measure* of investors across types $(q, \delta) \in \{0, 1\} \times [0, 1]$. Our analysis below reveals that the model becomes much more tractable when we study instead the *cumulative measure*; this allows for a closed-form solution for an arbitrary underlying distribution of types, both in and out of steady state.

Let $\Phi_{q,t}(\delta)$ denote the measure of investors at time $t \geq 0$ with asset holdings $q \in \{0, 1\}$ and utility type less than or equal to $\delta \in [0, 1]$. These joint distributions must satisfy the following accounting identities for all $t \geq 0$:

$$\Phi_{0,t}(\delta) + \Phi_{1,t}(\delta) = F_t(\delta) \tag{1}$$

$$\Phi_{1,t}(1) = s. \tag{2}$$

[3]All of our results apply mutatis mutandis to the case where types are persistent, in the sense that the distribution of an agent's new type δ' conditional on his old type δ, $F(\delta'|\delta)$, is first-order stochastically increasing in δ. The only caveat is that the equilibrium is then unique in the class of equilibria for which the reservation value function (defined below) is bounded, rather than globally unique as it is in our benchmark model.

Equation (1) requires that the cross-sectional distribution of utility types in the population is equal to $F_t(\delta)$ for all $t \geq 0$. Equation (2) is a market-clearing condition that equates the total measure of investors who own the asset and the total supply of assets in the economy.

2.2 *The frictionless benchmark: Centralized exchange*

Before analyzing the environment with search frictions, it is helpful to first characterize the equilibrium in a frictionless benchmark; this allows us to identify certain key parameters and, later, study the limiting properties of equilibria as search frictions vanish. To that end, consider an environment with a competitive, centralized market where investors can buy or sell the asset instantly at some price p_t for all $t \geq 0$. Since there is no aggregate uncertainty, the price path is necessarily a function of time in a deterministic equilibrium. We assume that this function is uniformly bounded and absolutely continuous with a uniformly bounded (almost everywhere) derivative $\dot{p}_t$.

Given the price path, the objective of an investor is to choose a finite variation asset-holding process $q_t \in \{0, 1\}$ that is progressively measurable with respect to the filtration generated by his utility-type process and that maximizes

$$\mathbb{E}_\delta\left[\int_0^\infty e^{-rt}\delta_t q_t\,dt - \int_0^\infty e^{-rt}p_t\,dq_t\right] = p_0 q_0 + \mathbb{E}_\delta\left[\int_0^\infty e^{-rt}q_t(\delta_t - rp_t + \dot{p}_t)\,dt\right],$$

where the equality follows from integration by parts. Maximizing pointwise on the right-hand side shows that an investor's optimal asset holdings satisfy

$$q_t^\star = \begin{cases} 0 & \text{if } \delta_t < rp_t - \dot{p}_t \\ \in \{0, 1\} & \text{if } \delta_t = rp_t - \dot{p}_t \\ 1 & \text{if } \delta_t > rp_t - \dot{p}_t. \end{cases}$$

Since the supply of the asset is s, in equilibrium the marginal type must belong to the set

$$\Delta_t^\star \equiv \left\{\delta \in [0, 1] : \lim_{y\uparrow\delta} F_t(y) \leq 1 - s \leq F_t(\delta)\right\}$$

at all times. The equilibrium distribution of utility types among investors who own 1 unit of the asset is accordingly given by

$$\Phi_{1,t}^\star(\delta) = \max\left\{0, F_t(\delta) - (1 - s)\right\}$$

and, from (1), the equilibrium distribution of utility types among investors who do not own the asset must be $\Phi_{0,t}^\star(\delta) = \min\{F_t(\delta), 1 - s\}$.

Finally, since the correspondence $\Delta_t^\star$ is compact-valued and upper hemicontinuous, it follows from the measurable selection theorem (Stokey and Lucas (1989, Theorem 7.6)) that there exists a measurable path of marginal types, and it is easily seen that given any such path of marginal types, the induced price path

$$p_t^\star = \int_t^\infty e^{-r(u-t)}\delta_u^\star\,du$$

implements the equilibrium asset allocation. In words, the frictionless price is the present value of the utility flows enjoyed by a hypothetical investor who holds the asset forever and whose utility type is marginal at all times. To guarantee that the frictionless price is uniquely defined in the steady state, from now on we will ignore the nongeneric case where the steady-state distribution of utility types $F(\delta)$ is flat at the level $1-s$.

3. Equilibrium with search frictions

We now characterize the equilibrium with search frictions in three steps. First, in Section 3.1, we derive investors' reservation value functions, which allows us to characterize the (unique) optimal trading rules and equilibrium asset prices given any path for the joint distribution of utility types and asset holdings, $\Phi_{0,t}$ and $\Phi_{1,t}$. Then, in Section 3.2, we show that the optimal trading rules imply a unique path for the joint distributions $\Phi_{0,t}$ and $\Phi_{1,t}$, which we derive explicitly. Finally, in Section 3.3, we combine these results to construct the unique equilibrium and show that it converges to a steady state from any initial allocation.

3.1 *Reservation values*

Let $V_{q,t}(\delta)$ denote the maximum attainable utility of an investor with $q \in \{0, 1\}$ units of the asset and utility type $\delta \in [0, 1]$ at time $t \geq 0$, and denote this investor's *reservation value* by[4]

$$\Delta V_t(\delta) \equiv V_{1,t}(\delta) - V_{0,t}(\delta).$$

In addition to considering an arbitrary distribution of utility types, our analysis of reservation values improves on the existing literature in several dimensions. First, in Section 3.1.1, we depart from the usual guess-and-verify approach by establishing elementary properties of reservation values directly, without making any *a priori* assumption on the direction of gains from trade. This allows us, down the road in Theorem 1, to claim a general uniqueness result for equilibrium. Second, in Section 3.1.2, we study a differential representation of reservation values that generalizes an earlier closed-form solution for the trading surplus in DGP's two-type model. Third, in Section 3.1.3, we study a sequential representation of reservation values that generalizes the concept of a marginal investor to an asset market with search-and-matching frictions.

3.1.1 *Necessary properties* Denote by

$$P_\tau(\delta, \delta') \equiv \theta_0 \Delta V_\tau(\delta) + \theta_1 \Delta V_\tau(\delta') \tag{3}$$

the Nash solution to the bargaining problem at time $\tau \geq 0$ between an asset owner of utility type δ and a non-owner of utility type δ'. An application of Bellman's principle of

[4]Note that the reservation value function is well defined for all $\delta \in [0, 1]$ and not only for those utility types in the support of the underlying distribution, $F_t(\cdot)$.

optimality shows that

$$V_{1,t}(\delta) = \mathbb{E}_t\bigg[\int_t^\tau e^{-r(u-t)}\delta\,du + e^{-r(\tau-t)}\bigg(\mathbf{1}_{\{\tau=\tau_1\}}V_{1,\tau}(\delta) + \mathbf{1}_{\{\tau=\tau_\gamma\}}\int_0^1 V_{1,\tau}(\delta')\,dF(\delta')$$

$$+ \mathbf{1}_{\{\tau=\tau_0\}}\int_0^1 \max\{V_{1,\tau}(\delta), V_{0,\tau}(\delta) + P_\tau(\delta,\delta')\}\frac{d\Phi_{0,\tau}(\delta')}{1-s}\bigg)\bigg], \tag{4}$$

where τ_γ is an exponential random variable with parameter γ that represents the arrival of a preference shock, τ_q is an exponential random variable with parameter λs if $q = 1$ and $\lambda(1-s)$ if $q = 0$ that represents the occurrence of a meeting with a randomly selected investor who owns q units of the asset, and the expectation is conditional on $\tau \equiv \min\{\tau_0, \tau_1, \tau_\gamma\} > t$.

Substituting the price (3) into (4) and simplifying shows that the maximum attainable utility of an asset owner satisfies

$$V_{1,t}(\delta) = \mathbb{E}_t\bigg[\int_t^\tau e^{-r(u-t)}\delta\,du + e^{-r(\tau-t)}\bigg(V_{1,\tau}(\delta)$$

$$+ \mathbf{1}_{\{\tau=\tau_\gamma\}}\int_0^1 \big(V_{1,\tau}(\delta') - V_{1,\tau}(\delta)\big)\,dF(\delta')$$

$$+ \mathbf{1}_{\{\tau=\tau_0\}}\int_0^1 \theta_1\big(\Delta V_\tau(\delta') - \Delta V_\tau(\delta)\big)^+\frac{d\Phi_{0,\tau}(\delta')}{1-s}\bigg)\bigg]. \tag{5}$$

The first term on the right-hand side of (5) accounts for the fact that an asset owner enjoys a constant flow of utility at rate δ until time τ. The remaining terms capture the three possible events for an asset owner at time τ: a preference shock ($\tau = \tau_\gamma$), in which case a new utility type is drawn from the distribution $F(\delta')$; meeting another asset owner ($\tau = \tau_1$), in which case there are no gains from trade and the continuation payoff is $V_{1,\tau}(\delta)$; or meeting a non-owner ($\tau = \tau_0$), who is of type δ' with probability $d\Phi_{0,\tau}(\delta')/(1-s)$, in which case the owner sells the asset if the payoff from doing so exceeds the payoff from keeping the asset and continuing to search.

Proceeding in a similar way for $q = 0$ shows that the maximum attainable utility of an investor who does not own an asset satisfies

$$V_{0,t}(\delta) = \mathbb{E}_t\bigg[e^{-r(\tau-t)}\bigg(V_{0,\tau}(\delta) + \mathbf{1}_{\{\tau=\tau_\gamma\}}\int_0^1 \big(V_{0,\tau}(\delta') - V_{0,\tau}(\delta)\big)\,dF(\delta')$$

$$+ \mathbf{1}_{\{\tau=\tau_1\}}\int_0^1 \theta_0\big(\Delta V_\tau(\delta) - \Delta V_\tau(\delta')\big)^+\frac{d\Phi_{1,\tau}(\delta')}{s}\bigg)\bigg], \tag{6}$$

and subtracting (6) from (5) shows that the reservation value function satisfies the autonomous dynamic programming equation

$$\Delta V_t(\delta) = \mathbb{E}_t\bigg[\int_t^\tau e^{-r(u-t)}\delta\,du + e^{-r(\tau-t)}\bigg(\Delta V_\tau(\delta)$$

$$+ \mathbf{1}_{\{\tau=\tau_\gamma\}}\int_0^1 \big(\Delta V_\tau(\delta') - \Delta V_\tau(\delta)\big)\,dF(\delta')$$

$$+ \mathbf{1}_{\{\tau=\tau_0\}} \int_0^1 \theta_1\big(\Delta V_\tau(\delta') - \Delta V_\tau(\delta)\big)^+ \frac{d\Phi_{0,\tau}(\delta')}{1-s}$$
$$- \mathbf{1}_{\{\tau=\tau_1\}} \int_0^1 \theta_0\big(\Delta V_\tau(\delta) - \Delta V_\tau(\delta')\big)^+ \frac{d\Phi_{1,\tau}(\delta')}{s} \Bigg)\Bigg]. \tag{7}$$

This equation reveals that an investor's reservation value is influenced by two distinct option values, which have opposing effects. On the one hand, an investor who owns an asset has the option to search and find a non-owner who will pay even more for the asset; as shown on the third line, this option increases her reservation value. On the other hand, an investor who does not own an asset has the option to search and find an owner who will sell at an even lower price; as shown on the fourth line, this option decreases her willingness to pay and, hence, her reservation value.

To guarantee the global optimality of the trading decisions induced by (5) and (6), we further require that the maximum attainable utilities of owners and non-owners, and hence the reservation values, satisfy the transversality conditions

$$\lim_{t\to\infty} e^{-rt} V_{q,t}(\delta) = \lim_{t\to\infty} e^{-rt} \Delta V_t(\delta) = 0, \quad (q,\delta) \in \{0,1\} \times [0,1]. \tag{8}$$

The following proposition establishes the existence, uniqueness, and some necessary properties of solutions to (5), (6), and (7) that satisfy (8).

Proposition 1. *There exists a unique function* $\Delta V : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ *that satisfies (7) subject to (8). This function is uniformly bounded, absolutely continuous in* $(t,\delta) \in \mathbb{R}_+ \times [0,1]$*, and strictly increasing in* $\delta \in [0,1]$*, with a uniformly bounded derivative with respect to utility type. Given* $\Delta V_t(\delta)$*, there are unique functions* $V_{0,t}(\delta)$ *and* $V_{1,t}(\delta)$ *that satisfy (5), (6), and (8).*

The fact that reservation values are strictly increasing in δ implies that when an asset owner of type δ meets a non-owner of type $\delta' > \delta$, they will always agree to trade. Indeed, these two investors face the same distributions of future trading opportunities and preference shocks. Thus, the only relevant distinction between them is the difference in utility flow enjoyed from the asset, which implies that the reservation value of an investor of type δ' is strictly larger than that of an investor of type $\delta < \delta'$. The monotonicity property holds regardless of the distributions $\Phi_{q,t}(\delta)$, which investors take as given when calculating their optimal trading strategy. Moreover, as we establish below, this property greatly simplifies the derivation of closed-form solutions for both reservation values and the equilibrium distribution of asset holdings and utility types.

3.1.2 *Differential representation* Integrating both sides of (7) with respect to the conditional distribution of τ, and using the fact that reservation values are strictly increasing in utility type, we obtain that the reservation value function satisfies the integral equation

$$\Delta V_t(\delta) = \int_t^\infty e^{-(r+\gamma+\lambda)(u-t)} \bigg(\delta + \lambda \Delta V_u(\delta) + \gamma \int_0^1 \Delta V_u(\delta')\, dF(\delta')$$

$$+ \lambda \int_{\delta}^{1} \theta_1 \big(\Delta V_u(\delta') - \Delta V_u(\delta)\big)\, d\Phi_{0,u}(\delta')$$
$$- \lambda \int_{0}^{\delta} \theta_0 \big(\Delta V_u(\delta) - \Delta V_u(\delta')\big)\, d\Phi_{1,u}(\delta')\bigg)\, du. \tag{9}$$

In addition, since Proposition 1 establishes that $\Delta V_t(\delta)$ is absolutely continuous in $(t, \delta) \in \mathbb{R}_+ \times [0, 1]$ with a bounded derivative with respect to utility type, we know that

$$\Delta V_t(\delta) = \Delta V_t(0) + \int_0^{\delta} \sigma_t(\delta')\, d\delta' \tag{10}$$

for some nonnegative and uniformly bounded function $\sigma_t(\delta)$ that is itself absolutely continuous in time for almost every $\delta \in [0, 1]$. We naturally interpret this function as a measure of the local surplus in the decentralized market, since the gains from trade between a seller of type δ and a buyer of type $\delta + d\delta$ are approximately given by $\sigma_t(\delta)\, d\delta$.

Substituting (10) into (9), changing the order of integration, and differentiating both sides of the resulting equation with respect to t and δ reveals that the local surplus satisfies

$$\big(r + \gamma + \lambda\theta_1\big(1 - s - \Phi_{0,t}(\delta)\big) + \lambda\theta_0\Phi_{1,t}(\delta)\big)\sigma_t(\delta) = 1 + \dot{\sigma}_t(\delta) \tag{11}$$

at almost every point of $\mathbb{R}_+ \times [0, 1]$. The local surplus characterized in (11) is the natural generalization of the trading surplus in DGP to nonstationary environments with arbitrary distributions of utility types. To see this precisely, recall that DGP characterized an equilibrium in a special case of our model: in a steady state with two utility types, $\delta_\ell \le \delta_h$. In that setting, the measures $1 - s - \Phi_0(\delta)$ and $\Phi_1(\delta)$ are constant over $[\delta_\ell, \delta_h)$, and correspond to the masses of buyers and sellers, respectively, denoted by μ_{hn} and $\mu_{\ell o}$ in DGP. Using this property, integrating both sides of (11), and restricting attention to the steady state gives

$$(r + \gamma + \lambda\theta_1\mu_{hn} + \lambda\theta_0\mu_{\ell o})\big(\Delta V(\delta_h) - \Delta V(\delta_\ell)\big) = \delta_h - \delta_\ell,$$

which is the surplus formula of DGP.

Given (11), we can now derive a closed-form solution for reservation values. A calculation provided in Appendix A.1 shows that, together with the requirements of boundedness and absolute continuity in time, (11) uniquely pins down the local surplus as

$$\sigma_t(\delta) = \int_t^{\infty} e^{-\int_t^u (r+\gamma+\lambda\theta_1(1-s-\Phi_{0,\xi}(\delta))+\lambda\theta_0\Phi_{1,\xi}(\delta))\, d\xi}\, du. \tag{12}$$

Combining this explicit solution for the local surplus with (9) and (10) allows us to derive the reservation value function in closed form.

Proposition 2. *For any distribution $\Phi_{0,t}(\delta)$ and $\Phi_{1,t}(\delta)$ satisfying (1) and (2), the unique solution to (7) and (8) is explicitly given by*

$$\Delta V_t(\delta) = \int_t^{\infty} e^{-r(u-t)}\bigg(\delta - \int_0^{\delta} \sigma_u(\delta')\big(\gamma F(\delta') + \lambda\theta_0\Phi_{1,u}(\delta')\big)\, d\delta'$$

$$+\int_{\delta}^{1}\sigma_u(\delta')\big(\gamma\big(1-F(\delta')\big)+\lambda\theta_1\big(1-s-\Phi_{0,u}(\delta')\big)\big)\,d\delta'\bigg)\,du, \tag{13}$$

where the local surplus $\sigma_t(\delta)$ is defined by (12).

We close this subsection with several intuitive comparative static results for reservation values.

Corollary 1. *For any $(t,\delta)\in\mathbb{R}_+\times[0,1]$, the reservation value $\Delta V_t(\delta)$ increases if an investor can bargain higher selling prices (larger θ_1), if he expects to have higher future valuations (a first-order stochastic dominant shift in $F(\delta')$), or if he expects to trade with higher valuation counterparts (a first-order stochastic dominance shift in the path of either $\Phi_{0,t'}(\delta')$ or $\Phi_{1,t'}(\delta')$).*

To complement these results, note that an increase in the search intensity, λ, can either increase or decrease reservation values. This is because of the two option values discussed above: an increase in λ increases an owner's option value of searching for a buyer who will pay a higher price, which drives the reservation value up, but it also increases a non-owner's option value of searching for a seller who will offer a lower price, which has the opposite effect. As we will see below in Section 4, the net effect is ambiguous and depends on all parameters of the model.

3.1.3 *Sequential representation* Differentiating both sides of (9) with respect to time shows that the reservation value function can be characterized as the unique bounded and absolutely continuous solution to the Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{aligned} r\Delta V_t(\delta)=\delta+\Delta\dot V_t(\delta)&+\gamma\int_0^1\big(\Delta V_t(\delta')-\Delta V_t(\delta)\big)\,dF(\delta')\\ &+\lambda\int_{\delta}^{1}\theta_1\big(\Delta V_t(\delta')-\Delta V_t(\delta)\big)\,d\Phi_{0,t}(\delta')\\ &+\lambda\int_0^{\delta}\theta_0\big(\Delta V_t(\delta')-\Delta V_t(\delta)\big)\,d\Phi_{1,t}(\delta'). \end{aligned} \tag{14}$$

The following proposition shows that the solution to this equation can be represented as the present value of utility flows from the asset to a hypothetical investor whose utility type process is adjusted to reflect the frictions present in the market.

Proposition 3. *The reservation value function can be represented as*

$$\Delta V_t(\delta)=\mathbb{E}_{t,\delta}\bigg[\int_t^{\infty}e^{-r(s-t)}\hat\delta_s\,ds\bigg], \tag{15}$$

where the market-valuation process, $\hat\delta_t$, is a pure jump Markov process on $[0,1]$ with infinitesimal generator defined by

$$\mathcal{A}_t[v](\delta)\equiv\int_0^1\big(v(\delta')-v(\delta)\big)\big(\gamma\,dF(\delta')+\mathbf{1}_{\{\delta'>\delta\}}\lambda\theta_1\,d\Phi_{0,t}(\delta')+\mathbf{1}_{\{\delta'\le\delta\}}\lambda\theta_0\,d\Phi_{1,t}(\delta')\big)$$

for any uniformly bounded function $v:[0,1]\to\mathbb{R}$.

Representations such as (15) are standard in frictionless asset pricing, where private values are obtained as the present value of cash flows under a probability constructed from marginal rates of substitution. The emergence of such a representation in a decentralized market is, to the best of our knowledge, new to this paper and can be viewed as generalizing the concept of the marginal investor. Namely, in the frictionless benchmark, the market valuation is equal to the utility flow of the marginal investor, $\delta_t^\star$, since investors can trade instantly at price $p_t^\star$. In a decentralized market, the market valuation differs from $\delta_t^\star$ for two reasons. First, since meetings are not instantaneous, an owner receives his private utility flow until finding a trading partner. Second, since investors do not always trade with the marginal type, the terms of trade are random and depend on the distribution of types among trading partners. Importantly, this second channel is only active if there are more than two utility types, because otherwise a single price gets realized in bilateral meetings.

3.2 *The joint distribution of asset holdings and types*

In this section, we provide a closed-form characterization of the joint equilibrium distribution of asset holdings and utility types, in and out of steady state. To the best of our knowledge, this characterization is new to the literature. In particular, even in their special two-type case, DGP did not derive an explicit characterization of out-of-steady-state dynamics. We then establish that this distribution converges to the steady state from any initial conditions satisfying (1) and (2). Finally, we discuss several properties of the steady-state distribution and explain how its shape depends on the arrival rates of preference shocks and trading opportunities.

Since reservation values are increasing in utility type, trade occurs between two investors if and only if one is an owner with utility type δ' and the other is a non-owner with utility type $\delta'' \geq \delta'$. Investors with the same utility type are indifferent between trading or not, but whether they trade is irrelevant since they effectively exchange ownership type. As a result, the rate of change in the measure of owners with utility type less than or equal to a given $\delta \in [0, 1]$ satisfies

$$\dot{\Phi}_{1,t}(\delta) = \gamma\big(s - \Phi_{1,t}(\delta)\big)F(\delta) - \gamma\Phi_{1,t}(\delta)\big(1 - F(\delta)\big) - \lambda\Phi_{1,t}(\delta)\big(1 - s - \Phi_{0,t}(\delta)\big). \tag{16}$$

The first term in (16) is the inflow due to type-switching: at each instant, a measure $\gamma(s - \Phi_{1,t}(\delta))$ of owners with utility type greater than δ draws a new utility type, which is less than or equal to δ with probability $F(\delta)$. A similar logic can be used to understand the second term, which is the outflow due to type-switching. The third term is the outflow due to trade. In particular, a measure $(\lambda/2)\Phi_{1,t}(\delta)$ of investors who own the asset and have utility type less than δ initiates contact with another investor, and with probability $1 - s - \Phi_{0,t}(\delta)$, that investor is a non-owner with utility type greater than δ, so that trade ensues. The same measure of trades occur when non-owners with utility type greater than δ initiate trade with owners with utility type less than δ, so that the sum equals the third term in (16).[5]

[5] Note that trading generates positive *gross* inflow into the set of owners with utility type less than δ, but zero *net* inflow. Indeed, a gross inflow arises when a non-owner with utility type $\delta' \leq \delta$ meets an owner

Using (1), we can rewrite (16) as a first-order ordinary differential equation for the measure of asset owners with utility type less than or equal to each δ:

$$\dot{\Phi}_{1,t}(\delta) = -\lambda\Phi_{1,t}(\delta)^2 - \Phi_{1,t}(\delta)\big(\gamma + \lambda\big(1 - s - F_t(\delta)\big)\big) + \gamma s F(\delta). \tag{17}$$

Importantly, this Riccati equation for $\Phi_{1,t}(\delta)$ is independent from $\Phi_{1,t}(\delta')$ for all $\delta' \neq \delta$, and holds without imposing any regularity conditions on the distribution of utility types $F_t(\delta)$: it works equally well for continuous distributions, discrete distributions, or mixtures of the two, with or without transient states.[6] Proposition 4 below establishes that there exists a unique solution to this equation and shows that it converges to a unique steady state.

Proposition 4. *Given* $(\delta, \Phi_{1,0}(\delta)) \in [0, 1]^2 \times [0, s]$, *there exists a unique solution* $\Phi_{1,t}(\delta)$ *to (17). This solution is defined for all* $t \geq 0$ *and converges to the steady-state measure*

$$\Phi_1(\delta) = F(\delta) - \Phi_0(\delta) \equiv -\frac{1}{2}\big(1 - s + \gamma/\lambda - F(\delta)\big) + \frac{1}{2}\Lambda(\delta), \tag{18}$$

where

$$\Lambda(\delta) \equiv \sqrt{\big(1 - s + \gamma/\lambda - F(\delta)\big)^2 + 4s(\gamma/\lambda)F(\delta)}.$$

In the proof of Proposition 4, in the Appendix, we derive the explicit solution for $\Phi_{1,t}(\delta)$ outside of the steady state. To illustrate the convergence of the equilibrium distributions to the steady state, we introduce a simple numerical example, which we will continue to use throughout the text. In this example, the discount rate is $r = 0.05$; the asset supply is $s = 0.5$; the meeting rate is $\lambda = 12$, so that a given investor meets others on average once a month; the arrival rate of preference shocks is $\gamma = 1$, so that investors change type on average once a year; the initial distribution of utility types among asset owners is $\Phi_{1,0}(\delta) = sF(\delta)$;[7] and the underlying distribution of utility types is $F_0(\delta) = F(\delta) = \delta^\alpha$ with $\alpha = 1.5$, so that $F_t(\delta) = F(\delta)$ at all times and the (constant) marginal type from the frictionless benchmark is given by $\delta^\star = 0.6299$.

Using this parameterization, the left panel of Figure 1 plots the equilibrium distributions among owners and non-owners at $t = 0$, after 1 month, after 6 months, and in the limiting steady state. As time passes, one can see that the assets are gradually allocated toward investors with higher valuations: the distribution of utility types among owners improves in the sense of first-order stochastic dominance (FOSD). Similarly, the distribution of utility types among non-owners deteriorates, in the FOSD sense, indicating that investors with low valuations are less and less likely to hold the asset over time.

with an even lower type $\delta'' < \delta'$. By trading, the previous owner of utility type δ'' leaves the set, but the new owner of utility type δ' enters the same set, resulting in zero net inflow.

[6] In contrast, differentiating with respect to δ reveals that the dynamic system for *measures* (instead of cumulative measures) does exhibit interdependence across values of δ, i.e., the equation characterizing the density (or point mass) at δ depends on the density at $\delta' \neq \delta$, making closed-form solutions more difficult to attain in all but the simplest cases.

[7] Note that, since $s = 0.5$, the initial distribution of utility types among asset owners and non-owners are the same, i.e., $\Phi_{1,0}(\delta) = sF(\delta) = (1 - s)F(\delta) = \Phi_{0,0}(\delta)$.

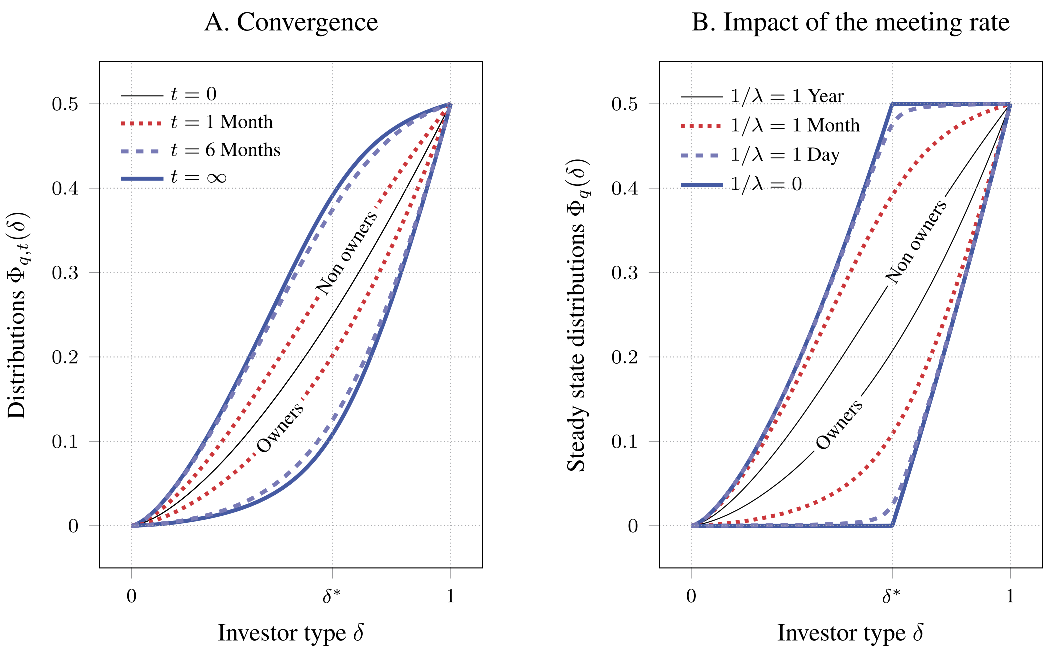

FIGURE 1. Equilibrium distributions. *Notes*: Panel A plots the cumulative distribution of types among non-owners (upper curves) and owners (lower curves) at different points in time. Panel B plots these distributions in the steady state, for different levels of search frictions, indexed by the average intercontact time, $1/\lambda$.

Focusing on the steady-state distributions, (18) offers several natural comparative statics that we summarize in the following corollary.

COROLLARY 2. *For any $\delta \in [0, 1]$, the steady-state measure $\Phi_1(\delta)$ of asset owners with utility type less than or equal to δ is increasing in γ and decreasing in λ.*

Intuitively, as preference shocks become less frequent (i.e., γ decreases) or trading opportunities become more frequent (i.e., λ increases), the asset is allocated to investors with higher valuations more efficiently, implying a FOSD shift in the distribution of types among owners. In the limit, where types are permanent ($\gamma \to 0$) or trading opportunities are constantly available ($\lambda \to \infty$), the steady-state distributions converge to their frictionless counterparts, as illustrated by the right panel of Figure 1, and the allocation is efficient. We return to this frictionless limit in Section 4.

3.3 *Equilibrium*

DEFINITION 1. An equilibrium is a reservation value function $\Delta V_t(\delta)$, and a pair of distributions $\Phi_{0,t}(\delta)$ and $\Phi_{1,t}(\delta)$ such that the distributions satisfy (1), (2), and (17), and the reservation value function satisfies (7) subject to (8) given the distributions.

Given the analysis above, a full characterization of the unique equilibrium is immediate. Note that uniqueness follows from the fact that we proved reservation values were strictly increasing directly, given arbitrary time paths for the distributions $\Phi_{0,t}(\delta)$ and $\Phi_{1,t}(\delta)$, rather than guessing and verifying that such an equilibrium exists.[8]

Theorem 1. *There exists a unique equilibrium. Moreover, given any initial conditions satisfying (1) and (2), this equilibrium converges to the steady state where reservation values are given by*

$$\begin{aligned} r\Delta V(\delta) = \delta &- \int_0^\delta \sigma(\delta')\big(\gamma F(\delta') + \lambda\theta_0\Phi_1(\delta')\big)\,d\delta' \\ &+ \int_\delta^1 \sigma(\delta')\big(\gamma\big(1-F(\delta')\big) + \lambda\theta_1\big(1-s-\Phi_0(\delta')\big)\big)\,d\delta' \end{aligned} \tag{19}$$

with the time-invariant local surplus

$$\sigma(\delta) = \frac{1}{r+\gamma+\lambda\theta_1\big(1-s-\Phi_0(\delta)\big)+\lambda\theta_0\Phi_1(\delta)},$$

and the cumulative distributions of utility types among asset owners and non-owners are given by (1) and (18).

4. The frictionless limit

We now study the equilibrium as trading frictions vanish, i.e., as $\lambda \to \infty$.[9] This is an important exercise for two reasons. First, this is the empirically relevant case in many financial markets, where trading speeds are becoming faster and faster (e.g., Pagnotta and Philippon (2018)). Second, this exercise reveals several new economic insights regarding the effects of heterogeneity in markets with search frictions. We highlight two specific results. First, we show that heterogeneity creates a large volume of trade, relative to the frictionless benchmark, that becomes increasingly concentrated among a small set of agents as the trading speed increases. Second, we show that heterogeneity magnifies the impact of frictions on equilibrium outcomes, and that this impact is more pronounced on price levels than on price dispersion and welfare. As a result, using observed price dispersion to quantify the effect of search frictions on price discounts or premia can be misleading, as price dispersion can essentially vanish while price levels are still far from their frictionless counterpart.

4.1 *Misallocation and trading volume*

We focus on the asymptotic properties of the steady-state equilibrium, in which the distribution of utility types is $F(\delta)$, and drop all time subscripts accordingly. To start, we

[8]Also note that uniqueness does not depend on the assumption of Nash bargaining, but rather extends to any method of price determination that achieves bilateral efficiency and preserves the monotonicity of reservation value functions.

[9]As will become clear, convergence is governed by a composite parameter, λ/γ, which may be interpreted as the ratio between the supply and the demand of transaction services. However, in many contexts, it is economically more meaningful to vary the supply holding the demand fixed, or vice versa. Correspondingly, we choose to present our results as the limiting case of $\lambda \to \infty$, holding γ fixed.

explore the equilibrium asset allocation—and the process of reallocation—as trading frictions vanish. As a first step, we establish that the allocation converges to its frictionless counterpart, $\Phi_q^\star(\delta)$, $q \in \{0, 1\}$, as $\lambda \to \infty$.

Lemma 1. *As search frictions vanish,* $\lim_{\lambda\to\infty} \Phi_q(\delta) = \Phi_q^\star(\delta)$ *for* $q \in \{0, 1\}$.

Misallocation While Lemma 1 is standard in models of this ilk, the nature of misallocation near the frictionless limit reveals new insights when there is rich heterogeneity in the distribution of valuations. To formalize the concept of misallocation, let

$$M(\delta) = \int_0^\delta \mathbf{1}_{\{\delta' < \delta^\star\}} \, d\Phi_1(\delta') + \int_0^\delta \mathbf{1}_{\{\delta' \geq \delta^\star\}} \, d\Phi_0(\delta'),$$

where the utility type $\delta^\star$ defined by the equality $F(\delta^\star) = 1 - s$ is the marginal type of the frictionless steady-state equilibrium. This measure is the sum of two types of misallocation: the measure of investors with utility type less than δ who would own the asset in a frictionless environment but do not own it in the presence of search frictions; and the measure of investors with utility type less than δ who would not own the asset in a frictionless environment but own it in the presence of search frictions.

To measure the extent of misallocation at a specific utility type, we study the ratio $\frac{M(\delta)}{M(1)}$, i.e., the cumulative distribution function (CDF) of misallocation across utility types. The following result establishes that misallocation becomes concentrated around the marginal type as trading frictions vanish.

Lemma 2. *For any* $\varepsilon > 0$, $\lim_{\lambda\to\infty} \frac{M(\delta^\star+\varepsilon)-M(\delta^\star-\varepsilon)}{M(1)} = 1$.

Intuitively, misallocation becomes highly concentrated around the marginal type because there is an equilibrium feedback loop between the intensity with which agents with utility type δ trade and the distribution of utility types among owners and non-owners. For example, since the selling intensity $\lambda(1 - s - \Phi_0(\delta))$ is decreasing in δ, an owner with a utility type $\delta' \approx 0$ sells relatively quickly. As a result, there is little misallocation among low utility types, i.e., most agents with $\delta' \approx 0$ are non-owners, which makes it easy to sell when an owner draws a low utility type. By contrast, an owner with utility type δ'' just below $\delta^\star$ sells much more slowly, since most agents with $\delta > \delta''$ already own the asset, thus reinforcing the fact that misallocation clusters in a neighborhood around the marginal type $\delta^\star$.

We emphasize that this property of misallocation arises in our decentralized market because trading intensities differ across utility types. Indeed, when all investors trade with equal intensity—as in frictionless models with centralized markets or in frictional models where all trades are executed by a set of dealers who have access to centralized markets—the measure of misallocation described above would be constant across utility types.

Trading volume Next, we show that the concentration of misallocation translates into a concentration of trading volume near the marginal type. To see this, let us first define trading volume as the flow rate of trades per unit time:

$$\vartheta = \lambda \int_{[0,1]^2} \mathbf{1}_{\{\delta_0 > \delta_1\}} \, d\Phi_0(\delta_0) \, d\Phi_1(\delta_1). \tag{20}$$

When the underlying distribution of utility types is continuous, we can use integration by parts to rewrite (20) as

$$\begin{aligned} \vartheta &= \lambda \Phi_1(\delta^\star)\big(1 - s - \Phi_0(\delta^\star)\big) \\ &\quad + \lambda \int_0^{\delta^\star} dM(\delta)\big(\Phi_0(\delta^\star) - \Phi_0(\delta)\big) + \lambda \int_{\delta^\star}^1 dM(\delta)\big(\Phi_1(\delta) - \Phi_1(\delta^\star)\big). \end{aligned} \tag{21}$$

The first term in (21) represents the volume generated by trades between owners with utility types in $[0, \delta^\star]$ and non-owners with utility types in $[\delta^\star, 1]$; these would be the only trades taking place in the equilibrium of a model with frictionless exchange. With search frictions, however, there are additional inframarginal trades, captured by the second and third terms. In particular, the second term accounts for inframarginal trades between owners with utility types $\delta < \delta^\star$ and non-owners with utility types in $[\delta, \delta^\star]$, while the third term accounts for inframarginal trades between non-owners with utility types $\delta > \delta^\star$ and owners with utility types in $[\delta^\star, \delta]$.

The formula highlights the role of misallocation in generating trading volume in excess of that in the frictionless benchmark.[10] It also suggests that near-marginal investors, who are characterized by greater misallocation, are likely to have a larger contribution to trading volume. This is confirmed in the next proposition.

Proposition 5. *Assume that the distribution of utility types is continuous. Then the steady-state trading volume is explicitly given by*

$$\vartheta \equiv \gamma s(1-s)\left[\left(1 + \frac{\gamma}{\lambda}\right)\log\left(1 + \frac{\lambda}{\gamma}\right) - 1\right]. \tag{22}$$

In particular, the steady-state trading volume ϑ is strictly increasing in the meeting rate λ, with $\lim_{\lambda\to\infty} \vartheta = \infty$ and

$$\lim_{\lambda\to\infty} \frac{\lambda}{\vartheta}\left(\int_{\delta^\star-\varepsilon}^{\delta^\star} \Phi_1(\delta)\, d\Phi_0(\delta) + \int_{\delta^\star}^{\delta^\star+\varepsilon} \big(1 - s - \Phi_0(\delta)\big)\, d\Phi_1(\delta)\right) = 1$$

for any $\varepsilon > 0$.

Proposition 5 establishes two key results. First, when the underlying distribution of utility types is continuous, the equilibrium trading volume is unbounded as $\lambda \to \infty$. By

[10] Note, however, that these additional trades are not an indication of inefficiency; the equilibrium with search frictions is constrained efficient.

contrast, the equilibrium trading volume is finite in the frictionless benchmark.[11] One can also show that volume remains bounded if the distribution of utility types is discrete, under the natural assumption that investors who are indifferent do not trade. Therefore, as long as search frictions are sufficiently small, our fully decentralized market can generate arbitrarily large excess volume relative to the frictionless benchmark and relative to a model in which heterogeneity is generated by a discrete distribution of utility types.[12]

Second, trading volume is, for the most part, generated by investors near the marginal type when the meeting rate is sufficiently large. To illustrate this phenomenon, Figure 2 plots the contribution of each owner non-owner pair to the equilibrium trading volume, defined as

$$\kappa(\delta_0, \delta_1) = \mathbf{1}_{\{\delta_0 > \delta_1\}} \frac{d\Phi_0}{dF}(\delta_0) \frac{d\Phi_1}{dF}(\delta_1).$$

The figure shows that investors with extreme utility types account for a small fraction of total trades and, therefore, lie at the periphery of the trading network. For example,

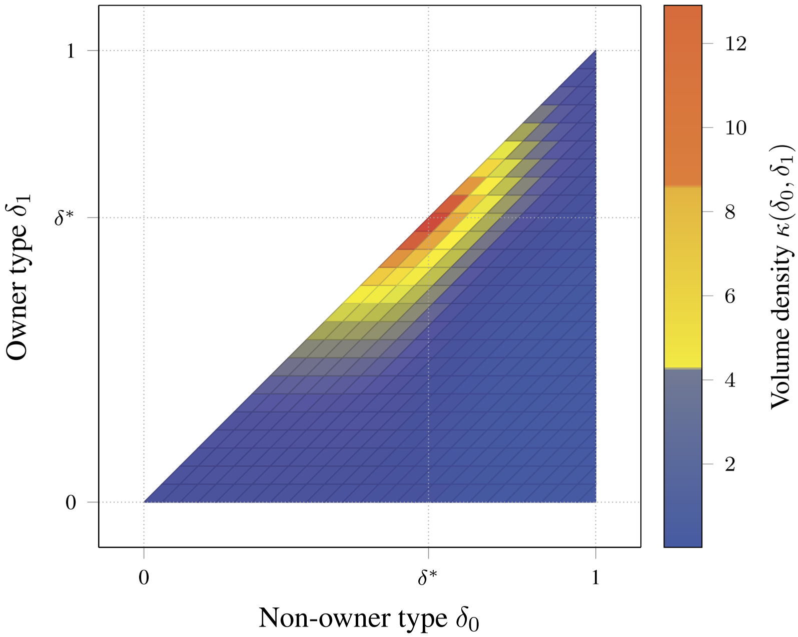

FIGURE 2. Contribution to trading volume. *Notes*: This figure plots the volume density as a function of the owner's and non-owner's type when meetings occur, on average, once a week. The parameters we use in this figure are otherwise the same as in Figure 1.

[11] In a frictionless equilibrium, a measure s of agents holds the asset, each with type $\delta > \delta^\star$. They sell as soon as they switch to a type $\delta < \delta^\star$, which occurs with intensity $\gamma(1 - F(\delta^\star)) = \gamma(1 - s)$ by the market-clearing condition. Hence, the trading volume is equal to $\gamma s(1 - s)$.

[12] Equation (22) also delivers several additional comparative statics. For example, it shows that trading volume peaks when the asset supply equates the number of potential buyers and sellers—which is well known from the monetary search literature (Kiyotaki and Wright (1993))—and that it increases when investors change type more frequently.

owners with low utility types may trade quickly, but there are very few such owners in equilibrium. Hence, these owners account for little trading volume. Likewise, there are many asset owners with high utility types, but these investors trade very slowly, so they do not account for many trades in equilibrium. Only in the cluster of investors with near-marginal utility types do we find a sufficiently large fraction of individuals who are both holding the "wrong" portfolio and able to meet suitable trading partners at a reasonably high rate—these are the investors that make up the core of the trading network.

4.2 *Prices*

We start with the intuitive, but important result that the reservation value of all investors converges to the frictionless equilibrium price.

Proposition 6. *As search frictions vanish,* $\lim_{\lambda\to\infty}\Delta V(\delta)=\delta^\star/r\equiv p^\star$ *for every* $\delta\in[0,1]$.

To understand this result, consider the market-valuation process of Proposition 3. Since the equilibrium asset allocation becomes approximately efficient as $\lambda\to\infty$, it becomes very easy for an investor with utility type $\delta<\delta^\star$ ($\delta>\delta^\star$) to sell (buy) an asset, but a lot more difficult to buy (sell) one. In particular, we show in Appendix A.2 that the trading intensities of non-owners and owners, respectively, satisfy

$$\lim_{\lambda\to\infty}\lambda\Phi_1(\delta)=\frac{\gamma sF(\delta)}{\big(F(\delta^\star)-F(\delta)\big)^+}\begin{cases}<\infty & \text{if } \delta<\delta^\star\\ =\infty & \text{if } \delta\geq\delta^\star\end{cases}\tag{23}$$

and

$$\lim_{\lambda\to\infty}\lambda\big(1-s-\Phi_0(\delta)\big)=\frac{\gamma(1-s)\big(1-F(\delta)\big)}{\big(F(\delta)-F(\delta^\star)\big)^+}\begin{cases}=\infty & \text{if } \delta\leq\delta^\star\\ <\infty & \text{if } \delta>\delta^\star,\end{cases}$$

Thus, it follows from Proposition 3 that, starting from below (above) the marginal type, the market-valuation process moves up (down) very quickly as the meeting frequency increases. Taken together, these observations imply that the market-valuation process $\hat{\delta}_t$ defined in Proposition 3 converges to the marginal type $\delta^\star$ as $\lambda\to\infty$, and it now follows from the sequential representation (15) that all reservation values converge to the frictionless equilibrium price.

Price level near the frictionless limit To analyze the behavior of reservation values and prices near the frictionless limit, we study the behavior of the market-valuation process near the marginal type, which yields the following result.

Proposition 7. *Assume that the distribution of utility types is twice continuously differentiable on supp*(F) *with a derivative that is bounded away from zero. Then*

$$\Delta V(\delta)=p^\star+\frac{\pi/r}{F'(\delta^\star)}\left(\frac{1}{2}-\theta_0\right)\left(\frac{\gamma s(1-s)}{\theta_0\theta_1}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\lambda}}+o\left(\frac{1}{\sqrt{\lambda}}\right)\tag{24}$$

for all utility types $\delta\in[0,1]$.

The first term in the expansion follows directly from Proposition 6, since all reservation values converge to the frictionless price $p^\star = \delta^\star/r$. The main result of the proposition is the second term in the expansion, which determines the deviation of reservation values from the frictionless price. To calculate this term, we center the market-valuation process around its frictionless limit and scale it by its convergence rate, $\sqrt{\lambda}$. This delivers an auxiliary process $\hat{x}_t = \sqrt{\lambda}(\hat{\delta}_t - \delta^\star)$ whose limit distribution can be characterized explicitly, and the second term of the expansion is then obtained by calculating the limit of

$$\sqrt{\lambda}\big(\Delta V(\delta) - p^\star\big) = \mathbb{E}_{\sqrt{\lambda}(\delta-\delta^\star)}\left[\int_0^\infty e^{-rt}\hat{x}_t\,dt\right].$$

We see from the proposition that the deviation from the frictionless price depends on three key features of our decentralized market model. The first key feature is the time it takes near-marginal investors on *both* sides of the market to find trading opportunities.[13] Specifically, since the asset is almost perfectly allocated in the limit, it takes a long search time—of the order $1/\sqrt{\lambda}$, instead of $1/\lambda$—for near-marginal investors to find a counterparty who is willing to trade.[14]

The second key feature is the relative bargaining powers of buyers and sellers, which determine whether the asset is traded at a discount or at a premium: if $\theta_0 > 1/2$, the asset is traded at a discount relative to the frictionless equilibrium price in all bilateral meetings, and vice versa if $\theta_0 < 1/2$. When buyers and sellers have equal bargaining powers, the correction term vanishes and all reservation values are well approximated by the frictionless price, irrespective of the other features of the market. The intuition is that, in this case, the bargaining positions of near-marginal buyers and sellers cancel out since they have equal bargaining power and they find counterparties willing to trade with approximately equal intensity $\sqrt{\lambda} \times \sqrt{\gamma s(1-s)}$.

The third feature of the market that matters for reservation values is the heterogeneity among investors near the marginal type, as measured by $F'(\delta^\star)$. Formally, consider the following ordering of distributions in terms of heterogeneity: we say that a distribution G is *more heterogenous* than F if G is obtained from F by way of a single-crossing spread, i.e., if there is a δ_0 such that $G(\delta) \geq F(\delta)$ for $\delta < \delta_0$ and $G(\delta) \leq F(\delta)$ for $\delta \geq \delta_0$. As Chateauneuf, Cohen, and Meilijson (2004) argue, the distribution G is more heterogenous than F in a very intuitive sense, since it is obtained by shifting probability mass to the left in the interval $[0, \delta_0]$ and to the right in the interval $[\delta_0, 1]$.[15] If, in addition, F

[13]Interestingly, the slow convergence that we characterize in Proposition 7 is not present in the special case of two types studied in DGP. The reason is that, with a continuous distribution of types, near-marginal investors on both sides of the market experience long search times as $\lambda \to \infty$. In contrast, in the two-type case studied in DGP, the search times are asymmetric—bounded away from zero for buyers and of order $1/\lambda$ for sellers. Since the price deviation is determined by the side of the market with shortest search time, convergence occurs in order $1/\lambda$.

[14]We thank a referee for noting that these long search times are also responsible for the concentration of misallocation near the marginal type, $\delta^\star$, and the large trading volume as $\lambda \to \infty$.

[15]If this operation is also mean-preserving, it makes the distribution more heterogenous in the sense of second-order stochastic dominance. Conversely, Chateauneuf, Cohen, and Meilijson (2004) note that any increase in heterogeneity in the second-order stochastic dominance sense can be obtained by successive mean-preserving single crossings.

and G satisfy the conditions of Proposition 7, and G has the same marginal investor as F, so that $F(\delta^\star) = G(\delta^\star)$, then $F'(\delta^\star) \geq G'(\delta^\star)$. Therefore, according to this definition, an increase in heterogeneity that preserves the marginal type reduces the derivative of the distribution at the marginal type, increases the leading term of the expansion (24), and thus induces larger deviations from the frictionless equilibrium price.[16]

To further emphasize the role of heterogeneity, consider what happens when the continuous distribution of utility types approximates a discrete distribution. In such a case, the cumulative distribution function will approach a step function that is vertical at the marginal type, where demand is perfectly elastic. As a result, the derivative $F'(\delta^\star)$ will approach infinity, and it follows from (24) that the corresponding deviation from the frictionless equilibrium price will be very small. This informal argument can be made precise by working out the convergence rate of reservation values with a discrete distribution of utility types (see Appendix A.2 for the explicit expression of the correction term).

Proposition 8. *When the distribution of utility types is discrete, the convergence rate of reservation values to the frictionless equilibrium price is generically equal to* $1/\lambda$.

To understand the different convergence rates in Propositions 7 and 8, consider a sequence of discrete distributions converging weakly to some continuous distribution. A simple argument shows that the corresponding allocations and prices converge to their continuous counterparts, but the asymptotic expansions of reservation values do not. Specifically, the proof of Proposition 8 reveals that, in the expansion with a discrete distribution, the coefficient multiplying $1/\lambda$ diverges as the discrete distribution approaches its continuous limit. This means that convergence is slower and slower. Proposition 7 makes this observation mathematically precise by showing that, in the continuous limit, the convergence rate switches from $1/\lambda$ to $1/\sqrt{\lambda}$.

To see that the difference in convergence rates is economically significant, let us compare the price deviation $p^\star - \Delta V(\delta^\star)$ implied by the continuous distribution of our baseline example with that implied by a two-point distribution, constructed to keep the marginal and average investors the same. The left panel of Figure 3 shows that when investors meet counterparties twice a day on average (i.e., $\lambda = 500$), the deviation is 60 percent for the continuous distribution and only about 2 percent for the corresponding discrete distribution. When meetings occur 20 times per day on average (i.e., $\lambda = 10,000$), the deviation is 15 percent for the continuous distribution, but it is now indistinguishable from 0 for the discrete distribution. Why is there such a quantitatively large difference in price impact? According to our analysis, the difference is driven by a fundamental economic difference between the two classes of distributions: the elasticity of asset demand is infinite with a discrete distribution and finite with a continuous one.

[16]Interestingly, a direct calculation shows that the derivative is proportional to the elasticity of the Walrasian demand at the frictionless price, $\frac{p^\star}{F(rp^\star)-1}\frac{d(1-F(rp))}{dp}|_{p=p^\star} = \frac{\delta^\star}{s}F'(\delta^\star)$, keeping in mind that $1-F(\delta^\star) = s$. Hence, holding the marginal investor and the supply the same, if the Walrasian demand is less elastic, price effects in the decentralized market will be larger. It is intuitive that a less elastic demand magnifies the bilateral monopoly effects at play in our search-and-matching market.

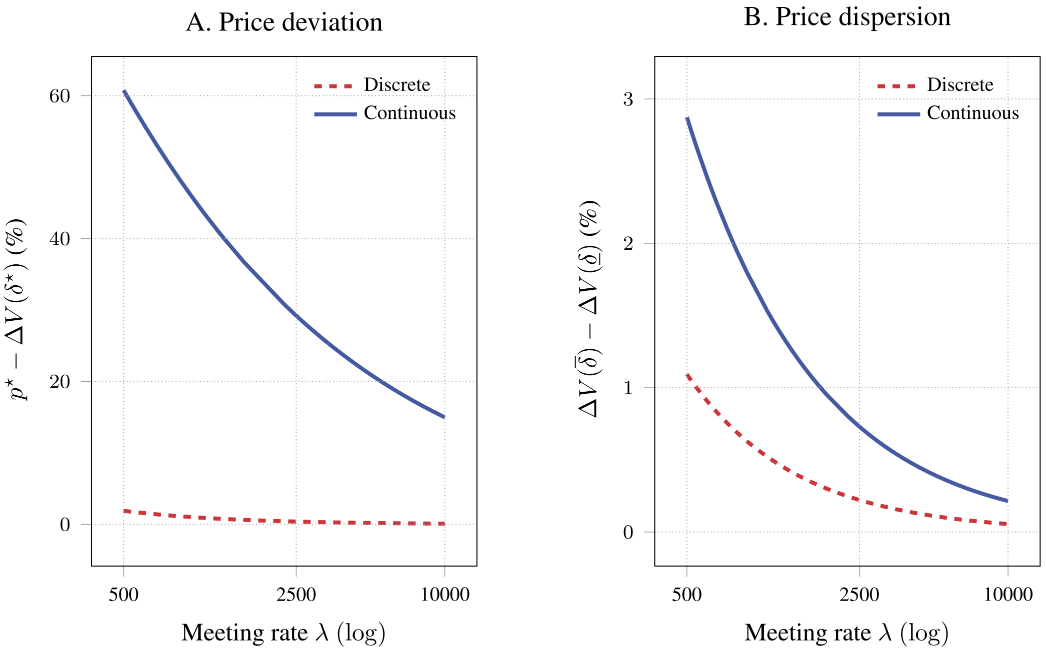

Figure 3. Continuous versus discrete distribution. *Notes*: These figures plots the price deviation relative to the frictionless equilibrium (panel A) and the price dispersion (panel B) as functions of the meeting rate for the base case model of Figure 1 with bargaining power $\theta_0 = 0.75$, and a model with a two-point distribution of types constructed to have the same mean and to induce the same marginal investor as the continuous distribution of the base case model.

Price dispersion near the frictionless limit An important implication of Proposition 7 is that, to a first-order approximation, there is no price dispersion. This can be seen by noting that the correction term in (24) does not depend on the investor's utility type. Hence, to obtain information about the impact of frictions on price dispersion, it is necessary to work out higher order terms. This is the content of our next result.

Proposition 9. *Assume that the distribution of utility types is twice continuously differentiable with a derivative that is bounded away from 0. Then*

$$\Delta V(1) - \Delta V(0) = \frac{1}{2\theta_0\theta_1 F'(\delta^\star)} \frac{\log(\lambda)}{\lambda} + O\left(\frac{1}{\lambda}\right).$$

By contrast, with a discrete distribution of utility types, the convergence rate of the price dispersion is generically equal to $1/\lambda$.

Comparing the results of Propositions 7 and 9 shows that, with a continuous distribution of utility types, the price dispersion induced by search frictions vanishes at rate $\log(\lambda)/\lambda$, which is much faster than the rate $1/\sqrt{\lambda}$ at which reservation values converge to the frictionless equilibrium price. This finding has important consequences for empirical analysis of decentralized markets, as it implies that inferring the impact of search frictions based on the observable level of price dispersion can be misleading. In particular, search frictions can have a very small impact on price dispersion and, yet, have a large impact on the equilibrium price level.

This finding is illustrated in Figure 3. Comparing the left and right panels, one sees that price dispersion induced by search frictions converges to 0 much faster than the price deviation. For instance, when investors meet counterparties twice a day on average, the price discount implied by our baseline model is about 60 percent, but the corresponding price dispersion is about 20 times smaller. One can also see from the figure that, in accordance with the result of Proposition 9, price dispersion is larger with a continuous distribution of utility types than with a discrete distribution.

4.3 *Welfare*

In the analysis above, we established that the asymptotic behavior of two liquidity measures—the deviation of price from its frictionless limit and the dispersion of prices—provides quantitatively different signals about market liquidity. We now ask how these two measures are related to the welfare cost of frictions, defined as

$$w(\lambda) \equiv \int_{\delta^\star}^{1} \delta \, d\Phi_0(\delta) - \int_0^{\delta^\star} \delta \, d\Phi_1(\delta).$$

In words, this cost is the difference between the collective flow utility of investors in the market with and without frictions: the first term accounts for the forgone utility of those investors who do not hold an asset in the frictional market when they should (according to the frictionless benchmark), while the second term accounts for the extra utility attributed to those investors who hold an asset in the frictional market when they should not.

Proposition 10. *Assume that the distribution of utility types is twice continuously differentiable with a derivative that is bounded away from 0. Then*

$$w(\lambda) = \frac{\gamma s(1-s)}{F'(\delta^\star)} \frac{\log(\lambda)}{\lambda} + O\left(\frac{1}{\lambda}\right).$$

By contrast, with a discrete distribution of utility types, the convergence rate of the welfare cost to 0 is generically equal to $1/\lambda$.

Proposition 10 establishes that search frictions have a larger welfare impact when the distribution of utility types is continuous than they do when the distribution is discrete—as was the case for price levels and price dispersion. The proposition also reveals that as trading gets faster, the welfare cost of frictions is accurately measured by the observed amount of price dispersion, since the two quantities converge to their frictionless counterparts at the same speed. At an intuitive level, both price dispersion and welfare depend on the allocation of the asset among investors with valuations *away* from the marginal type, which approaches the frictionless limit relatively quickly. Trading volume and price levels, however, depend on the allocation of the asset among *inframarginal* investors, which approaches the frictionless limit more slowly.

5. Conclusion

We analyze a search and bargaining model of an asset market in which investors' valuations are periodically drawn from an arbitrary distribution. The main contribution is methodological: we develop a solution technique that allows for a full characterization of the equilibrium, in closed form, both in and out of steady state. The result is a framework that is much richer than the popular workhorse model with only two valuations, yet equally tractable. As such, the model offers a variety of novel implications and can be used to confront newer, transaction-level data emerging from a variety of OTC markets.

Appendix

A.1 *Proofs omitted in Section 3*

We start by showing that (8) is equivalent to the seemingly stronger requirement of boundedness and that any such solution to (7) must be strictly increasing in utility types.

Lemma A.1. *Any solution to (7) that satisfies (8) is uniformly bounded on* $\mathcal{S} \equiv \mathbb{R}_+ \times [0, 1]$ *and strictly increasing in* $\delta \in [0, 1]$ *for any fixed* $t \geq 0$.

Proof. Integrating with respect to the conditional distribution of the stopping time τ shows that the set of solutions to (7) is the set of fixed points of the operator defined by

$$T_t[f](\delta) = \int_t^\infty e^{-\rho(u-t)}\big(\delta + (\gamma+\lambda) f_u(\delta) + \mathcal{O}_u[f](\delta)\big)\,du \tag{25}$$

with

$$\mathcal{O}_t[f](\delta) = \int_0^1 \big(f_t(\delta') - f_t(\delta)\big)\left(\gamma\, dF(\delta') + \sum_{q=0}^{1} \mathbf{1}_{\{(2q-1)(f_t(\delta')-f_t(\delta))\geq 0\}} \lambda\theta_q\, d\Phi_{1-q,t}(\delta')\right).$$

Assume that $\Delta V_t(\delta) = T_t[\Delta V](\delta)$ is a fixed point that satisfies (8). Since the right-hand side of (25) is absolutely continuous in time, we have that the solution inherits this property, and it thus follows from Lebesgue's differentiation theorem that we have $\Delta\dot{V}_t(\delta) = r\Delta V_t(\delta) - \delta - \mathcal{O}_t[\Delta V](\delta)$ for every $\delta \in [0, 1]$ and almost every $t \geq 0$. Integrating by parts then shows that

$$\Delta V_t(\delta) = e^{-r(H-t)}\Delta V_H(\delta) + \int_t^H e^{-r(u-t)}\big(\delta + \mathcal{O}_u[\Delta V](\delta)\big)\,du \tag{26}$$

$$= \lim_{H\to\infty} \int_t^H e^{-r(u-t)}\big(\delta + \mathcal{O}_u[\Delta V](\delta)\big)\,du \tag{27}$$

for all $(\delta, t) \in \mathcal{S}$ and any constant $t \leq H < \infty$, where the second equality follows from (8). Now assume toward a contradiction that the given solution fails to be nondecreasing in space so that $\Delta V_t(\delta) > \Delta V_t(\delta')$ for some $(t, \delta) \in \mathcal{S}$ and $1 \geq \delta' > \delta$. Because the right-hand side of (25) is absolutely continuous in time, this assumption implies that

$$H^\star \equiv \inf\big\{u \geq t : \Delta V_u(\delta) \leq \Delta V_u(\delta')\big\} > t.$$

By definition we have that

$$\Delta V_u(\delta) \geq \Delta V_u(\delta'), \quad t \leq u \leq H^\star, \tag{28}$$

and because the continuous functions $x \mapsto (y-x)^+$ and $x \mapsto -(x-y)^+$ are both nonincreasing for every fixed $y \in \mathbb{R}$, it follows that

$$\mathcal{O}_u[\Delta V](\delta) \leq \mathcal{O}_u[\Delta V](\delta'), \quad t \leq u \leq H^\star. \tag{29}$$

To proceed further, we distinguish two cases depending on whether the constant $H^\star$ is finite or not. Assume first that it is finite. In this case, it follows from (26) that we have

$$\Delta V_t(\delta) = \int_t^{H^\star} e^{-r(u-t)}\big(\delta + \mathcal{O}_u[\Delta V](\delta)\big)\,du + e^{-r(H^\star - t)}\Delta V_{H^\star}(\delta),$$

and combining this identity with (29) then gives

$$\Delta V_t(\delta) = \int_t^{H^\star} e^{-r(u-t)}\big(\delta + \mathcal{O}_u[\Delta V](\delta')\big)\,du + e^{-r(H^\star - t)}\Delta V_{H^\star}(\delta') < \Delta V_t(\delta'), \tag{30}$$

where the equality follows by continuity and the second inequality follows from $\delta < \delta'$. Now assume that $H^\star = \infty$ so that (28) and (29) hold for all $u \geq t$. In this case, (27) implies that

$$\Delta V_t(\delta) \leq \lim_{H\to\infty} \int_t^{H} e^{-r(u-t)}\big(\delta + \mathcal{O}_u[\Delta V](\delta')\big)\,du < \Delta V_t(\delta')$$

and combining this inequality with (30) delivers the required contradiction. To see that the solution is strictly increasing, rewrite (25) as

$$T_t[f](\delta) = \int_t^{\infty} e^{-\rho(u-t)}\big(\delta + \mathcal{M}_u[f](\delta)\big)\,du \tag{31}$$

with the operator

$$\begin{aligned}\mathcal{M}_u[f](\delta) = {} & \lambda\eta f_u(\delta) + \gamma \int_0^1 f_u(\delta')\,dF(\delta') + \lambda\theta_0 \int_0^1 \min\{f_u(\delta'), f_u(\delta)\}\,d\Phi_{1,u}(\delta') \\ & + \lambda\theta_1 \int_0^1 \max\{f_u(\delta'), f_u(\delta)\}\,d\Phi_{0,u}(\delta'),\end{aligned}$$

and the constants $\rho \equiv r + \gamma + \lambda$ and $\eta \equiv 1 - s\theta_0 - (1-s)\theta_1$. Because $\mathcal{M}_u[f](\delta)$ is increasing in $f_u(\delta)$ and the given solution is nondecreasing in δ, we have that

$$\Delta V_t(\delta') - \Delta V_t(\delta) = \int_t^{\infty} e^{-\rho(u-t)}\big(\delta' - \delta + \mathcal{M}_u[\Delta V](\delta') - \mathcal{M}_u[\Delta V](\delta)\big)\,du \geq \frac{\delta' - \delta}{\rho}$$

for any $0 \leq \delta \leq \delta' \leq 1$ and strict monotonicity follows. To conclude the proof, it remains to establish boundedness. Because the given solution is increasing we have

$$\sup_{t\geq 0} \mathcal{O}_t[\Delta V](1) \leq 0 \leq \inf_{t \geq 0} \mathcal{O}_t[\Delta V](0)$$

and it now follows from (27) that $0 \leq \Delta V_t(0) \leq \Delta V_t(\delta) \leq \Delta V_t(1) \leq 1/r$ on $\mathcal{S}$. □

Proof of Proposition 1. By Lemma A.1, we have that the existence, uniqueness, and strict (positive) monotonicity of a solution to (7) such that (8) holds is equivalent to the existence and uniqueness of a fixed point of the operator T in the space $\mathcal{X}$ of uniformly bounded, measurable functions from $\mathcal{S}$ to $\mathbb{R}$ equipped with the sup norm. As is easily seen from (31), we have that T maps $\mathcal{X}$ into itself. Moreover, using the definition of $\mathcal{M}_u[f](\delta)$, one easily sees that T satisfies Blackwell's sufficient conditions for a contraction (see Theorem 3.3 in Stokey and Lucas (1989)) with modulus $\frac{\gamma+\lambda}{r+\gamma+\lambda}$. The existence of a unique fixed point in $\mathcal{X}$ now follows from the contraction mapping theorem because $r>0$ by assumption.

To establish the second part, let $\mathcal{X}_k$ denote the subset of functions $f\in\mathcal{X}$ that are nonnegative and nondecreasing in space with

$$0\le f_t(\delta')-f_t(\delta)\le\frac{\delta'-\delta}{r+\gamma}\equiv k(\delta'-\delta) \tag{32}$$

for all $0\le\delta\le\delta'\le 1$ and $t\ge 0$. Further, let $\mathcal{X}_k^\star$ denote the set of functions $f\in\mathcal{X}_k$ that are strictly increasing in space and absolutely continuous with respect to time and space, and observe that because $\mathcal{X}_k$ is closed in $\mathcal{X}$, it suffices to prove that T maps $\mathcal{X}_k$ into $\mathcal{X}_k^\star$.

Fix an arbitrary $f\in\mathcal{X}_k$. Since $f\ge 0$, it follows from (31) that $T_t[f](\delta)\ge 0$. On the other hand, using (32), the definition of η, the increase of $f_t(\delta)$, and the fact that the nondecreasing functions $x\mapsto\min\{a;x\}$ and $x\mapsto\max\{a;x\}$ are 1-Lipschitz continuous, we deduce that $0\le\mathcal{M}_t[f](\delta'')-\mathcal{M}_t[f](\delta)\le\lambda k(\delta''-\delta)$ for all $0\le\delta\le\delta''\le 1$ and $t\ge 0$. Combining this with (31) and the definition of k then shows that

$$\frac{\delta''-\delta}{\rho}\le T_t[f](\delta'')-T_t[f](\delta)\le\frac{(1+\lambda k)(\delta''-\delta)}{\rho}=k(\delta''-\delta)$$

for all $0\le\delta\le\delta''\le 1$ and $t\ge 0$. These bounds imply that $T_t[f](\delta)$ is strictly increasing in space and lies in $\mathcal{X}_k$, so it now only remains to establish absolute continuity. By definition of $\mathcal{X}_k$, we have that $f_t(\delta)=f_t(\delta')+\int_{\delta'}^{\delta}\phi_t(x)\,dx$ for all $t\ge 0$, almost every $\delta,\delta'\in[0,1]^2$, and some $0\le\phi_t(x)\le k$. Substituting this identity into (25) and changing the order of integration shows that

$$\begin{aligned}T_t[f](\delta)=\int_t^\infty e^{-\rho(u-t)}\bigg(&\delta+(\lambda+\gamma)f_u(\delta)-\int_0^\delta\phi_u(\delta')\big(\gamma F(\delta')+\lambda\theta_0\Phi_{1,u}(\delta')\big)\,d\delta'\\&+\int_\delta^1\phi_u(\delta')\big(\gamma\big(1-F(\delta')\big)+\lambda\theta_1\big(1-s-\Phi_{0,u}(\delta')\big)\big)\,d\delta'\bigg)\,du,\end{aligned} \tag{33}$$

and absolute continuity now follows from Sremr (2010, Theorem 3.1). □

Lemma A.2. *Given the reservation value function, there exists a unique pair of functions $V_{1,t}(\delta)$ and $V_{0,t}(\delta)$ that satisfy (4) and (6) subject to (8).*

Proof. Assume that $V_{1,t}(\delta)$ and $V_{0,t}(\delta)$ satisfy (4) and (6) subject to (8). Integrating on both sides of (4) and (6) with respect to the conditional distribution of τ shows that

$$V_{q,t}(\delta)=\int_t^\infty e^{-\rho(u-t)}\bigg(\lambda V_{q,u}(\delta)+\mathcal{C}_{q,u}(\delta)+\gamma\int_0^1 V_{q,u}(\delta')\,dF(\delta')\bigg)\,du \tag{34}$$

with the uniformly bounded functions

$$\mathcal{C}_{q,t}(\delta) = q\delta + \int_0^1 \lambda\theta_q\big((2q-1)\big(\Delta V_t(\delta') - \Delta V_t(\delta)\big)\big)^+ \, d\Phi_{1-q,t}(\delta'). \tag{35}$$

Because the right-hand side of (34) is absolutely continuous in time, we have that $V_{q,t}(\delta)$ inherits this property, and it thus follows from Lebesgue's differentiation theorem that

$$\dot{V}_{q,t}(\delta) = rV_{q,t}(\delta) - \mathcal{C}_{q,t}(\delta) - \gamma \int_0^1 \big(V_{q,t}(\delta') - V_{q,t}(\delta)\big)\, dF(\delta') \tag{36}$$

for all $\delta \in [0, 1]$ and almost every $t \geq 0$. Combining this differential equation with the assumed transversality condition then implies that

$$\begin{aligned} V_{q,t}(\delta) &= e^{-r(H-t)} V_{q,H}(\delta) \\ &\quad + \int_t^H e^{-r(u-t)} \left(\mathcal{C}_{q,u}(\delta) + \gamma \int_0^1 \big(V_{q,u}(\delta') - V_{q,u}(\delta)\big)\, dF(\delta')\right) du \\ &= \lim_{H\to\infty} \int_t^H e^{-r(u-t)} \left(\mathcal{C}_{q,u}(\delta) + \gamma \int_0^1 \big(V_{q,u}(\delta') - V_{q,u}(\delta)\big)\, dF(\delta')\right) du, \end{aligned} \tag{37}$$

and because $\mathcal{C}_{q,t}(\delta)$ is increasing in space by Lemma A.4 below, the same arguments as in the proof of Lemma A.1 show that $V_{q,t}(\delta)$ is increasing in space and uniformly bounded. Combining these properties with (36) then shows that $e^{-rt}V_{q,t}(\delta_t) + \int_0^t e^{-ru}\mathcal{C}_{q,u}(\delta_u)\, du$ is a bounded martingale in the filtration of the investor's utility type process, and it follows that

$$V_{q,t}(\delta) = \mathbb{E}_{t,\delta}\left[\int_t^\infty e^{-r(u-t)} \mathcal{C}_{q,u}(\delta_u)\, du\right]. \tag{38}$$

This establishes uniqueness of the solutions and it now only remains to show that these solutions are consistent with the reservation value function. Applying the law of iterated expectations in (38) shows that $V_{1,t}(\delta) - V_{0,t}(\delta)$ is a bounded fixed point of

$$U_t[f](\delta) = \int_t^\infty e^{-\rho(u-t)} \left(\lambda f_u(\delta) + \mathcal{C}_{1,u}(\delta) - \mathcal{C}_{0,u}(\delta) + \gamma \int_0^1 f_u(\delta')\, dF(\delta')\right) du.$$

A direct calculation shows that U is a contraction on $\mathcal{X}$ and, therefore, admits a unique fixed point in $\mathcal{X}$. Because the reservation value function is increasing, we have

$$\mathcal{C}_{1,t}(\delta) - \mathcal{C}_{0,t}(\delta) + \gamma \int_0^1 \Delta V_t(\delta')\, dF(\delta') = \delta + \gamma \Delta V_t(\delta) + \mathcal{O}_t[\Delta V](\delta)$$

and it follows that this fixed point coincides with the reservation value function. □

Lemma A.3. *For any fixed $\delta \in [0, 1]$, the unique solution to (11) that is both absolutely continuous in time and uniformly bounded is explicitly given by*

$$\sigma_t(\delta) = \int_t^\infty e^{-\int_t^u R_\xi(\delta)\, d\xi}\, du, \tag{39}$$

with the effective discount rate $R_t(\delta) = r + \gamma + \lambda\theta_1(1 - s - \Phi_{0,t}(\delta)) + \lambda\theta_0\Phi_{1,t}(\delta)$.

Proof. Fix an arbitrary $\delta \in [0, 1]$ and assume that $\sigma_t(\delta)$ is a uniformly bounded solution to (11) that is absolutely continuous in time. Using integration by parts, we easily obtain that

$$\sigma_t(\delta) = e^{-\int_t^T R_\xi(\delta)\,d\xi}\sigma_T(\delta) + \int_t^T e^{-\int_t^u R_\xi(\delta)\,d\xi}\,du, \quad 0 \le t \le T < \infty.$$

Since $\sigma \in \mathcal{X}$ and $R_t(\delta) > 0$, we have that $\lim_{T\to\infty} e^{-\int_t^T R_\xi(\delta)\,d\xi}\sigma_T(\delta) = 0$ and, therefore,

$$\sigma_t(\delta) = \lim_{T\to\infty}\left(e^{-\int_t^T R_\xi(\delta)\,d\xi}\sigma_T(\delta) + \int_t^T e^{-\int_t^u R_\xi(\delta)\,d\xi}\,du\right) = \int_t^\infty e^{-\int_t^s R_u(\delta)\,du}\,ds$$

by monotone convergence. $\square$

Lemma A.4. *The functions $\mathcal{C}_{q,t}(\delta)$ are increasing in $\delta \in [0, 1]$.*

Proof. For $q = 0$, the result follows from (35) and the fact that the reservation value function is increasing in δ. Assume now that $q = 1$. Using the monotonicity of the reservation value function is increasing and integrating by parts on the right of (35) gives

$$\mathcal{C}_{1,t}(\delta) = \delta + \int_\delta^1 \lambda\theta_1\sigma_t(\delta')\big(1 - s - \Phi_{1,t}(\delta')\big)\,d\delta',$$

and differentiating this expression shows that

$$\mathcal{C}'_{1,t}(\delta) = 1 - \lambda\sigma_t(\delta)\theta_1\big(1 - s - \Phi_{1,t}(\delta)\big) \ge 1 - \frac{\lambda\theta_1(1-s)}{r + \gamma + \lambda\big(\theta_0 s + \theta_1(1-s)\big)} > 0,$$

where the inequalities follow from (39), the definition of $R_t(\delta)$, and the fact that $r > 0$. $\square$

Proof of Proposition 2. Let $\sigma_t(\delta)$ be as above and consider the absolutely continuous function

$$f_t(\delta) = \int_t^\infty e^{-r(u-t)}\left(\delta - \int_0^\delta \sigma_u(\delta')\big(\gamma F(\delta') + \lambda\theta_0\Phi_{1,u}(\delta')\big)\,d\delta' + \int_\delta^1 \sigma_u(\delta')\big(\gamma\big(1 - F(\delta')\big) + \lambda\theta_1\big(1 - s - \Phi_{0,u}(\delta')\big)\big)\,d\delta'\right)du.$$

Using the boundedness of $\sigma_t(\delta)$, $F(\delta)$, and $\Phi_{q,t}(\delta)$, we deduce that $f \in \mathcal{X}$. On the other hand, Lebesgue's differentiation theorem implies that $f_t(\theta)$ is almost everywhere differentiable in both time and space with

$$\dot{f}_t(\delta) = rf_t(\delta) - \delta + \int_0^\delta \sigma_t(\delta')\big(\gamma F(\delta') + \lambda\theta_0\Phi_{1,t}(\delta')\big)\,d\delta' - \int_\delta^1 \sigma_t(\delta')\big(\gamma\big(1 - F(\delta')\big) + \lambda\theta_1\big(1 - s - \Phi_{0,t}(\delta')\big)\big)\,d\delta' \tag{40}$$

for all $\delta \in [0, 1]$ and almost every $t \geq 0$, and

$$f_t'(\delta) = \int_t^\infty e^{-r(u-t)}\big(1 - \sigma_u(\delta)\big(\gamma + \lambda\theta_1\big(1 - s - \Phi_{0,u}(\delta)\big) + \lambda\theta_0\Phi_{1,u}(\delta)\big)\big)\,du$$
$$= \int_t^\infty e^{-r(u-t)}\big(r\sigma_u(\delta) - \dot{\sigma}_u(\delta)\big)\,du = \sigma_t(\delta)$$

for all $t \geq 0$ and almost every $\delta \in [0, 1]$, where the second equality follows from (11) and the third follows from integration by parts and the boundedness of $\sigma_t(\delta)$. Therefore,

$$f_t(\delta') - f_t(\delta) = \int_\delta^{\delta'} \sigma_t(\delta'')\,d\delta'', \quad (\delta, \delta') \in [0, 1]^2, \tag{41}$$

and it follows that $f_t(\delta)$ is strictly increasing in space. Using this monotonicity in conjunction with (41) and integrating by parts on the right-hand side of (40) shows that $\dot{f}_t(\delta) = rf_t(\delta) - \delta - \mathcal{O}_t[f](\delta)$ for all $\delta \in [0, 1]$ and almost every $t \geq 0$. Solving that equation shows

$$f_t(\delta) = e^{-\rho(H-t)}f_H(\delta) + \int_t^H e^{-\rho(u-t)}\big(\delta + (\gamma + \lambda)f_u(\delta) + \mathcal{O}_u[f](\delta)\big)\,du$$

for any $t \leq H < \infty$, and it now follows from the dominated convergence theorem and the uniform boundedness of the function $f_t(\delta)$ that

$$f_t(\delta) = \int_t^\infty e^{-\rho(u-t)}\big(\delta + (\gamma + \lambda)f_u(\delta) + \mathcal{O}_u[f](\delta)\big)\,du.$$

Comparing this expression with (25), we conclude that $f_t(\delta) = T_t[f](\delta) \in \mathcal{X}$, and the result now follows from the uniqueness established in the proof of Proposition 1. □

Proof of Corollary 1. As shown in the proof of Proposition 1, we have that $\Delta V_t(\delta)$ is the unique fixed point of $T : \mathcal{X}_k \to \mathcal{X}_k$ defined by (31) and, by inspection, this mapping is increasing in $f_t(\delta)$ and decreasing in r. Furthermore, it follows from (33) that T is increasing in θ_1 and decreasing in θ_0, $F(\delta)$ and $\Phi_{q,t}(\delta)$, and the desired monotonicity now follows from Lemma A.5 below. □

Lemma A.5. *Let $\mathcal{C} \subseteq \mathcal{X}$ be closed and assume that $A[\cdot; \alpha] : \mathcal{C} \to \mathcal{C}$ is a contraction that is increasing in f and increasing in α. Then its fixed point is increasing in α.*

Proof. Denote by $f_t(\delta; \alpha) \in \mathcal{C}$ the fixed point of $A[\cdot; \alpha]$. Combining the assumed monotonicity with the fixed-point property shows that we have $f_t(\delta; \alpha) \leq A_t[f(\cdot; \alpha); \beta](\delta)$ for all $(t, \delta) \in \mathcal{S}$ and $\beta \geq \alpha$. Iterating this relation shows that $f_t(\delta; \alpha) \leq A_t^n[f; \beta](\delta)$ for all $n \geq 1$ and the result now follows by letting $n \to \infty$ and using the fact that the mapping $A[\cdot; \beta]$ is a contraction. □

Proof of Proposition 3. Using (14) together with the notation of the statement shows that the reservation value function is the unique bounded and absolutely continuous solution to

$$r\Delta V_t(\delta) = \Delta\dot{V}_t(\delta) + \delta + \mathcal{A}_t[\Delta V](\delta).$$

Therefore, it follows from Itô's lemma that $e^{-rt}\Delta V_t(\hat{\delta}_t)+\int_0^t e^{-ru}\hat{\delta}_u\,du$ is a local martingale and this implies that we have

$$\Delta V_t(\delta)=\mathbb{E}_{t,\delta}\big[e^{-r(\tau_n-t)}\Delta V_{\tau_n}(\hat{\delta}_{\tau_n})\big]+\mathbb{E}_{t,\delta}\bigg[\int_0^{\tau_n} e^{-r(u-t)}\hat{\delta}_u\,du\bigg]$$

for a nondecreasing sequence of stopping times that converges to infinity. Since the reservation value function is uniformly bounded, the first term on the right-hand side converges to zero as $n\to\infty$, and the desired result now follows by monotone convergence. □

Proof of Proposition 4. Given an initial condition satisfying $\Phi_{0,0}(\delta)+\Phi_{1,0}(\delta)=F_0(\delta)$, it follows from textbook results (see, e.g., Reid (1972)) that the Riccati equation (17) admits a unique solution that can be expressed in terms of the confluent hypergeometric function of the first kind $M_1(a,b;x)$ (see Abramowitz and Stegun (1964)) as

$$\lambda\Phi_{1,t}(\delta)=\lambda\big(F_t(m)-\Phi_{0,t}(\delta)\big)=\frac{\dot{Y}_{+,t}(\delta)-A(\delta)\dot{Y}_{-,t}(\delta)}{Y_{+,t}(\delta)-A(\delta)Y_{-,t}(\delta)} \tag{42}$$

with

$$\begin{aligned}
Y_{\pm,t}(\delta)&=e^{-\lambda Z_\pm(\delta)t}W_{\pm,t}(\delta)\\
Z_\pm(\delta)&=\frac{1}{2}\big(1-s+\gamma/\lambda-F(\delta)\big)\pm\frac{1}{2}\Lambda(\delta)\\
W_{\pm,t}(\delta)&=M_1\bigg(\frac{\lambda}{\gamma}Z_\pm(\delta),1\pm\frac{\lambda}{\gamma}\Lambda(m);e^{-\gamma t}\frac{\lambda}{\gamma}\big(F(\delta)-F_0(\delta)\big)\bigg)
\end{aligned} \tag{43}$$

and

$$A(\delta)=\frac{\dot{Y}_{+,0}(\delta)-\lambda\Phi_{1,0}(\delta)Y_{+,0}(\delta)}{\dot{Y}_{-,0}(\delta)-\lambda\Phi_{1,0}(\delta)Y_{-,0}(\delta)}.$$

Straightforward algebra shows that (42) can be rewritten as

$$\lambda\Phi_{1,t}(\delta)=\frac{\lambda Z_+(\delta)W_{+,t}(\delta)-\dot{W}_{-,t}(\delta)+e^{\lambda\Lambda(\delta)t}A(\delta)\big(\dot{W}_{+,t}(\delta)-\lambda Z_-(\delta)W_{-,t}(\delta)\big)}{e^{\lambda\Lambda(\delta)t}A(\delta)W_{-,t}(\delta)-W_{+,t}(\delta)}.$$

Well known properties of $M_1(a,b;x)$ imply that $\lim_{t\to\infty}\dot{W}_{\pm,t}(\delta)=1-\lim_{t\to\infty}W_{\pm,t}(\delta)=0$ and combining these limits with the above expression of the equilibrium distribution finally shows that we have $\lim_{t\to\infty}\Phi_{1,t}(\delta)=-Z_-(\delta)=\Phi_1(\delta)$, where the last equality follows from (43). □

Lemma A.6. *The steady-state cumulative distribution of types among owners $\Phi_1(\delta)$ is increasing in the asset supply, and increasing and concave in $\phi=\gamma/\lambda$ with*

$$\lim_{\phi\to 0}\Phi_1(\delta)=sF(\delta)\quad\textit{and}\quad\lim_{\phi\to\infty}\Phi_1(\delta)=\big(F(\delta)-1+s\big)^+.$$

In particular, the steady-state cumulative distribution functions converge to their frictionless counterparts as the meeting rate $\lambda\to\infty$.

Proof. Monotonicity in s follows by noting that $\frac{\partial \Phi_1(\delta)}{\partial s} = (\Phi_1(\delta) + \phi F(\delta))/\Lambda(\delta)$. On the other hand, using the definition of the steady-state distribution, it can be shown that

$$\frac{\partial \Phi_1(\delta)}{\partial \phi} = \frac{sF(\delta) - \Phi_1(\delta)}{\Lambda(\delta)} = \frac{s(1-s)F(\delta)\big(1-F(\delta)\big)}{\big(\phi + \Phi_1(\delta) + (1-s)\big(1-F(\delta)\big)\big)\Lambda(\delta)} \tag{44}$$

and the desired monotonicity follows by observing that all the terms on the right-hand side are nonnegative. Knowing that $\Phi_1(\delta)$ is increasing in ϕ, we deduce that $\Lambda(\delta) = 2\Phi_1(\delta) + 1 - s + \phi - F(\delta)$ is also increasing in ϕ and it now follows from the first equality in (44) that

$$\frac{\partial^2 \Phi_1(\delta)}{\partial \phi^2} = -\frac{1}{\Lambda(\delta)} \frac{\partial \Phi_1(\delta)}{\partial \phi}\left(1 + \frac{\partial \Lambda(\delta)}{\partial \phi}\right) \le 0.$$

The limiting values follow by sending ϕ to zero and ∞ in the definition of $\Phi_1(\delta)$. □

The proof of Corollary 2 follows directly from Lemma A.6.

The proof of Theorem 1 follows directly from the definition of an equilibrium, Proposition 1, and Proposition 4. We omit the details.

A.2 *Proofs omitted in Section 4*

Proof of Lemma 1. The proof is contained in the proof of Lemma A.6. □

To simplify the notation, let $\phi \equiv \gamma/\lambda$. The following lemma follows immediately from the equation defining the steady-state distribution of utility types among asset owners.

Lemma A.7. *The steady-state distributions of types satisfy* $\Phi_1(\delta) = F(\delta) - \Phi_0(\delta) = \ell(F(\delta))$, *where the bounded function*

$$\ell(x) \equiv -\frac{1}{2}(1 - s + \phi - x) + \frac{1}{2}\sqrt{(1 - s + \phi - x)^2 + 4s\phi x}$$

is the unique positive solution to $\ell^2 + (1 - s + \phi - x)\ell - s\phi x = 0$. *Moreover, the function* $\ell(x)$ *is strictly increasing and convex, and strictly so if* $s \in (0, 1)$.

Proof. It is obvious that $\ell(x)$ is the unique positive solution of the second-order polynomial shown above, and the implicit function theorem implies that $\ell(x)$ is strictly increasing. In particular we have that $\ell(x) > 0$ for $x > 0$ so that the second-order polynomial must be strictly increasing in ℓ and strictly decreasing in x. Convexity follows by direct calculation of $\ell''(x)$. □

Convergence rates of the distributions. To derive the rates at which the equilibrium distributions converge to their frictionless counterparts, recall the inflow–outflow equation that characterizes the steady-state equilibrium distributions:

$$\gamma F(\delta)\big(s - \Phi_1(\delta)\big) = \gamma \Phi_1(\delta)\big(1 - F(\delta)\big) + \lambda \Phi_1(\delta)\big(1 - s - \Phi_0(\delta)\big). \tag{45}$$

By Proposition 6, we have that $\Phi_1(\delta) \to 0$ and $\Phi_0(\delta) \to F(\delta) < 1 - s$ for all $\delta < \delta^\star$ as $\lambda \to \infty$, and it thus follows from (45) that for $\delta < \delta^\star$, the distribution of utility types among asset owners admits the approximation

$$\Phi_1(\delta) = \frac{\gamma F(\delta) s}{1 - s - F(\delta)}\left(\frac{1}{\lambda}\right) + o\left(\frac{1}{\lambda}\right). \tag{46}$$

Similarly, by Proposition 6 we have that $\Phi_1(\delta) \to F(\delta) - 1 + s > 0$ and $\Phi_0(\delta) \to 1 - s$ for all utility types $\delta > \delta^\star$ as the meeting frequency becomes infinite, and it thus follows from (45) that for $\delta > \delta^\star$, the distribution of utility types among non-owners admits the approximation

$$1 - s - \Phi_0(\delta) = \frac{\gamma(1-s)\big(1 - F(\delta)\big)}{F(\delta) - (1-s)}\left(\frac{1}{\lambda}\right) + o\left(\frac{1}{\lambda}\right). \tag{47}$$

To derive the convergence rate at the point $\delta = \delta^\star$, assume first that the distribution of utility types crosses the level $1 - s$ continuously and observe that in this case we have

$$1 - s - \Phi_0\big(\delta^\star\big) = 1 - s - F\big(\delta^\star\big) + \Phi_1\big(\delta^\star\big) = \Phi_1\big(\delta^\star\big).$$

Substituting these identities into (45) evaluated at the marginal type and letting $\lambda \to \infty$ on both sides shows that we have

$$\Phi_1\big(\delta^\star\big) = 1 - s - \Phi_0\big(\delta^\star\big) = \sqrt{\gamma s(1-s)}\left(\frac{1}{\sqrt{\lambda}}\right) + o\left(\frac{1}{\sqrt{\lambda}}\right). \tag{48}$$

If the distribution of utility types crosses $1 - s$ by a jump, we have $F(\delta^\star) > 1 - s$, and it follows that the approximation (47) also holds at the marginal type. □

Proof of Lemma 2. The total measure of misallocated assets is

$$M(1) = \Phi_1\big(\delta^\star\big) + \Phi_0(1) - \Phi_0\big(\delta^\star\big) = \Phi_1\big(\delta^\star\big) + (1-s) - F\big(\delta^\star\big) + \Phi_1\big(\delta^\star\big) = 2\Phi_1\big(\delta^\star\big).$$

Therefore, for any $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}\frac{M\big(\delta^\star + \varepsilon\big) - M\big(\delta^\star - \varepsilon\big)}{M(1)} &= \frac{\Phi_1\big(\delta^\star\big) + \Phi_0\big(\delta^\star + \varepsilon\big) - \Phi_0\big(\delta^\star\big) - \Phi\big(\delta^\star - \varepsilon\big)}{2\Phi_1\big(\delta^\star\big)} \\ &= 1 - \frac{1 - s - \Phi_0^\star\big(\delta^\star + \varepsilon\big) + \Phi_1\big(\delta^\star - \varepsilon\big)}{2\Phi_1\big(\delta^\star\big)},\end{aligned}$$

where we used that $\Phi_0(\delta) + \Phi_1(\delta) = F(\delta)$ and $F(\delta^\star) = 1 - s$ for all δ. It then follows from (46), (47), and (48) that the second term goes to zero as $\lambda \to \infty$. □

Trading volume If a meeting between a buyer and a seller with the same type results in trade with some probability $\pi \in [0, 1]$, then we can express the steady-state trading volume as

$$\vartheta(\pi) = \lambda \int_{[0,1]^2} \mathbf{1}_{\{\delta_0 > \delta_1\}}\, d\Phi_0(\delta_0)\, d\Phi_1(\delta_1) + \pi\lambda \sum_{\delta \in [0,1]} \Delta\Phi_0(\delta)\Delta\Phi_1(\delta),$$

where $\Delta\Phi_q(\delta) = \Phi_q(\delta) - \lim_{y\uparrow\delta}\Phi_q(y) \geq 0$ denotes the discrete mass of investors who hold $q \in \{0, 1\}$ units of the asset and have a utility type exactly equal to δ.

LEMMA A.8. *If the distribution of utility types is continuous, then*

$$\vartheta(\pi) = \vartheta_c \equiv \gamma s(1-s)\left[(1+\gamma/\lambda)\log\left(1+\frac{\lambda}{\gamma}\right)-1\right] \tag{49}$$

for all $\pi \in [0, 1]$*, and is strictly increasing in both the meeting rate* λ *and the arrival rate of preference shocks* γ*. Otherwise, if the distribution of utility types has atoms, then the steady-state trading volume is strictly increasing in* $\pi \in [0, 1]$ *with* $\vartheta(0) < \vartheta_c < \vartheta(1)$*.*

PROOF. Consider the continuous functions $G_1(x) = \ell(x)/s$ and $G_0(x) = (x - \ell(x))/(1-s)$ Rearranging the equation for $\ell(x)$ in Lemma A.7 shows that

$$G_1(x) = \frac{\phi G_0(x)}{1+\phi - G_0(x)}. \tag{50}$$

Since the functions $G_q(x)$ are continuous, strictly increasing, and map $[0, 1]$ onto itself, they admit continuous and strictly increasing inverses $G_q^{-1}(y)$, and it follows from (50) that

$$G_1\big(G_0^{-1}(y)\big) = \frac{\phi y}{1+\phi - y}. \tag{51}$$

Consider the class of tie-breaking rules whereby a fraction $\pi \in [0, 1]$ of the meetings between an owner and a non-owner of the same type lead to a trade. By definition, the trading volume associated with such a tie-breaking rule can be computed as

$$\vartheta(\pi) = \lambda s(1-s)\big(\mathbb{P}[\delta_0 > \delta_1] + \pi\mathbb{P}[\delta_0 = \delta_1]\big),$$

where the random variables $(\delta_0, \delta_1) \in [0, 1]^2$ are distributed according to $\Phi_0(\delta)/(1-s) = G_0(F(\delta))$ and $\Phi_1(\delta)/s = G_1(F(\delta))$ independently of each other. A direct calculation shows that the quantile functions of these random variables are given by

$$\inf\big\{x \in [0, 1] : G_q\big(F(x)\big) \geq u\big\} = \inf\big\{x \in [0, 1] : F(x) \geq G_q^{-1}(u)\big\} = \Delta\big(G_q^{-1}(u)\big),$$

where $\Delta(y)$ denotes the quantile function of the underlying distribution of types, and it thus follows from Embrechts and Hofert (2013, Proposition 2) below that

$$\frac{\vartheta(\pi)}{\lambda s(1-s)} = \mathbb{P}\big[\Delta\big(G_0^{-1}(u_0)\big) > \Delta\big(G_1^{-1}(u_1)\big)\big] + \pi\mathbb{P}\big[\Delta\big(G_0^{-1}(u_0)\big) = \Delta\big(G_1^{-1}(u_1)\big)\big],$$

where u_0 and u_1 denote a pair of i.i.d. uniform random variables. If the distribution is continuous, then its quantile function is strictly increasing, and the above identity simplifies to

$$\begin{aligned}\frac{\vartheta(\pi)}{\lambda s(1-s)} &= \mathbb{P}\big[G_0^{-1}(u_0) > G_1^{-1}(u_1)\big] = \mathbb{P}\big[u_1 < G_1\big(G_0^{-1}(u_0)\big)\big] \\ &= \mathbb{E}\big[G_1\big(G_0^{-1}(u_0)\big)\big] = \int_0^1 G_1\big(G_0^{-1}(x)\big)\,dx = \int_0^1 \frac{\phi x}{1+\phi - x}\,dx = \frac{\vartheta^\star}{\lambda s(1-s)},\end{aligned}$$

where we used formula (51) for $G_1(G_0^{-1}(y))$. If the distribution fails to be continuous, then its quantile function will have flat spots that correspond to the levels across which the distribution jumps, but it will remain weakly increasing. Therefore,

$$\{\Delta(G_0^{-1}(u_0)) > \Delta(G_1^{-1}(u_1))\} \subset \{G_0^{-1}(u_0) > G_1^{-1}(u_1)\} \subset \{\Delta(G_0^{-1}(u_0)) \geq \Delta(G_1^{-1}(u_1))\}$$

and it follows that

$$\begin{aligned}\frac{\vartheta(0)}{\lambda s(1-s)} &= \mathbb{P}[\Delta(G_0^{-1}(u_0)) > \Delta(G_1^{-1}(u_1))] \leq \frac{\vartheta^\star}{\lambda s(1-s)} \\ &= \mathbb{P}[G_0^{-1}(u_0) \geq G_1^{-1}(u_1)] < \mathbb{P}[\Delta(G_0^{-1}(u_0)) \geq \Delta(G_1^{-1}(u_1))] = \frac{\vartheta(1)}{\lambda s(1-s)}.\end{aligned}$$

Since the function $\vartheta(\pi)$ is continuous and strictly increasing in π, this further implies that there exists a unique tie-breaking probability $\pi^\star$ such that $\vartheta^\star = \vartheta(\pi^\star)$. □

Proof of Proposition 5. The first part follows directly from Lemma A.8. To establish the second part, let ε be as in the statement and assume that the distribution of utility types is continuous. In this case the equilibrium trading volume can be decomposed as

$$\begin{aligned}\vartheta_c = {}& \lambda\Phi_1(\delta^\star)(1-s-\Phi_0(\delta^\star)) + \lambda\int_0^{\delta^\star-\varepsilon}\Phi_1(\delta)\,d\Phi_0(\delta) + \lambda\int_{\delta^\star+\varepsilon}^1 (1-s-\Phi_0(\delta))\,d\Phi_1(\delta) \\ & + \lambda\int_{\delta^\star-\varepsilon}^{\delta^\star}\Phi_1(\delta)\,d\Phi_0(\delta) + \lambda\int_{\delta^\star}^{\delta^\star+\varepsilon}(1-s-\Phi_0(\delta))\,d\Phi_1(\delta).\end{aligned} \tag{52}$$

We show that all the terms on the first line remain bounded as $\lambda \to \infty$. Since $F(\delta^\star) = 1-s$ when the distribution of type is continuous, we have that the first term is equal to

$$\lambda\Phi_1(\delta^\star)(1-s-F(\delta^\star)+\Phi_1(\delta^\star)) = \lambda\Phi_1(\delta^\star)^2$$

and we know from Lemma A.7 that the measure $\Phi_1(\delta^\star)$ of owners below the marginal type solves $\lambda\Phi_1(\delta^\star)^2 + \gamma\Phi_1(\delta^\star) - \gamma s(1-s) = 0$. This immediately implies that $\lambda\Phi_1(\delta^\star)^2 \leq \gamma s(1-s)$ and it follows that the first term on the first line of (52) remains bounded as $\lambda \to \infty$. Turning to the second term, we note that

$$\lambda\int_0^{\delta^\star-\varepsilon}\Phi_1(\delta)\,d\Phi_0(\delta) \leq \lambda\Phi_1(\delta^\star-\varepsilon)F(\delta^\star-\varepsilon), \tag{53}$$

where the inequality follows (1) and the increases of $\Phi_1(\delta)$. From Lemma A.7, we have that the steady-state measure of owners with valuations below $\delta^\star-\varepsilon$ solves

$$\lambda\Phi_1(\delta^\star-\varepsilon)^2 + \left(1-s-F(\delta^\star-\varepsilon)+\frac{\gamma}{\lambda}\right)\lambda\Phi_1(\delta^\star-\varepsilon) - \gamma sF(\delta^\star-\varepsilon) = 0.$$

This immediately implies that

$$\lambda\Phi_1(\delta^\star-\varepsilon) \leq \frac{\gamma sF(\delta^\star-\varepsilon)}{1-s-F(\delta^\star-\varepsilon)}$$

and combining this inequality with (53) shows that the second term on the first line of (52) remains bounded as $\lambda \to \infty$. Proceeding similarly, one can show that the third term also remains bounded as frictions vanish, and the result now follows by observing that $\lim_{\lambda\to\infty} \vartheta_c = \infty$. □

Proof of Proposition 6. By Theorem 1, we have that

$$r\Delta V(\delta) = \delta - \int_0^{\delta} k_0(\delta')\, d\delta' + \int_{\delta}^{1} k_1(\delta')\, d\delta'$$

with the uniformly bounded functions defined by

$$k_0(\delta') = \frac{\gamma F(\delta') + \lambda\theta_0\Phi_1(\delta')}{r + \gamma + \lambda\theta_1(1 - s - \Phi_0(\delta')) + \lambda\theta_0\Phi_1(\delta')}$$

$$k_1(\delta') = \frac{\gamma(1 - F(\delta')) + \lambda\theta_1(1 - s - \Phi_0(\delta'))}{r + \gamma + \lambda\theta_1(1 - s - \Phi_0(\delta')) + \lambda\theta_0\Phi_1(\delta')}.$$

Using Lemma 1 and the assumption that $\theta_q > 0$, we obtain

$$\lim_{\lambda\to\infty} k_q(\delta') = \frac{\theta_q\Phi^\star_{1-q}(\delta')}{\theta_0\Phi^\star_1(\delta') + \theta_1\Phi^\star_0(\delta')} = \mathbf{1}_{\{q=0\}}\mathbf{1}_{\{\delta\geq\delta^\star\}} + \mathbf{1}_{\{q=1\}}\mathbf{1}_{\{\delta<\delta^\star\}},$$

and the required result now follows from an application of the dominated convergence theorem because the functions $k_q(\delta')$ take values in $[0, 1]$. □

Proof of Proposition 7. Assume without loss of generality that $\operatorname{supp}(F) = [0, 1]$. Evaluating (19) at $\delta^\star$ and making the change of variable $x = \sqrt{\lambda}(\delta' - \delta^\star)$ in the two integrals shows that

$$r\sqrt{\lambda}(\Delta V(\delta^\star) - p^\star) = P(\lambda) - D(\lambda), \tag{54}$$

where the functions on the right-hand side are defined by

$$D(\lambda) \equiv \int_{-\infty}^{0} \mathbf{1}_{\{x\geq-\delta^\star\sqrt{\lambda}\}} \frac{\gamma F(\delta^\star + x/\sqrt{\lambda}) + \theta_0\sqrt{\lambda}g_1(x)}{r + \gamma + \theta_0\sqrt{\lambda}g_1(x) + \theta_1\sqrt{\lambda}g_0(x)}\, dx$$

$$P(\lambda) \equiv \int_{0}^{\infty} \mathbf{1}_{\{x\leq(1-\delta^\star)\sqrt{\lambda}\}} \frac{\gamma(1 - F(\delta^\star + x/\sqrt{\lambda})) + \theta_1\sqrt{\lambda}g_0(x)}{r + \gamma + \theta_0\sqrt{\lambda}g_1(x) + \theta_1\sqrt{\lambda}g_0(x)}\, dx$$

with the functions

$$g_q(x) \equiv \sqrt{\lambda}(1 - q)(1 - s - F(\delta^\star + x/\sqrt{\lambda})) + \sqrt{\lambda}\Phi_1(\delta^\star + x/\sqrt{\lambda}).$$

Letting the meeting rate $\lambda \to \infty$ on both sides of (54) and using the convergence result established by Lemma A.11 below we obtain that

$$\lim_{\lambda\to\infty} r\sqrt{\lambda}(\Delta V(\delta^\star) - p^\star)$$
$$= \int_0^{\infty} \frac{\theta_1 g(-x)\, dx}{\theta_0 g(x) + \theta_1 g(-x)} - \int_{-\infty}^{0} \frac{\theta_0 g(z)\, dz}{\theta_0 g(z) + \theta_1 g(-z)}$$

$$= \int_0^\infty \frac{(1-2\theta_0)g(x)g(-x)\,dx}{\big(\theta_0 g(x)+\theta_1 g(-x)\big)\big(\theta_0 g(-x)+\theta_1 g(x)\big)}\,dx$$

$$= \int_0^\infty \frac{\gamma s(1-s)(1-2\theta_0)\,dx}{\gamma s(1-s)+\theta_0\theta_1\big(xF'(\delta^\star)\big)^2} = \frac{\pi}{F'(\delta^\star)}\left(\frac{1}{2}-\theta_0\right)\left(\frac{\gamma s(1-s)}{\theta_0\theta_1}\right)^{\frac{1}{2}},$$

where the function

$$g(x) = \frac{1}{2}xF'(\delta^\star) + \frac{1}{2}\sqrt{\big(xF'(\delta^\star)\big)^2 + 4\gamma s(1-s)}$$

is the unique positive solution to (55), the second equality follows by making the change of variable $-z = x$ in the second integral, the third equality follows from the definition $g(x)$, and the last equality follows from the fact that

$$\int_0^\infty \frac{dx}{a+x^2} = \frac{\arctan(x/\sqrt{a})}{\sqrt{a}}\bigg|_0^\infty = \frac{\pi}{2\sqrt{a}}, \quad a>0.$$

This shows that the asymptotic expansion holds at the marginal type and the desired result now follows from the fact that $\Delta V(\delta) = \Delta V(\delta^\star) + o(1/\sqrt{\lambda})$ by Proposition 9. □

Lemma A.9. *Assume that the conditions of Proposition 7 hold and denote by $g(x)$ the positive solution to the quadratic equation*

$$g^2 - gF'(\delta^\star)x - \gamma s(1-s) = 0. \tag{55}$$

Then we have that $g_1(x) \to g(x)$ and $g_0(x) \to g(-x)$ for all $x \in \mathbb{R}$ as $\lambda \to \infty$.

Proof. Evaluating (17) at the steady state shows that the function $g_1(x)$ is the unique positive solution to the quadratic equation given by

$$g^2 + \left[\frac{\gamma}{\sqrt{\lambda}} + \sqrt{\lambda}\left(F(\delta^\star) - F\left(\delta^\star + \frac{x}{\sqrt{\lambda}}\right)\right)\right]g - \gamma s F\left(\delta^\star + \frac{x}{\sqrt{\lambda}}\right) = 0. \tag{56}$$

Because the left-hand side of this quadratic equation is negative at the origin and positive at $g=1$, we have that $g_1(x)$ is the unique positive root of the polynomial. Notice that the coefficients have well defined limits as $\lambda \to \infty$:

$$\lim_{\lambda\to\infty} \sqrt{\lambda}\left(F(\delta^\star) - F\left(\delta^\star + \frac{x}{\sqrt{\lambda}}\right)\right) = -F'(\delta^\star)x$$

$$\lim_{\lambda\to\infty} -\gamma s F\left(\delta + \frac{x}{\sqrt{\lambda}}\right) = -\gamma s(1-s).$$

Since the positive root of the quadratic equation (56) can be written as a continuous function of the coefficient, it follows that $g_1(x)$ has a well defined limit as $\lambda \to \infty$ and that the limit is the positive root of (55). Next, we note that

$$g_0(x) = g_1(x) + \sqrt{\lambda}\big(F(\delta^\star) - F(\delta^\star + x/\sqrt{\lambda})\big).$$

Substituting this expression into (56) then shows that the function $g_0(x)$ is the unique positive solution to the quadratic equation given by

$$g^2+\left[\frac{\gamma}{\sqrt{\lambda}}-\sqrt{\lambda}\big(F(\delta^\star)-F\big(\delta^\star+x/\sqrt{\lambda}\big)\big)\right]g-\gamma(1-s)\big(1-F\big(\delta^\star+x/\sqrt{\lambda}\big)\big),$$

and the desired result follows from the same arguments as above. □

LEMMA A.10. *Assume that the conditions of Proposition 7 hold.*

(a) There exists a finite $K\geq 0$ such that

$$\begin{aligned} g_1(x)&\leq K/|x|, \quad x\in I_-^\lambda\equiv\left[-\delta^\star\sqrt{\lambda},0\right]\\ g_0(x)&\leq K/|x|, \quad x\in I_+^\lambda\equiv\left[0,\left(1-\delta^\star\right)\sqrt{\lambda}\right]. \end{aligned}\tag{57}$$

(b) For any given $\bar{x}\in I_+^\lambda\cap(-I_-^\lambda)$, there exists a strictly positive k such that

$$\begin{aligned} g_1(x)&\geq k|x|, \quad x\in I_+^\lambda\cap\left[\bar{x},\infty\right)\\ g_0(x)&\geq k|x|, \quad x\in I_-^\lambda\cap\left(-\infty,-\bar{x}\right] \end{aligned}\tag{58}$$

for all sufficiently large λ.

PROOF. (a) Because $g_1(x)$ is the positive root of (56), we have that (57) holds if and only if

$$\min_{x\in I_-^\lambda}\left\{\frac{K^2}{x^2}+\frac{K}{|x|}\left(\frac{\gamma}{\sqrt{\lambda}}+\sqrt{\lambda}\left[F(\delta^\star)-F\left(\delta^\star+\frac{x}{\sqrt{\lambda}}\right)\right]\right)-\gamma sF\left(\delta^\star+\frac{x}{\sqrt{\lambda}}\right)\right\}\geq 0,$$

and a sufficient condition for this to be the case is that

$$\min_{x\in I_-^\lambda}\left\{\frac{K}{|x|}\sqrt{\lambda}\left[F(\delta^\star)-F\left(\delta^\star+\frac{x}{\sqrt{\lambda}}\right)\right]-\gamma s(1-s)\right\}\geq 0.\tag{59}$$

By the mean value theorem, we have that for any $x\in I_-^\lambda\cup I_+^\lambda$, there exists $\hat{\delta}(x)\in[0,1]$ such that

$$F(\delta^\star)-F\left(\delta^\star+\frac{x}{\sqrt{\lambda}}\right)=-\frac{xF'\big(\hat{\delta}(x)\big)}{\sqrt{\lambda}},\tag{60}$$

and substituting this expression into (59) shows that a sufficient condition for the validity of equation (57) is that we have $K\geq K^\star\equiv\max_{\delta\in[0,1]}\frac{\gamma s(1-s)}{F'(\delta)}$. Because the derivative of the distribution of utility types is assumed to be bounded away from zero on the whole interval $[0,1]$, we have that $K^\star$ is finite and (57) follows. One obtains the same constant when applying the same calculations to the function $g_0(x)$ over the interval I_+^λ.

(b) Fix an arbitrary $\bar{x}\in I_+^\lambda\cap(-I_-^\lambda)$. Because $g_1(x)$ is the positive root of (56), we have that (58) holds if and only if

$$\max_{x\in I_+^\lambda\cap[\bar{x},\infty)}\left\{k^2x^2+kx\left(\frac{\gamma}{\sqrt{\lambda}}+\sqrt{\lambda}\left[F(\delta^\star)-F\left(\delta^\star+\frac{x}{\sqrt{\lambda}}\right)\right]\right)-\gamma sF\left(\delta^\star+\frac{x}{\sqrt{\lambda}}\right)\right\}\leq 0.$$

Combining this inequality with (60) then shows that a sufficient condition for (58) is that $k \le k^\star \equiv \inf_{\delta\in[0,1]}(F'(\delta) - \frac{\gamma}{\bar{x}\sqrt{\lambda}})$, and the desired result now follows by noting that because the derivative of the distribution of types is strictly positive on $[0, 1]$, we can pick the meeting rate λ large enough for the constant $k^\star$ to be strictly positive. One obtains the same constant when applying the same calculations to the function $g_0(x)$ over the interval $I_-^\lambda \cap (-\infty, -\bar{x}]$. $\square$

LEMMA A.11. *Assume that the conditions of Proposition 7 hold.*

$$\lim_{\lambda\to\infty} D(\lambda) = \int_{-\infty}^{0} \frac{\theta_0 g(x)\,dx}{\theta_0 g(x) + \theta_1 g(-x)} \quad \textit{and} \quad \lim_{\lambda\to\infty} P(\lambda) = \int_{0}^{\infty} \frac{\theta_1 g(-x)\,dx}{\theta_0 g(x) + \theta_1 g(-x)},$$

where the function $g(x)$ is defined as in Lemma A.9.

PROOF. By Lemma A.9 we have that the integrand $H(x; \lambda)$ in the definition of $D(\lambda)$ satisfies

$$\lim_{\lambda\to\infty} H(x; \lambda) = \frac{\theta_0 g(x)}{\theta_0 g(x) + \theta_1 g(-x)}. \tag{61}$$

Now fix an arbitrary $\bar{x} \in I_+^\lambda \cap (-I_-^\lambda)$ and let the meeting rate λ be large enough. On the interval $[-\bar{x}, 0]$, we can bound the integrand above by 1 and below by 0, while on the interval $I_-^\lambda \backslash [-\bar{x}, 0]$, we can use the bounds provided by Lemma A.10 to show that

$$0 \le H(x; \lambda) \le \frac{\gamma|x| + \theta_0\sqrt{\lambda}K}{\sqrt{\lambda}(\theta_0 K + \theta_1 k|x|^2)} \le \frac{\gamma\delta^\star + \theta_0 K}{\theta_0 K + \theta_1 k|x|^2},$$

where the inequality follows from the definition of I_-^λ. Combining these bounds shows that the integrand is bounded by a function that is integrable on $\mathbb{R}_-$ and does not depend on λ. This allows us to apply the dominated convergence theorem, and the result for $D(\lambda)$ now follows from (61). The result for the other integral follows from identical calculations. We omit the details. $\square$

PROOF OF PROPOSITION 8. Assume that there are $I \ge 2$ utility types $\delta_1 < \delta_2 < \cdots < \delta_I$, identify the marginal type with index $m \in \{1, \ldots, I\}$ so that $1 - F(\delta_m) \le s < 1 - F(\delta_{m-1})$, and set $\delta_0 \equiv 0$ and $\delta_{I+1} \equiv 1$. Assume further that $1 - F(\delta_m) < s$, which occurs generically when the distribution of utility types is restricted to be discrete. Under these assumptions, the same algebraic manipulations that we used to establish (46) and (47) show that we have

$$\Phi_1(\delta) = \Phi_1(\delta_i) = \begin{cases} \dfrac{1}{\lambda}\dfrac{\gamma s F(\delta_i)}{1 - s - F(\delta_i)} + o\left(\dfrac{1}{\lambda}\right) & \text{if } i < m \\ F(\delta_i) - (1-s) + \dfrac{1}{\lambda}\dfrac{\gamma(1 - F(\delta_i))(1-s)}{F(\delta_i) - (1-s)} + o\left(\dfrac{1}{\lambda}\right) & \text{if } i \ge m \end{cases} \tag{62}$$

for all $\delta \in [\delta_i, \delta_{i+1})$ and $i \in \{1, \dots, I\}$. Likewise, we have that the local surplus satisfies

$$\sigma(\delta) = \sigma(\delta_i) = \begin{cases} \dfrac{1}{\lambda\theta_1\big(1-s-F(\delta_i)\big)} + o\left(\dfrac{1}{\lambda}\right) & \text{if } i < m \\ \dfrac{1}{\lambda\theta_0\big(F(\delta_i)-(1-s)\big)} + o\left(\dfrac{1}{\lambda}\right) & \text{if } i \geq m \end{cases}$$

for all $\delta \in [\delta_i, \delta_{i+1})$ and $i \in \{1, \dots, I\}$, and it follows that

$$\Delta V(\delta_m) - \Delta V(\delta_i) = \sum_{j=i}^{m-1} (\delta_{j+1}-\delta_j)\sigma(\delta_j) = \frac{1}{\lambda}\sum_{j=i}^{m-1} \frac{\delta_{j+1}-\delta_j}{\theta_1\big(1-s-F(\delta_j)\big)} + o\left(\frac{1}{\lambda}\right), \quad i > m$$

$$\Delta V(\delta_i) - \Delta V(\delta_m) = \sum_{j=m}^{i-1} (\delta_{j+1}-\delta_j)\sigma(\delta_j) = \frac{1}{\lambda}\sum_{j=m}^{i-1} \frac{\delta_{j+1}-\delta_j}{\theta_0\big(F(\delta_j)-(1-s)\big)} + o\left(\frac{1}{\lambda}\right), \quad i < m.$$

This calculation shows that with a discrete distribution of utility types, price dispersion converges to 0 in order $1/\lambda$. To complete the proof, we calculate the steady-state reservation value $\Delta V(\delta_m)$ of the marginal investor using formula (13). This gives

$$\begin{aligned} r\Delta V(\delta_m) &= \delta_m + \sum_{i=m}^{I} (\delta_{i+1}-\delta_i) \frac{\gamma\big(1-F(\delta_i)\big) + \lambda\theta_1\big(1-s-\Phi_0(\delta_i)\big)}{r+\gamma+\lambda\theta_0\Phi_1(\delta_i)+\lambda\theta_1\big(1-s-\Phi_0(\delta_i)\big)} \\ &\quad - \sum_{i=0}^{m-1} (\delta_{i+1}-\delta_i) \frac{\gamma F(\delta_i) + \lambda\theta_0\Phi_1(\delta_i)}{r+\gamma+\lambda\theta_0\Phi_1(\delta_i)+\lambda\theta_1\big(1-s-\Phi_0(\delta_i)\big)} \\ &= \delta_m + \frac{1}{\lambda}\sum_{i=m}^{I} (\delta_{i+1}-\delta_i) \frac{\gamma\big(1-F(\delta_i)\big)\big(F(\delta_i)-(1-s)(1-\theta_1)\big)}{\big(F(\delta_i)-(1-s)\big)^2} \\ &\quad - \frac{1}{\lambda}\sum_{i=0}^{m-1} (\delta_{i+1}-\delta_i) \frac{\gamma F(\delta_i)\big(1-F(\delta_i)-s(1-\theta_0)\big)}{\big(F(\delta_i)-(1-s)\big)^2} + o\left(\frac{1}{\lambda}\right), \end{aligned}$$

where the second equality follows from (1) and (62). □

Proof of Proposition 9. The result follows from Lemmas A.12 and A.13 below. To simplify the presentation we assume without loss of generality that supp$(F) = [0, 1]$ in the statement and proofs of these lemmas. □

Lemma A.12. *Under the conditions of Proposition 9, we have that*

$$A(\lambda) \equiv \lambda \int_0^{\delta^\star} \sigma(\delta)\, d\delta - \int_0^{\delta^\star} \frac{d\delta}{\dfrac{r+\gamma}{\lambda} + \theta_1 F'(\delta^\star)(\delta^\star - \delta) + \Phi_1(\delta)} = O(1) \tag{63}$$

$$B(\lambda) \equiv \lambda \int_{\delta^\star}^1 \sigma(\delta)\, d\delta - \int_{\delta^\star}^1 \frac{d\delta}{\dfrac{r+\gamma}{\lambda} + \theta_0 F'(\delta^\star)(\delta^\star - \delta) + 1 - s - \Phi_0(\delta)} = O(1) \tag{64}$$

as the meeting rate $\lambda \to \infty$.

Proof. To establish (63) we start by noting that

$$\lambda\sigma(\delta) = \frac{1}{\frac{r+\gamma}{\lambda} + \theta_1\big(F(\delta^\star) - F(\delta)\big) + \Phi_1(\delta)}, \tag{65}$$

where we used the facts that $\Phi_0(\delta) = F(\delta) - \Phi_1(\delta)$ and $F(\delta^\star) = 1 - s$ due to the assumed continuity of the distribution. Substituting this identity into (63), we obtain

$$|A(\lambda)| \leq \int_0^{\delta^\star} \frac{\big|F'(\delta^\star)(\delta^\star - \delta) - \big(F(\delta^\star) - F(\delta)\big)\big|}{\theta_1 F'(\delta^\star)(\delta^\star - \delta)\big(F(\delta^\star) - F(\delta)\big)}\, d\delta.$$

Under our assumption that the distribution of utility types is twice continuously differentiable, we can use Taylor's theorem to extend the integrand by continuity at $\delta^\star$, with value

$$\lim_{\delta\to\delta^\star} \frac{\big|F'(\delta^\star)(\delta^\star - \delta) - \big(F(\delta^\star) - F(\delta)\big)\big|}{\theta_1 F'(\delta^\star)(\delta^\star - \delta)\big(F(\delta^\star) - F(\delta)\big)} = \frac{|F''(\delta^\star)|}{2\theta_1\big(F'(\delta^\star)\big)^2}.$$

Since the derivative is bounded away from 0, this shows that the integrand is bounded and (63) follows. Turning to (64), we start by observing that due to (1) and the continuity of the distribution, we have $\Phi_1(\delta) = F(\delta) - F(\delta^\star) + 1 - s - \Phi_0(\delta)$. Substituting into (65) shows that

$$\lambda\sigma(\delta) = \frac{1}{\frac{r+\gamma}{\lambda} + \theta_0\big(F(\delta) - F(\delta^\star)\big) + 1 - s - \Phi_0(\delta)}$$

and the desired result now follows from the same argument as above. ☐

Lemma A.13. *Under the conditions of Proposition 9, we have that*

$$\begin{aligned} A_0(\lambda) &\equiv \int_0^{\delta^\star} \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_1 F'(\delta^\star)(\delta^\star - \delta) + \Phi_1(\delta)} = \frac{\log(\lambda)}{2\theta_1 F'(\delta^\star)} + O(1) \\ B_0(\lambda) &\equiv \int_{\delta^\star}^{1} \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_0 F'(\delta^\star)(\delta - \delta^\star) + 1 - s - \Phi_0(\delta)} = \frac{\log(\lambda)}{2\theta_0 F'(\delta^\star)} + O(1) \end{aligned} \tag{66}$$

as the meeting rate $\lambda \to \infty$.

Proof. To establish a lower bound, we start by noting that $\Phi_1(\delta) \leq \Phi_1(\delta^\star)$ for all $\delta \leq \delta^\star$. Substituting this into (66) and integrating, we find that

$$\begin{aligned} A_0(\lambda) &\geq \int_0^{\delta^\star} \frac{d\delta}{\frac{r+\gamma}{\lambda} + \theta_1 F'(\delta^\star)(\delta^\star - \delta) + \Phi_1(\delta^\star)} \\ &= \frac{-1}{\theta_1 F'(\delta^\star)} \log\left(\frac{r+\gamma}{\lambda} + \theta_1 F'(\delta^\star)(\delta^\star - \delta) + \Phi_1(\delta^\star)\right)\Bigg|_0^{\delta^\star} \end{aligned}$$

$$= \frac{-1}{\theta_1 F'(\delta^\star)} \log\left(\sqrt{(\gamma/\lambda)s(1-s)} + o\left(\frac{1}{\sqrt{\lambda}}\right)\right) + O(1) = \frac{\log(\lambda)}{2\theta_1 F'(\delta^\star)} + O(1),$$

where the second equality follows from the asymptotic expansion of $\Phi_1(\delta^\star)$ given in (48) above. To establish the reverse inequality, let us break down the integral into an integral over the interval $[0, \delta^\star - 1/\sqrt{\lambda}]$, and an integral over the interval $[\delta^\star - 1/\sqrt{\lambda}, \delta^\star]$. A direct calculation shows that the first integral can be bounded above by

$$\int_0^{\delta^\star - 1/\sqrt{\lambda}} \frac{d\delta}{\theta_1 F'(\delta^\star)(\delta^\star - \delta)} = \frac{1}{\theta_1 F'(\delta^\star)} \log(\delta^\star \sqrt{\lambda}) = \frac{\log(\lambda)}{2\theta_1 F'(\delta^\star)} + O(1).$$

On the other hand, noting that $\inf_{\delta \in [\delta^\star - 1/\sqrt{\lambda}, \delta^\star]} \Phi_1(\delta) \geq \frac{g_1(-1)}{\sqrt{\lambda}}$ and integrating, we find that the second integral can be bounded from above by

$$\int_{\delta^\star - 1/\sqrt{\lambda}}^{\delta^\star} \frac{d\delta}{\theta_1 F'(\delta^\star)(\delta^\star - \delta) + g_1(-1)/\sqrt{\lambda}} = \frac{1}{\theta_1 F'(\delta^\star)} \log\left(1 + \frac{\theta_1 F'(\delta^\star)}{g_1(-1)}\right) = O(1),$$

where the last equality follows Lemma A.9. The proof of the asymptotic expansion for the second integral is similar. We omit the details. □

Proof of Proposition 10. Integrating by parts shows that

$$w(\lambda) = \int_0^{\delta^\star} \Phi_1(\delta)\, d\delta + \int_{\delta^\star}^1 \big(1 - s - \Phi_0(\delta)\big)\, d\delta.$$

The quadratic equation for the equilibrium distribution and the assumed continuity of the distribution of utility types jointly imply that $\lambda \Phi_1(\delta) = \gamma s F(\delta)/(\gamma/\lambda + \Phi_1(\delta) + F(\delta^\star) - F(\delta))$, and combining this identity with arguments similar to those of the proof of Lemma A.12 shows that the first integral in the definition of the welfare cost satisfies

$$\left| \int_0^{\delta^\star} \left(\lambda \Phi_1(\delta) - \frac{\gamma s F(\delta^\star)}{\gamma/\lambda + \Phi_1(\delta) + F'(\delta^\star)(\delta^\star - \delta)} \right) d\delta \right| = O(1). \tag{67}$$

On the other hand, the same arguments as in the proof of Lemma A.13 imply that

$$\int_{\delta^\star - 1/\sqrt{\lambda}}^{\delta^\star} \frac{\gamma s F(\delta^\star)\, d\delta}{\gamma/\lambda + \Phi_1(\delta) + F'(\delta^\star)(\delta^\star - \delta)} \leq \frac{\gamma s F(\delta^\star)}{F'(\delta^\star)} \log\left(1 + \frac{F'(\delta^\star)}{g_1(-1)}\right) = O(1),$$

and combining this inequality with (67) gives

$$\int_0^{\delta^\star} \lambda \Phi_1(\delta)\, d\delta = \int_0^{\delta^\star - 1/\sqrt{\lambda}} \frac{\gamma s F(\delta^\star)}{\gamma/\lambda + \Phi_1(\delta) + F'(\delta^\star)(\delta^\star - \delta)}\, d\delta + O(1).$$

To obtain a lower bound for the integral, we can bound $\Phi_1(\delta)$ above by $\Phi_1(\delta^\star - 1/\sqrt{\lambda})$, and to obtain an upper bound, we can bound $\Phi_1(\delta)$ below by 0. In both cases, we can compute the resulting integral explicitly and we find that the bounds can both be written as

$$\frac{\gamma s F(\delta^\star)}{2F'(\delta^\star)}\log(\lambda)+O(1)=\frac{\gamma s(1-s)}{2F'(\delta^\star)}\log(\lambda)+O(1).$$

Going through the same steps shows that the second integral satisfies

$$\int_{\delta^\star}^{1}\lambda\bigl(1-s-\Phi_0(\delta)\bigr)\,d\delta=\frac{\gamma s(1-s)}{2F'(\delta^\star)}\log(\lambda)+O(1)$$

and the desired result now follows by adding the two asymptotic expansions. To complete the proof, assume that the distribution of utility types is discrete. Using the same notation as in the proof of Proposition 8, we find that

$$w(\lambda)=\sum_{i=0}^{m-1}(\delta_{i+1}-\delta_i)\Phi_1(\delta_i)+\sum_{i=m}^{I}(\delta_{i+1}-\delta_i)\bigl(1-s-F(\delta_i)+\Phi_1(\delta_i)\bigr)$$

and the conclusion follows from the expansion of $\Phi_1(\delta_i)$ in (62). □

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