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A GENERAL FORMULA FOR VALUING DEFAULTABLE SECURITIES

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Previous research has shown that under a suitable no-jump condition, the price of a defaultable security is equal to its risk-neutral expected discounted cash flows if a modified discount rate is introduced to account for the possibility of default. Below, we generalize this result by demonstrating that one can always value defaultable claims using expected risk-adjusted discounting provided that the expectation is taken under a slightly modified probability measure. This new probability measure puts zero probability on paths where default occurs prior to the maturity, and is thus only absolutely continuous with respect to the risk-neutral probability measure. After establishing the general result and discussing its relation with the existing literature, we investigate several examples for which the no-jump condition fails. Each example illustrates the power of our general formula by providing simple analytic solutions for the prices of defaultable securities.

KEYWORDS: Defaultable securities, risk-adjusted discounting, absolutely continuous change of measures, counterparty risk, flight to quality.

1. INTRODUCTION

THE REDUCED FORM APPROACH to modelling default risk has become a popular framework for the valuing of defaultable securities, arguably for two reasons. First, the difference between reduced-form models and the more economically intuitive structural models of default[2] becomes moot when one includes realistic frictions in the structural models, such as imperfect information about the asset or liability structure (see Duffie and Lando (2001)).[3] Second, the reduced-form framework often provides tractable formulas for the valuation of defaultable claims, in turn facilitating empirical implementation. Indeed, under a suitable no-jump condition, the price of a defaultable security is equal to the risk-neutral expectation of its discounted future cash flows, where the discount rate is no longer the risk-free rate, but rather a rate that

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[2]Structural models of default explicitly specify firm value dynamics, and typically relate default to asset value dropping below the liabilities of the firm. In contrast, reduced-form models abstract from the firm value process, and instead directly specify the default process as a totally inaccessible stopping time.

[3]Duffie and Lando (2001) show that the structural model effectively becomes a reduced-form model when the firm's asset value is imperfectly observed, because investors can no longer predict the occurrence of the default event perfectly.

has been modified to reflect default risk. Hence, this framework allows standard default-free term structure machinery to be used for pricing securities subject to default risk. As such, if the risk-neutral dynamics of this modified discount rate are chosen judiciously (e.g., specified as affine), then simple analytic solutions can be obtained for the price of defaultable securities.[4]

Unfortunately, when this no-jump condition fails to hold, the simple approach of risk-adjusted discounting is no longer valid. Furthermore, while Duffie, Schroder, and Skiadas (1996) provide a formula for the pricing of defaultable securities in this more general case, their solution requires a rather involved two-step procedure that loses the natural economic interpretation of risk-adjusted discounting.[5]

In this paper, we show that it is always possible to value defaultable claims as the expectation of future discounted cash flows, even when the no-jump condition is violated. However, in order to do so, the expectation needs to be computed under a new probability measure, which we identify below. This new probability measure is characterized by the fact that it puts zero probability on those paths for which the default event occurs prior to the maturity of the security. As such, this measure is not equivalent to, but rather only absolutely continuous with respect to the risk-neutral probability measure. Our formula retains the economically appealing structure of expected risk-adjusted discounting. Furthermore, since our approach is valid in all cases, it eliminates the need to check the validity of the no-jump condition. Finally, and as we demonstrate below using a number of examples, our formula typically leads to tractable solutions even when alternatives appear to fail.

More specifically, assume that the intensity of the default time is given by some $\mathbb{F}$-adapted process λ, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ represents the information available to agents. Consider a defaultable security that pays the random amount X at maturity T conditional upon no default and assume for simplicity that the recovery is zero if default occurs prior to maturity. In order to price such a security, it is convenient to first compute the value of an otherwise identical default-free security in a fictitious economy where the interest rate is $(r+\lambda)$. The value of this pseudo-security is given by

$$V_t := E\left[e^{-\int_t^T (r_s+\lambda_s)\,ds} X \mid \mathcal{F}_t\right],$$

where the expectation is taken under the risk-neutral probability measure. Duffie, Schroder, and Skiadas (1996) and Duffie and Singleton (1999) demon-

[4]See, for example, Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Lando (1998), Duffie and Singleton (1999), and Bélanger, Shreve, and Wong (2002).

[5]In fact, we know of no paper that has been able to successfully implement their formula when the no-jump condition fails.

strate that, at any time prior to maturity, the ex-dividend value of the defaultable security is given by

$$S_t = \mathbf{1}_{\{\tau>t\}}\{V_t - E[e^{-\int_t^\tau r_s\,ds}\Delta V_\tau|\mathcal{F}_t]\},$$

where ΔV_τ denotes the jump in the pseudo-value process at the time of default. Hence, if there is no jump in V at the default date, then the valuation of a defaultable security reduces to the computation of its expected discounted cash flows using a discount rate that has been modified to reflect default risk. Because of this simplification, most of the existing literature has chosen to restrict investigation to models of default that satisfy this no-jump condition. Indeed, most papers adopt the so-called Cox process framework[6] where the intensity is generated by a vector process whose filtration is conditionally independent from the default event.[7]

However, for those models of default that do not satisfy the no-jump condition,[8] the valuation formula above loses its simple economic interpretation as a stochastic version of risk-adjusted discounting. Indeed, it requires a two-step procedure: first compute the process V, then subtract from it the present value of its jump at the default time. As a consequence of this more complicated procedure, it is no longer obvious how to apply standard default-free term structure machinery to price defaultable securities.

To circumvent this difficulty, we demonstrate below that even when the no-jump condition fails, the price of a defaultable security can still be written as an expectation of discounted cash flows. In particular, we demonstrate that, at any time before maturity, the ex-dividend value of the defaultable security is given by $S_t = \mathbf{1}_{\{\tau>t\}}V'_t$, where

$$V'_t := E'[e^{-\int_t^T (r_s+\lambda_s)\,ds}X|\mathcal{F}'_t].$$

Here, the symbol E' indicates that the conditional expectation is to be computed under the probability measure P', and $\mathbb{F}' := (\mathcal{F}'_t)_{t\geq 0}$ denotes the augmentation of $\mathbb{F}$ under P'. This probability measure is characterized by the property that it puts zero mass on the paths for which default occurs prior to the maturity of the security and is thus only absolutely continuous with respect

[6]See Brémaud (1981) for the definition and basic properties of Cox processes.

[7]See, for example, Duffie and Singleton (1999), Jarrow and Turnbull (1995), Lando (1998), Elliott, Jeanblanc, and Yor (2000), Jeanblanc and Rutkowski (2001), and Schonbücher (1997). These models can be distinguished by their specification of the recovery process. Standard approaches include: recovery of face value, recovery of pre-default market value, and recovery of a fraction of Treasury.

[8]As we demonstrate below in a number of examples, the no-jump condition is violated as soon as the default event is allowed to have a direct impact on either market prices or the default-free interest rate.

to the risk-neutral probability measure. Not surprisingly, if the no-jump condition is satisfied for the claim under consideration, then the processes V and V' coincide.

Below, we consider several examples that illustrate the power of our general formula. Specifically we investigate the pricing of defaultable securities in the presence of (i) flight to quality, (ii) counterparty credit risk, and (iii) systematic jump risk. Each of these examples illustrates an economically plausible scenario where the no-jump condition fails.

(i) *Flight to Quality*: Flights to quality typically refer to downward jumps in the risk-free interest rate as a consequence of unexpected defaults by large institutions. Examples of flights to quality include LTCM and the Russian default crises (see Chang and Sundaresan (1999) and Longstaff (2001)). Interestingly, such a situation would imply a violation of the no-jump condition even if the default arrival times and intensities are modeled by a Cox process.

(ii) *Counterparty Risk*: Counterparty risk arises when the default of one firm triggers a jump in the probabilities of default of other firms. One situation where this occurs is when firms have economic ties that render one firm vulnerable to the default of another. Note that, due to the interdependence between the intensities and default arrival times, such a model falls outside of the Cox process framework. Such a model has recently been proposed by Jarrow and Yu (2001), who showed that in such a case the survival probabilities cannot be computed using the standard approach. Below, however, we demonstrate that such a model admits simple analytic solutions that are readily determined within our framework. Further, our approach permits us to investigate the impact of counterparty risk on the valuation of more complex structures, such as collateralized debt obligations, which depends crucially on the default correlation structure.

(iii) *Systematic Jump Risk*: Recently there has been some debate in the literature as to whether event risk should be priced (see Jarrow, Lando, and Yu (2000), Driessen (2002), and Collin-Dufresne, Goldstein, and Helwege (2002)). If in fact unexpected default events of large corporations generate a marketwide impact, then such events should command a risk premium. This in turn implies that the pricing kernel should also jump at the event time. As a result, if one models the pricing kernel directly and uses it as a deflator for valuation purposes (see, e.g., Constantinides (1992)), then the no-jump condition will in general be violated.

We also provide an example that nests the three previous deviations from the no-jump condition into a single model, which falls in the class of affine jump-diffusion models (see, e.g., Duffie, Pan, and Singleton (2000)). Combining our general formula with well-known results from the theory of affine jump diffusions, we show that the price of a defaultable bond in such a model is an exponential-affine function of the state variables which can be computed by solving a fairly simple system of ordinary differential equations.

The remainder of the paper is organized as follows. In Section 2 we introduce the model. In Section 3 we derive the general formula and relate it to results in the existing literature. In Section 4 we present several examples that illustrate the failure of the standard approach and demonstrate the usefulness of our general formula. Section 5 concludes. All technical results and proofs are collected in two appendices.

2. THE MODEL

2.1. *Information Structure*

We consider an infinite horizon economy where the uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ models the arrival of information over time. It will be assumed throughout the paper that it satisfies the usual conditions of right continuity and completeness with respect to the null sets of the probability measure P.

2.2. *Default Time*

Following the so-called reduced-form approach to default risk, we model the stochastic structure of the default time through a totally inaccessible[9] $\mathbb{F}$-stopping time $\tau : \Omega \to (0, \infty]$. As is well known (see, e.g., Dellacherie and Meyer (1980, VI.78)), this assumption is equivalent to the existence of a continuous, increasing process of finite variation A with initial value equal to zero such that

$$M_t := \mathbf{1}_{\{\tau \leq t\}} - A_t \tag{2.1}$$

is a uniformly integrable $(\mathbb{F}, P)$-martingale. The process A is uniquely defined up to the occurrence of the default time and is referred to as the compensator of the default indicator function. In order to simplify the exposition, we assume throughout that[10]

$$A_t = \int_0^{\tau\wedge t} \lambda_s\, ds = \int_0^t \mathbf{1}_{\{\tau > s\}} \lambda_s\, ds \tag{2.2}$$

for some strictly positive, adapted process λ where $a \wedge b = \min(a, b)$. The process λ is referred to as the $(\mathbb{F}, P)$-intensity of the default time and will be assumed to satisfy the integrability condition

$$E\left[\exp\left(q \int_0^{\vartheta} \lambda_t\, dt\right)\right] < \infty \tag{2.3}$$

[9] A stopping time θ is *predictable* if there exists an increasing sequence $(\theta_n)_{n\geq 1}$ of stopping times such that $\theta = \lim_{n\to\infty} \theta_n$. A stopping time σ is *totally inaccessible* if $P(\sigma = \theta < \infty) = 0$ holds for all predictable stopping times.

[10] The extension of our results to the case where the compensator of the default time is not of the form (2.2) is available upon request.

for some strictly positive nonrandom time ϑ and some nonrandom constant $q \geq 1$ where $E[\cdot]$ denotes expectation under P.

2.3. *Financial Market*

Although we do not explicitly model it, we assume throughout the paper that there exists an underlying frictionless financial market in which a number of securities are traded. One of these traded securities is a locally riskless savings account whose price process is given by

$$B_t := \exp\left(\int_0^t r_s \, ds\right)$$

for some positive, adapted interest rate process $(r_t)_{t\geq 0}$. Furthermore, taking the savings account price process as numéraire, we assume that the probability measure P is an *equivalent martingale measure* for the securities market in the sense of Harrison and Kreps (1979).[11]

3. MAIN RESULTS

3.1. *Valuation of Defaultable Securities*

By the definition of a martingale measure, the price process of an arbitrary cumulative dividend process $(D_t)_{t\geq 0}$ of integrable variation satisfies the fundamental valuation formula

$$S_t = E\left[\frac{B_t S_T}{B_T} + \int_t^T \frac{B_t \, dD_s}{B_s} \middle| \mathcal{F}_t\right], \quad t \leq T. \tag{3.1}$$

Following Duffie, Schroder, and Skiadas (1996), henceforth (DSS), we assume throughout the paper that the value of a security at date t is zero if all dividend payments after that date are equal to zero. In other words, all securities prices are taken to be ex-dividend (note that this assumption is not implied by the valuation formula (3.1)).

Consider a defaultable security which matures at some nonrandom time $T \leq \vartheta$ yielding an $\mathcal{F}_T$-measurable payoff X provided that default has not occurred. The security's payment upon default is described by the value at the default time of a predictable process R. In the sequel, the pair (X, R) will be

[11] As shown by Delbaen and Schachermayer (1994) the existence of such a *risk-neutral probability measure* is essentially equivalent to the absence of arbitrage opportunities from the market. In our setting, however, uniqueness of the equivalent martingale measure, and the ensuing market completeness, is not guaranteed since one cannot necessarily hedge against jumps that may occur at default.

referred to as the characteristics of the defaultable security and will be assumed to satisfy

$$E[|X|^p] + E\left[\sup_{t\geq 0} |R_t|^p\right] < \infty \tag{3.2}$$

for some $p \in [1, \infty)$. With this specification, the cumulative dividend process associated with the defaultable security is given by

$$\begin{aligned} D_t &:= X\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T\leq t\}} + R_\tau \mathbf{1}_{\{\tau\leq t\wedge T\}} \\ &= X\mathbf{1}_{\{\tau>T\}}\mathbf{1}_{\{T\leq t\}} + \int_0^{\tau\wedge t\wedge T} R_s \lambda_s\, ds + \int_0^{t\wedge T} R_s\, dM_s. \end{aligned}$$

Using the result of Lemma A.1 in the Appendix, the last term on the right-hand side of the above equation is a uniformly integrable $(\mathbb{F}, P)$-martingale. Hence, it follows from (3.1) that the ex-dividend price process of the security is uniquely given by

$$S_t = \mathbf{1}_{\{t<T\}} E\left[\frac{B_t X}{B_T}\mathbf{1}_{\{\tau>T\}} + \int_t^T \mathbf{1}_{\{\tau>s\}} \frac{B_t R_s}{B_s}\lambda_s\, ds \middle| \mathcal{F}_t\right]. \tag{3.3}$$

As was originally noted by (DSS), the above valuation formula has the undesirable feature of involving the default time explicitly, in turn making it difficult to implement. To circumvent this difficulty, we propose an alternative valuation formula that does not explicitly involve the default time. The following constitutes our main result.

THEOREM 1: *Assume that conditions* (2.3) *and* (3.2) *hold and define a non-negative* $\mathbb{F}$*-adapted increasing process by setting*

$$\Lambda_t := \exp\left(\int_0^t \lambda_s\, ds\right). \tag{3.4}$$

Then the ex-dividend price process of the defaultable security associated with the characteristics (X, R) *is uniquely given by*

$$S_t = \mathbf{1}_{\{t<T\}}\mathbf{1}_{\{t<\tau\}} E'\left[\frac{\Lambda_t B_t X}{\Lambda_T B_T} + \int_t^T \frac{\Lambda_t B_t R_s}{\Lambda_s B_s}\lambda_s\, ds \middle| \mathcal{F}'_t\right], \tag{3.5}$$

where we denote by $E'[\cdot]$ *the expectations operator under the absolutely continuous probability measure defined by*

$$\left.\frac{dP'}{dP}\right|_{\mathcal{F}_t} = Z_t := \mathbf{1}_{\{\tau>t\wedge T\}}\Lambda_{t\wedge T}, \tag{3.6}$$

and the filtration $\mathbb{F}' := (\mathcal{F}'_t)_{t\geq 0}$ *is the augmentation of the original filtration* $\mathbb{F}$ *by the null sets of the probability measure* P'.

The intuition behind the result of Theorem 1 is twofold. First, since the possibility of default increases the risk associated with the security, one naturally expects that the discount rate has to be increased relative to the default-free case. Second, even though the economic factors that influence the security's value may themselves be affected by the default event, one should ignore these feedback effects for valuation purposes because the ex-dividend value process of the security depends only on those events that happen prior to default. Theorem 1 makes both of these intuitions precise and shows that the value of an arbitrary defaultable security can always be written as an expectation of discounted cash flows. However, the discount rate has to be modified from r to $r + \lambda$ to reflect default risk and the expectation has to be computed under a modified probability measure to reflect the possibility that the no-jump condition has been violated.

In order to better understand the impact of the change of measure described in Theorem 1, we now briefly investigate the filtration $\mathbb{F}'$ that it induces and the structure of $(P', \mathbb{F}')$-local martingales. Because the new probability measure is only absolutely continuous with respect to the original probability measure, the collection

$$\mathcal{N} := \{A \in \mathcal{F}_\infty : P(A) \neq 0 \text{ and } P'(A) = 0\}$$

of null sets that must be added to the original filtration when transforming to the new measure is nonempty. Furthermore, it follows from Lemma A.2 in the Appendix that $A \in \mathcal{N}$ if and only if $A \subseteq \{\tau \leq T\}$. In other words, the new filtration is obtained by adding to the original filtration the knowledge that the default time will not occur before the maturity date of the security under consideration.

REMARK 1: Note that the probability measure P' does not coincide with the conditional probability measure $P(\cdot|\{\tau > T\})$ in general. To see this, let $A \in \mathcal{F}_\infty$ and observe that

$$P(A|\{\tau > T\}) = \frac{P(A \cap \{\tau > T\})}{P(\{\tau > T\})} = E'\left[\frac{\mathbf{1}_{\{A\}}/\Lambda_T}{E'[1/\Lambda_T]}\right],$$

where the first equality follows from Bayes' rule and the second follows from the definition of the probability measure P'. Interestingly, we see from the above expression that P' coincides with the conditional probability $P(\cdot|\{\tau > T\})$ if and only if the intensity of the default time is a deterministic function of time.

In view of the above discussion and the fact that the principal source of risk driving the density process Z is the default time, it seems natural to expect that if a given process is, in some sense, independent from the occurrence of the default time, then its local characteristics should not change when passing to the probability measure P'. This intuition is confirmed by the following.

LEMMA 1: *Assume that condition* (2.3) *holds true and let L be an arbitrary* $(\mathbb{F}, P)$*-local martingale. Then the process defined by*

$$L_t - \int_0^t \frac{d\langle L, Z\rangle_s}{Z_{s-}} = L_t - \langle L, M\rangle_t,$$

where $\langle \cdot, \cdot \rangle$ denotes the quadratic covariation process, is a local martingale with respect to $(\mathbb{F}', P')$. *In particular*:

(i) *The default intensity and the default indicator function are both equal to zero almost surely under P' on the interval* $[0, T]$.

(ii) *If the process L does not jump at the default time, then it is a local martingale with respect to both* $(\mathbb{F}', P')$ *and* $(\mathbb{F}, P)$.

As will become clear in Section 4, the results of the above lemma are crucial for applications since they allow one to compute the local characteristics of an arbitrary process under the modified probability measure P'. In particular, it follows from (ii) that a process that is a Brownian motion under the probability measure P remains a Brownian motion under the probability measure P'.

The last result in this section provides an alternative representation of the ex-dividend price process for the defaultable security in terms of expectations under the probability measure P, and connects our main result to that of (DSS).

PROPOSITION 1: *Assume that conditions* (2.3) *and* (3.2) *hold true for some conjugate exponents* (p, q) *with* $p > 1$ *and define*

$$V_t := E\left[\frac{\Lambda_t B_t X}{B_T \Lambda_T} + \int_t^T \frac{\Lambda_t B_t R_s}{\Lambda_s B_s} \lambda_s \, ds \,\middle|\, \mathcal{F}_t\right].$$

Then the ex-dividend price process of the defaultable security associated with the characteristics (X, R) *is uniquely given by*

$$S_t = \mathbf{1}_{\{t<T\}} \mathbf{1}_{\{\tau>t\}} \left\{ V_t - E\left[\frac{B_t \Delta V_\tau}{B_\tau} \,\middle|\, \mathcal{F}_t\right]\right\},$$

where ΔV_τ denotes the jump in the process V at the time of default. In particular, if the process V is predictable, then we have that $S_t = V_t$ holds almost surely on the event $\{\tau > t\} \cap \{t < T\}$.

3.2. *Recursive Valuation of Defaultable Securities*

In this section we generalize the model of the previous section to allow for a possible dependence of the default payoff process on the pre-default value of the security under consideration. To model this dependence, we assume throughout this section that the default payoff is given by $R(S_{-})$ for some $\mathbb{F}$-adapted mapping $R: [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$. In order to guarantee that a security that defaults immediately after it has lost all its value yields a zero payoff, we shall further assume that the no-arbitrage condition $R(0) = 0$ holds almost everywhere.

The following theorem gives an explicit formula for the valuation of the security in terms of the solution to a $(\mathbb{F}', P')$-recursive stochastic integral equation, and constitutes the main result of this section.

THEOREM 2: *Assume that conditions* (2.3) *and* (3.2) *hold true for some conjugate exponents* (p, q) *with* $p > 1$ *and that*

$$\lambda_t \left| \frac{R_t(x) - R_t(y)}{x - y} \right| \leq C \tag{3.7}$$

holds P'*-almost everywhere for some positive constant* C*. Then there exists a unique solution to the recursive stochastic integral equation*

$$V'_t = E'\left[\frac{\Lambda_t B_t X}{\Lambda_T B_T} + \int_t^T \frac{\Lambda_t B_t}{\Lambda_s B_s} R_s(V'_{s-}) \lambda_s \, ds \Big| \mathcal{F}'_t \right] \tag{3.8}$$

in the space of processes that are $\mathbb{F}'$*-adapted and integrable with respect to* $dt \otimes dP'$*. In particular, the ex-dividend price process of the defaultable security is uniquely given by* $S_t = \mathbf{1}_{\{t<T\}} \mathbf{1}_{\{\tau>t\}} V'_t$.

Although it covers a number of situations of interest, the above result is not flexible enough to cover the case of fractional recovery of market value unless the intensity process of the default time is assumed to be bounded. Indeed, in this case the default payoff is given by $R = (1 - \delta)S_{-}$ for some $\mathbb{F}$-predictable process δ with values in $[0, 1]$ and the validity of the uniform Lipschitz condition (3.7) is equivalent to a boundedness assumption on the intensity process. To circumvent this, and in turn demonstrate that the risk-adjusted discounting valuation formula of Duffie and Singleton (1999) is always valid provided that the expectation is computed under the probability measure P' rather than under the probability measure P, we treat this particular case separately.

PROPOSITION 2: *Assume that conditions* (2.3) *and* (3.2) *hold true for some conjugate exponents* (p, q) *with* $p > 1$ *and that*

$$R_t(x) = (1 - \delta_t)x$$

for some predictable process δ *with value in* $[0,1]$. *Then the ex-dividend price process of the defaultable security is uniquely given by*

$$S_t = \mathbf{1}_{\{\tau>t\}} E'\left[e^{-\int_t^T \delta_s \lambda_s \, ds} \frac{B_t X}{B_T} \middle| \mathcal{F}'_t \right]$$

for all $t < T$, *and zero otherwise*.

In certain settings, it may be reasonable to assume that both the default payoff and the default intensity (and hence the default time itself) depend on the value of the security under consideration. This additional layer of recursiveness can be modelled by assuming that the intensity of the default time is given by $\lambda(S_-)$ for some $\mathbb{F}$-adapted mapping $\lambda : [0,T] \times \Omega \times \mathbb{R} \to (0,\infty)$. In this case, however, the basic valuation formula (3.1) no longer makes sense by itself. Indeed, and contrary to the case where the intensity is independent of the security's value, the risk-neutral probability measure P, the default stopping time τ and the security's value process now need to be constructed simultaneously since the risk-neutral distribution of the default time depends on the value of the security.

In order to simplify the presentation of our results in this case, we will assume that both the no-default payment and the default intensity mapping are essentially bounded.

PROPOSITION 3: *Assume that the intensity* $\lambda = \lambda(S_-)$ *is a bounded mapping, that the recovery* $R = R(S_-)$ *satisfies condition* (3.7), *and that* X *is bounded. Then the following assertions hold*:

(i) *There exists a triple* (τ, P, S) *such that the totally inaccessible stopping time* τ *has intensity* $\lambda = \lambda(S_-)$ *under the probability measure* P *and the basic valuation equation* (3.1) *holds true*.

(ii) *The ex-dividend price of the defaultable security associated with the characteristics* (X, R) *is given by* $S_t = \mathbf{1}_{\{t<T\}}\mathbf{1}_{\{\tau>t\}}V'_t$, *where the process* V' *is the unique solution to the recursive stochastic integral equation* (3.8) *with* $\lambda = \lambda(V'_-)$.

Although the result of assertion (ii) is similar to that of Theorem 2, the mathematical construction that underlies it is very different and thus deserves a few comments. As mentioned before the proposition, the fact that the default time depends on the value of the defaultable security forces us to construct τ, P, and S simultaneously. In order to do so, we rely on a change of measure argument similar to that of Kusuoka (1999). More precisely, we start from an exogenously specified probability measure under which the intensity of the default time is constant, then construct the value process S, and finally define P by a suitably chosen equivalent change of probability measure. Note that while we chose the starting intensity of the default time to be constant, the choice of the starting probability measure is really unconstrained. In particular, if one chooses this probability measure in such a way that it is a martingale measure for all default insensitive securities, then the resulting probability measure P is a martingale measure for all securities.

4. EXAMPLES

In this section we investigate several examples that demonstrate the power and tractability of the general formula derived in the previous section. In particular, in each example below, the standard approach of discounting risky cash flows at the default risk-adjusted rate under the risk-neutral probability measure fails. However, discounting cash flows at the default risk-adjusted rate under the probability measure P' generates the correct solution. Furthermore, the calculations are simplified considerably by using our approach even when alternative approaches appear to fail.

4.1. *Flight to Quality*

Consider an economy where the information filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is generated by a totally inaccessible stopping time τ with constant intensity $\lambda > 0$. In contrast to most reduced-form models, we allow the risk-free rate to be affected by the default event, which we interpret here as a flight to quality. Specifically, we assume that the risk-neutral dynamics for the risk-free rate are given by

$$r_t = r_0 + J\mathbf{1}_{\{\tau\leq t\}}. \tag{4.1}$$

In order to guarantee that the risk-free rate is nonnegative, we assume that $r_0 \geq 0$ and that the jump parameter satisfies $J \geq -r_0$. To interpret the jump in the interest rate as a flight to quality, we will assume for most of this paragraph that the jump-size parameter is negative.

4.1.1. *Zero Recovery*

Consider a defaultable bond that pays one dollar conditional upon no default, and zero otherwise. As a result of Theorem 1, the ex-dividend value of this defaultable security is given by $S_t = \mathbf{1}_{\{\tau>t\}}V'_t$ for all $t < T$, where

$$V'_t = E'\big[e^{-\int_t^T (r_s+\lambda)ds}|\mathcal{F}'_t\big]. \tag{4.2}$$

The expectation in the above equation is taken under the probability measure P' and with respect to the filtration $\mathbb{F}'$, both of which are defined as in the previous section. Using Lemma 1, it follows that the risk-free rate is constant under the probability measure P'. Thus the expectation in (4.2) can be readily computed as

$$V'_t = \exp[-(r_0 + \lambda)(T - t)]. \tag{4.3}$$

Note that the flight-to-quality jump-size parameter does not affect the value of the risky bond. Intuitively, this happens because the jump in the risk-free rate occurs precisely at the date of default and the recovery rate is zero. This can be readily seen from the definition of the ex-dividend price process

$$S_t = E\big[e^{-\int_t^T r_s\,ds}\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\big] = E\big[e^{-r_0(T-t)}\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\big],$$

since the interest rate does not jump on the set $\{\tau > T\}$. This simple example with zero recovery specification provides some intuition for our general formula. We now investigate more realistic recovery specifications that have been proposed in the literature.

4.1.2. *Recovery of Treasury*

Jarrow and Turnbull (1995) and Longstaff and Schwartz (1995) model the default recovery so that the owner of the defaultable bond receives a fraction $(1-\delta)$ of an otherwise identical default-free bond. Within our flight-to-quality framework, the price of the default-free zero-coupon bond is given by

$$\text{(4.4)}\qquad \begin{aligned} P_t &:= E\big[e^{-\int_t^T r_s\,ds}|\mathcal{F}_t\big] \\ &= \mathbf{1}_{\{\tau\le t\}}e^{-(r_0+J)(T-t)} + \mathbf{1}_{\{\tau>t\}}e^{-r_0(T-t)}\left[\frac{Je^{-\lambda(T-t)} - \lambda e^{-J(T-t)}}{J-\lambda}\right]. \end{aligned}$$

Let us now turn to the valuation of the defaultable bond under recovery of Treasury. Using the valuation formula (3.1) in conjunction with the above expression and the result from the zero-recovery solution (4.3), we find that at any time prior to maturity the value process of the defaultable bond is given by

$$\begin{aligned} S_t &= E\big[e^{-\int_t^T r_s\,ds}\mathbf{1}_{\{\tau>T\}} + e^{-\int_t^\tau r_s\,ds}(1-\delta)P_\tau\mathbf{1}_{\{\tau\le T\}}|\mathcal{F}_t\big] \\ &= \mathbf{1}_{\{\tau>t\}}(1-\delta)P_t + \delta E\big[e^{-\int_t^T r_s\,ds}\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\big] \\ &= \mathbf{1}_{\{\tau>t\}}\big[(1-\delta)P_t + \delta e^{-(r_0+\lambda)(T-t)}\big]. \end{aligned}$$

Not surprisingly, the value of the defaultable bond is always less than that of the corresponding default-free bond provided that the loss rate upon default is positive, i.e., $(S < P) \Leftrightarrow (\delta > 0)$. As demonstrated in the next example, this is not necessarily the case under all recovery specifications.

4.1.3. *Recovery of Market Value*

Duffie and Singleton (1999) specify the default recovery so that the owner of the defaultable bond receives a fraction of the bond's pre-default value. The corresponding default payoff process is given by $R = (1-\delta)S_-$ for some constant $\delta \in [0, 1]$. Applying Proposition 2, we find that the ex-dividend value of such a defaultable bond is given by

$$S_t = \mathbf{1}_{\{\tau>t\}} E'\left[e^{-\int_t^T (r_s+\delta\lambda)\,ds} | \mathcal{F}'_t\right] = \mathbf{1}_{\{\tau>t\}} e^{-(r_0+\delta\lambda)(T-t)},$$

where the second equality follows from the fact that the interest rate is almost surely constant under the probability measure P'. In contrast, and as can be seen from equation (4.4), the price of the corresponding default-free bond satisfies

$$\mathbf{1}_{\{\tau>t\}} P_t = \mathbf{1}_{\{\tau>t\}} e^{-r_0(T-t)} \left[\frac{Je^{-\lambda(T-t)} - \lambda e^{-J(T-t)}}{J-\lambda}\right].$$

Interestingly, we see that even in the limit where the fractional loss coefficient tends to zero, the default-free bond and defaultable bond do not obtain the same value. In fact, for $J \geq 0$ (which is contrary to the flight-to-quality specification), one can find a loss fraction δ such that the defaultable bond is actually worth more than the risk-free bond. While this feature smacks of arbitrage, in actuality it merely reflects the fact that the risky bond pays off a fraction of its pre-default market value at the default event. Hence, in the limit where the fractional loss coefficient goes to zero, the risky bond is unaffected by this jump risk. In contrast, the risk-free bond would be negatively affected by this positive jump in interest rates.

4.2. *Systematic Jump Risk*

As a second example, we investigate a continuous-time generalization of the Lucas (1978) pure exchange economy in which the interest rate and risk-neutral intensity experience jumps in equilibrium. This simple economy provides a theoretical justification for the flight-to-quality dynamics assumed above. It also suggests that in many equilibrium models the no-jump condition will be generically violated, and that our approach is uniquely suited for solving this class of models.[12]

We assume there is a representative agent with a constant relative risk-aversion utility function given by

$$U(C) := \frac{C^{1-\gamma} - 1}{1-\gamma}$$

[12]Unlike in the previous example where alternative approaches could have been used to obtain the solution, here we cannot think of any alternative that would lead to the solution straightforwardly.

for some $\gamma > 0$. Under the objective probability measure P_o, we specify the dynamics of the economy's aggregate output process on a standard filtered probability space $(\Omega, \mathbb{F}, P_o)$ by

$$\begin{aligned} d\log(D_t) &= \mu(X_t)\,dt + \sigma(X_t)\,dB_t - \phi(X_{t-})\,dN_t, \\ dX_t &= \kappa(X_t)\,dt + \nu(t, \omega, X_{t-})\,dN_t. \end{aligned}$$

Here B is a standard Brownian motion, N is a point Poisson process with constant intensity λ and $\{\mu, \sigma, \phi, \kappa\}$ are deterministic functions. The state variable experiences jumps at the event dates. For the purpose of our discussion, the specification of the size of these jumps can be arbitrary as long as the process X is well defined. In order to simplify the exposition, we will further assume that the deterministic functions $\{\mu, \sigma, \phi, \kappa\}$ are continuous, bounded, and such that the two technical conditions $\kappa(X_0) = 0$ and $\phi \geq 0$ hold.[13]

The negative jumps in the aggregate output capture marketwide events, such as natural catastrophes or the default of some large companies. The dynamics of the aggregate output are specified to reflect the fact that these events could have more than just a level effect; they could affect the investment opportunity set by generating changes in the expectation and volatility of future output changes. Within this context, we determine the equilibrium price of a security that pays one dollar at maturity T contingent on no catastrophe occurring, and zero otherwise.

As is well known, the equilibrium pricing kernel in this representative agent economy is given by (see, e.g., Lucas (1978))

$$\begin{aligned} \Pi_t &:= e^{-\int_0^t r_s\,ds}\xi_t = \left(\frac{D_t}{D_0}\right)^{-\gamma} \\ &= \exp\left[-\gamma\int_0^t \big(\mu(X_s)\,ds + \sigma(X_s)\,dB_s - \phi(X_{s-})\,dN_s\big)\right], \end{aligned}$$

where ξ is the density of the equilibrium risk-neutral measure P with respect to the objective probability measure P_o, defined by

$$\left.\frac{dP}{dP_o}\right|_{\mathcal{F}_T} = \xi_T.$$

[13]These conditions are purely technical and are sufficient for the stochastic differential equations (SDEs) to have a solution and for the existence of a risk-neutral probability measure. The condition $\kappa(X_0) = 0$ insures that X does not change until the first jump, which simplifies the results. Note that this condition would, for example, be satisfied by a mean reverting process starting at its long-term mean.

Using this relationship and applying Itô's lemma, we find that the equilibrium risk-free rate in this economy is given by

$$r_t := \gamma\mu(X_t) - \frac{1}{2}\gamma^2\sigma(X_t)^2 + \lambda\big[1 - e^{\gamma\phi(X_t)}\big].$$

Furthermore, if we set

$$\theta_t := \gamma\sigma(X_t), \tag{4.5}$$

$$\lambda_t^* := \lambda\exp(\gamma\phi(X_t)), \tag{4.6}$$

then θ identifies the Brownian risk premium and λ^* is the risk-neutral intensity of the point process. Indeed, it follows from an application of Girsanov's theorem that the processes defined by $W_t := B_t + \int_0^t \theta_s\,ds$ and $M_t := N_t - \int_0^t \lambda_s^*\,ds$ are respectively a Brownian motion and a uniformly integrable martingale under the risk-neutral measure.

Given our assumptions on the functions $\{\mu, \sigma, \phi, \kappa\}$, and since the state variable experiences jumps at the event dates, the equilibrium risk-free rate is a discontinuous process. Further, even though the intensity is constant under the objective probability measure, it is a function of the state variable under the risk-neutral probability measure. Thus the no-jump condition will be generically violated under the risk-neutral probability measure P because of jumps in both the intensity and the risk-free rate.

Denoting by τ the first jump of the Poisson process, the value of the catastrophe bond is given by

$$S_t := E_{P_o}\left[\frac{\Pi_T}{\Pi_t}\mathbf{1}_{\{\tau>T\}}\middle|\mathcal{F}_t\right] = E_P\big[e^{-\int_t^T r_s\,ds}\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\big].$$

We can use the result of Theorem 1 to value the catastrophe bond via the pricing kernel approach or the risk-neutral valuation approach. Prior to the event, we find that the price of the catastrophe bond is given by

$$S_t = e^{-\lambda(T-t)}E_{P'_o}\left[\frac{\Pi_T}{\Pi_t}\middle|\mathcal{F}'_t\right] = E_{P'}\big[e^{-\int_t^T(r_s+\lambda_s^*)\,ds}|\mathcal{F}'_t\big], \tag{4.7}$$

where the probability measures (P'_o, P') and corresponding augmented filtration are defined from the probability measures (P_o, P) as in the previous section. Using the result of Lemma 1, it follows that the point process is almost surely equal to zero up to the maturity under both probability measures, and that the process B (resp. W) remains a Brownian motion when passing from the probability measure P_o (resp. P) to the probability measure P'_o (resp. P').

Thus, the computation of the expectation in (4.7) gives

$$\begin{aligned} S_t &= \mathbf{1}_{\{\tau_1>t\}} \exp\Big[-\Big(\lambda(X_0)+\gamma\mu(X_0)-\frac{1}{2}\gamma^2\sigma(X_0)^2\Big)(T-t)\Big] \\ &= \mathbf{1}_{\{\tau_1>t\}} \exp\big[-\big(r(X_0)+\lambda^*(X_0)\big)(T-t)\big]. \end{aligned}$$

Using either of the probability measures P'_o and P' thus leads to a very simple expression for the price of the catastrophe bond which does not depend on the jumps in the pricing kernel. However, we emphasize that even though both of these changes of measure imply ignoring the possibility of a jump, catastrophe risk does carry a risk premium in the above model. Indeed, as illustrated in (4.6), the risk-neutral intensity λ^* in general differs from its historical counterpart. In particular, with constant downward jumps in the aggregate output ($\phi > 0$) and a risk-averse representative agent, there is a positive risk premium for systematic jump risk ($\lambda^* > \lambda$).

REMARK 2: Alternative approaches for pricing the catastrophe bond requires the evaluation of a conditional expectation of the form

$$\widetilde{S}_t = E_P\big[e^{-\int_t^T (r_s+\lambda_s^*)\,ds}|\mathcal{F}_t\big].$$

Note that for arbitrary functions $\{\mu, \sigma, \phi, \kappa\}$ and jump-size distributions ν there is in general no closed-form solution for this pseudo-value process. In turn, the nonavailability of a closed-form expression makes it hard to derive the actual value of the security since this would require the computation of the present value of the jump in $\widetilde{S}$ as described in Proposition 1. Our approach circumvents this difficulty.

This example demonstrates that jumps in the risk-free rate and jumps in the risk-neutral intensity of default occur generically in an economy where aggregate consumption is affected by the jump events. Below, in Section 4.5, we consider a more general setup that extends the two previous examples to a multivariate affine framework.

4.3. *Counterparty Credit Risk*

As a third example, we revisit the counterparty credit risk model that was first investigated by Jarrow and Yu (2001), henceforth (JY), and Kusuoka (1999).

Counterparty risk exists if firms have economic ties that render one firm vulnerable to the default of another. Mathematically, (JY) capture this notion by having the intensity of one firm jump at the default date of another firm.

They investigate a framework in which the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is generated by a pair $(\tau^i)_{i=a,b}$ of totally inaccessible stopping times with intensities

$$\begin{aligned} \lambda_t^a &:= a_1 + a_2 \mathbf{1}_{\{\tau^b \leq t\}}, \\ \lambda_t^b &:= b_1 + b_2 \mathbf{1}_{\{\tau^a \leq t\}}, \end{aligned} \tag{4.8}$$

for some $a_1 > 0$, $a_1 + a_2 > 0$, $b_1 > 0$, $b_1 + b_2 > 0$. Because of the looping nature of the model, the conditional survival probabilities

$$S_t^i(T) := P[\{\tau^i > T\}|\mathcal{F}_t], \quad i \in \{a, b\},$$

are difficult to compute. Therefore, (JY) modified their original model so that one of the two default arrival times is independent from the other and has constant intensity. This latter model breaks the recursiveness in the definition of the default times and thus belongs to the class of models that satisfy the no-jump condition.

Because of the intricate dependence between the default arrival time of firm a and that of firm b, the counterparty-risk model specified by (4.8) does not belong to the Cox process framework and thus constitutes a good candidate for the failure of the no-jump condition. In fact, Kusuoka (1999) used a similar example to demonstrate that the no-jump condition is not invariant under an equivalent change of probability measure.[14]

4.3.1. *Survival Probabilities*

Using the results of the previous section, here we demonstrate that the conditional survival probabilities possess simple analytic solutions. Define the probability measures $(P^i)_{i=a,b}$ by

$$\left.\frac{dP^i}{dP}\right|_{\mathcal{F}_\infty} = Z_T^i := \mathbf{1}_{\{\tau^i > T\}} \exp\left[\int_0^T \lambda_s^i \, ds\right], \tag{4.9}$$

and let $(\mathbb{F}^i)_{i=a,b}$ denote the corresponding completed filtrations. Using the expectation formula (B.1), we find that the conditional survival probabilities of the two firms are given by

$$S_t^i(T) = E\big[\mathbf{1}_{\{\tau^i > T\}}|\mathcal{F}_t\big] = \mathbf{1}_{\{\tau^i > t\}} E^i\big[e^{-\int_t^T \lambda_s^i \, ds}|\mathcal{F}_t^i\big], \tag{4.10}$$

where the superscript on the expectation operator refers to the measures defined in (4.9). From Lemma 1 it follows that $\{\tau^a \leq T\}$ is a null set of the probability measure P^a. This implies that the intensity of firm b is almost surely

[14] Note that the model of Section 4.2 also provides such an example since the default intensity is constant under the objective probability measure, but jumps at event dates under the risk-neutral measure.

constant under this probability measure. As a result, the conditional expectation in (4.10) can be easily computed for firm a (a symmetric expression holds for firm b) as

$$S_t^a(T) = \mathbf{1}_{\{\tau^a>t\}}\big[\mathbf{1}_{\{\tau^b>t\}}V_t^{1,a}(T) + \mathbf{1}_{\{\tau^b\le t\}}V_t^{2,a}(T)\big], \quad \text{where} \tag{4.11}$$

$$V_t^{2,a}(T) := e^{-(a_1+a_2)(T-t)}, \tag{4.12}$$

$$V_t^{1,a}(T) := \frac{a_2 e^{-(a_1+b_1)(T-t)} - b_1 e^{-(a_1+a_2)(T-t)}}{a_2 - b_1}. \tag{4.13}$$

The interpretation of the expression for $V^{2,a}$ is straightforward since, conditional on the event $\{\tau^b \le t\}$, the default arrival time of firm a has intensity $a_1 + a_2$. On the other hand, the expression for $V^{1,a}$ corresponds to computing the survival probability of firm a conditional on the event $\{\tau^b > t\}$, but effectively ignoring the potential impact of a jump in the intensity of firm b on the intensity of firm a. Again, the intuition is that since we are only interested in those paths where firm a does not default, we can ignore those paths where the intensity of firm b jumps before the survival horizon.

4.3.2. *Joint Distribution*

Our approach can also be used to derive the joint distribution of default times in the counterparty credit risk model. To this end, we start by observing that

$$P[\{\tau^a \le T, \tau^b \le U\}|\mathcal{F}_t] = 1 - S_t^a(T) - S_t^b(U) + S_t^{a,b}(T,U),$$

where we have set

$$S_t^{a,b}(T,U) := P[\{\tau^a > T, \tau^b > U\}|\mathcal{F}_t].$$

Thus, to obtain the joint distribution of default times we only need to compute the joint probability of survival $S^{a,b}$. Assuming, without loss of generality, that $U \ge T$, we find

$$\begin{aligned}
S_t^{a,b}(T,U) &= E\big[\mathbf{1}_{\{\tau^a>T\}}\mathbf{1}_{\{\tau^b>U\}}|\mathcal{F}_t\big] \\
&= E\big[\mathbf{1}_{\{\tau^a>T\}}\mathbf{1}_{\{\tau^b>T\}}V_T^{1,b}(U)|\mathcal{F}_t\big] \\
&= \mathbf{1}_{\{\tau^a>t\}}\mathbf{1}_{\{\tau^b>t\}}E^{a,b}\big[e^{-\int_t^T(\lambda_s^a+\lambda_s^b)\,ds}V_T^{1,b}(U)|\mathcal{F}_t^{a,b}\big] \\
&= \mathbf{1}_{\{\tau^a>t\}}\mathbf{1}_{\{\tau^b>t\}}e^{-(a_1+b_1)(T-t)}V_T^{1,b}(U),
\end{aligned}$$

where we have used the law of iterated expectations, (4.11), and the absolutely continuous change of probability measure

$$\left.\frac{dP^{a,b}}{dP}\right|_{\mathcal{F}_\infty} = Z_T^a Z_T^b$$

with augmented filtration ($\mathbb{F}^{a,b}$). Knowledge of the joint distribution can be particulary useful for analyzing securities that depend on the correlation structure of default events. Many credit derivatives such as first to default contracts, collateralized debt or loan obligations, or basket default options, for example, fall into that category. The next example shows how the counterparty structure could have a sizeable impact on the valuation of such a credit derivative.

4.3.3. *Collateralized Debt Obligation*

Here we investigate the impact of counterparty risk on the valuation of a collateralized debt obligation (CDO). Such an investment vehicle typically consists of an underlying pool of risky bonds, whose cash flows are repackaged into tranches of different seniority levels, which are then sold to investors. These tranches offer investors different levels of default-risk exposure.[15]

As an example, consider a CDO backed with a pool of two risky bonds $i \in \{a, b\}$, each with terminal payoff given by

$$\mathbf{1}_{\{\tau^i > T\}} + R\mathbf{1}_{\{\tau^i \le T\}}$$

for some constant recovery rate $R \le 1$. We specify that the risk-free rate is constant and that the risk-neutral default intensities for the default arrival times are given by (4.8). Using results from the previous paragraph, we find that the price of firm i's risky bond at date $t = 0$ is given by

$$B^i = e^{-rT}[R + (1 - R)V_t^{1,i}(T)].$$

We assume that the CDO is structured with two tranches: a safe tranche T^1 that pays one dollar unless both firms in the pool default, in which case it will pay R, and a 'toxic waste' tranche T^2 that is a claim to the residual cash flows of the underlying pool of bonds. Using the results from the previous section, we find that the initial values of these two tranches are given by

$$\begin{aligned} T^1 &= B^a + B^b - T^2, \\ T^2 &= e^{-rT} E\big[R + (1 - R)\mathbf{1}_{\{\tau^a > T\}}\mathbf{1}_{\{\tau^b > T\}}\big] \\ &= e^{-rT}\big[R + (1 - R)e^{-(a_1 + b_1)T}\big]. \end{aligned}$$

[15] Duffie and DeMarzo (1999) provide an analysis of the economic motives behind such structures.

In contrast to the pricing of individual bonds, the pricing of CDO's depends critically on the nature of default event risk correlation. For example, tranche T^1 is safer than either of the bonds in the pool since it experiences a short-fall only if both firms default. The likelihood of this occurring is clearly a function of any contagious response between the two firms. Hence, an incorrect assessment of that correlation could have detrimental consequences for investors. Indeed, suppose that investors assume that the risk-neutral default intensities of bonds a and b are constants (ℓ^a, ℓ^b). In such an i.i.d. world the prices of the risky bonds and various tranches would be:

$$B^i_{\text{iid}} = e^{-rT}[R + (1-R)e^{-\ell^i T}],$$
$$T^2_{\text{iid}} = e^{-rT}\left[R + (1-R)e^{-(\ell^a+\ell^b)T}\right],$$
$$T^1_{\text{iid}} = B^a_{\text{iid}} + B^b_{\text{iid}} - T^2_{\text{iid}}.$$

Since bond prices are observable, investors must value them correctly and hence must calibrate the risk-neutral default intensities (ℓ^a, ℓ^b) in such a way that $B^k_{\text{iid}} = B^k$ holds for both bonds. Using this calibration, it follows immediately from the definition of the tranches that

$$T^1_{\text{iid}} - T^1 = T^2 - T^2_{\text{iid}} = e^{-rT}(1-R)[S^{a,b}_0(T,T) - S^a_0(T)S^b_0(T)].$$

Now assume that the real model exhibits contagion in the sense that both a_2 and b_2 are positive. Using the results of the previous paragraph, we find that in this case

$$S^{a,b}_0(T,T) - S^a_0(T)S^b_0(T) = S^{a,b}_0(T,T)\left[1 - S^b_0(T)\frac{a_2 - b_1 e^{-(a_2-b_1)T}}{a_2 - b_1}\right] > 0.$$

Thus, investors are over-valuing the safe tranche and under-valuing the risky tranche if they assume an i.i.d. structure. The intuition behind this result is that the difference between the safer tranche T^1 and the riskier tranche T^2 can be viewed as a credit derivative that pays $(1-R)$ if and only if *exactly* one firm defaults before maturity. In a world with contagion, the probability that both firms will default by date T is larger than in the i.i.d. world, in turn reducing the value of that credit derivative.

Under the i.i.d. assumption, even though investors value risky bonds correctly, they are ignoring the counterparty-risk correlation structure which affects the valuation of CDOs.[16] Alternatively, this example suggests that if CDO tranches are correctly priced (i.e., if markets are efficient), then the joint

[16]It is interesting to observe that in practice investment banks tend to hold on to the so-called risky tranches and sell off the safe tranches to investors.

observation of CDO prices and corporate bond prices provides a mechanism for determining the correlation structure underpinning the risk-neutral default arrival times.

4.4. *Affine Jump Diffusions*

For our last example, we investigate a more general framework where both the risk-free rate r and default intensities are functions of an affine state vector process. Because this state process can jump at both default and nondefault dates, this example nests all previous deviations from the no-jump condition.

Following Duffie, Pan, and Singleton (2000), we start from a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and assume that $X : \mathbb{R}_+ \times \Omega \to D \subseteq \mathbb{R}^n$ is a strong Markov process solving the stochastic differential equation:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dY_t + \sum_{i=1}^{m} dQ_t^i, \tag{4.14}$$

where $\{\mu, \sigma\} : D \to \mathbb{R}^n \times \mathbb{R}^{n\times n}$ are deterministic functions, Y is an n-dimensional standard Brownian motion, and $(Q^i)_{i=1}^m$ are m strongly independent pure jump processes whose jumps have some probability distributions $(\nu^i)_{i=1}^m$ on $\mathbb{R}^n$ and arrive with intensities $(\lambda^i(X_t))_{i=1}^m$ for some deterministic functions $\lambda^i : D \to \mathbb{R}_+ \backslash \{0\}$.

The affine structure is imposed on the deterministic functions driving the stochastic differential equation (4.14) and on the risk-free interest rate function $r : D \to \mathbb{R}_+$ as follows:

(i) $\mu(x) = K_0 + K_1 x$, for some $(K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$;
(ii) $[\sigma\sigma^\top](x) = (H_0) + (H_1)\cdot x$, for some $(H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$;
(iii) $\lambda^i(x) = \ell_0^i + \ell_1^i x$, for some $(\ell_0^i, \ell_1^i) \in \mathbb{R} \times \mathbb{R}^{1\times n}$;
(iv) $r(x) = \rho_0 + \rho_1 x$, for some $(\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^{1\times n}$.

Now let $\mathbb{C}^n$ denote the set of n-dimensional complex numbers. We specify the jump distributions $(\nu^i)_{i=1}^m$ indirectly through their Fourier transforms:

$$\theta^i(c) := \int_{\mathbb{R}^n} e^{c\cdot z}\,d\nu^i(z), \quad c \in \mathbb{C}^n.$$

Sufficient restrictions on the coefficients (K, H, ℓ, ρ) that guarantee the existence and regularity of the process X for a given initial condition can be found in Duffie and Kan (1996), Duffie, Pan, and Singleton (2000), Duffie, Filipoviċ, and Schachermayer (2003) among others. For simplicity, we assume that P is the risk-neutral pricing measure.

Assume that default of firm 1 is triggered by the first increase of the counting process N^1 associated with the pure jump process Q^1 (see Brémaud (1981) for the definition of counting processes),

$$\tau^1 := \inf\{t \geq 0 : N_t^1 > 0\}.$$

Consider a defaultable bond issued by firm 1 whose recovery rate is equal to a constant fraction $(1-\delta)$ of its pre-default market value. Using Proposition 2, we find that prior to maturity the ex-dividend value of this defaultable bond is given by $S_t = \mathbf{1}_{\{\tau_1>t\}} V'_t$, where

$$V'_t := E'\left[e^{-\int_t^T [r+\delta\lambda^1](X_s)\,ds} | \mathcal{F}'_t\right].$$

Using the results of Lemma 1 in conjunction with the definition of the process X and well-known properties of affine models, we obtain an explicit expression for V':

$$V'_t = \exp[\alpha(t) + \beta(t)X_t], \tag{4.15}$$

where the deterministic functions $\alpha : [0, T] \to \mathbb{R}$ and $\beta : [0, T] \to \mathbb{R}^{1\times n}$ solve the system of ordinary differential equations (ODE's):

$$\partial_t\beta = \rho_1 + \delta\ell_1^1 - K_1^*\beta - \frac{1}{2}\beta^* H_1\beta - \sum_{i=2}^m \ell_1^i[\theta^i(\beta) - 1],$$

$$\partial_t\alpha = \rho_0 + \delta\ell_0^1 - K_0\beta - \frac{1}{2}\beta^* H_0\beta - \sum_{i=2}^m \ell_0^i[\theta^i(\beta) - 1],$$

subject to the boundary conditions $\alpha(T) = \beta(T) = 0$. The proof of this result, as well as sufficient conditions for the existence of solutions to the above system, can be found in Duffie, Pan, and Singleton (2000) and Duffie, Filipovič, and Schachermayer (2003) and thus is omitted.[17]

Due to the affine specification under consideration, we can explicitly characterize the impact of passing to the modified probability measure P' by computing the conditional expectation

$$V_t := E\left[e^{-\int_t^T [r+\delta\lambda^1](X_s)\,ds} | \mathcal{F}_t\right].$$

The latter maintains the exponential-affine structure, and is given by

$$V_t := \exp[A(t) + B(t)X_t],$$

[17]Note that if we set $R(x) := [r+\delta\lambda^1](x) = \rho_0 + \delta\ell_0^1 + (\rho_1 + \delta\ell_1^1)x$, then the valuation of our defaultable bond amounts to the computation of the conditional expectation

$$E'\left[e^{-\int_t^T R(X_s)\,ds} | \mathcal{F}'_t\right]$$

which is similar to the solution of a risk-free zero-coupon bond for some modified affine risk-free rate.

where the deterministic functions $A:[0,T]\to\mathbb{R}$ and $B:[0,T]\to\mathbb{R}^{1\times n}$ satisfy the system of ODE's:

$$\partial_t B = \rho_1 + \delta\ell_1^1 - K_1^* B - \frac{1}{2}B^* H_1 B - \sum_{i=1}^{m} \ell_1^i[\theta^i(B) - 1],$$

$$\partial_t A = \rho_0 + \delta\ell_0^1 - K_0 B - \frac{1}{2}B^* H_0 B - \sum_{i=1}^{m} \ell_0^i[\theta^i(B) - 1],$$

with the boundary condition $A(T) = B(T) = 0$. Comparing V' and V, we see that the two solutions are identical if and only if

$$\ell_1^1[\theta^1(B) - 1] = \mathbf{0} \quad \text{and} \quad \ell_0^1[\theta^1(B) - 1] = 0.$$

This condition is satisfied either if there is no default ($\ell^1 = \mathbf{0}$), or if the jump in V at the date of default is zero, which corresponds to the no-jump condition discussed previously. In some sense, the incorrect solution accounts twice for the jump to default. First, through the increased discount rate; second, through the impact of the jump to default on the distribution of interest rate and intensity used in the computation of the expectation. When using the modified probability measure P', only the former effect remains.

An interesting implication of the above results is that it is possible to obtain an exponential-affine expression for defaultable bonds even if the dynamics of the state vector are not affine. Indeed, suppose we modify slightly the model to allow the Fourier transform characterizing the jump distribution to be some function of the state:

$$\theta^1(t, c, \omega) := \theta^1(c, X_{t-}), \quad c \in \mathbb{C}^n.$$

Then, the solution of the defaultable bond is unchanged and thus is still given by (4.15). In contrast, V in general will not maintain its simple exponential-affine structure and may not possess an analytic solution at all.

REMARK 3: In the analysis of the counterparty credit risk model (equation (4.8)) used in the last example we have assumed that the intensity coefficients $\{a_1, a_2, b_1, b_2\}$ are constant. Note that it would be straightforward to relax this assumption by using the generalized structure of this section. For example, by specifying that the credit events of firm $i = 1, \ldots, m$ are triggered by the increases in the counting process N^i associated with the jump process Q^i, we obtain a model with m firms that face counterparty risk. Indeed, each firm's intensity will be affected by another firm's credit event through the jump in the state vector X. Our approach thus delivers closed-form solutions even in this more general case with time-varying parameters and an arbitrary number of firms sharing in the contagion.

5. CONCLUSION

It has been shown previously that, under a suitable no-jump condition, the price of a defaultable security is equal to its risk-neutral expected discounted cash flows if a modified discount rate is used to account for the possibility of default. In this paper, we generalize this result by demonstrating that even in cases where the no-jump condition is violated, the same risk-adjusted valuation formula obtains provided that the expectation is taken under a new probability measure. This probability measure is only absolutely continuous relative to the risk-neutral probability measure because it puts zero mass on paths where default occurs prior to the maturity. We investigate several examples where the no-jump condition is violated in order to illustrate the power and usefulness of our general formula.

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APPENDIX A: SOME TECHNICAL RESULTS

This appendix gathers technical results that either were discussed in the main part of the paper or are needed in the proof of our main results. The first of these results establishes a sufficient condition for the martingale property of a stochastic integral with respect to M and was used in passing from (3.1) to (3.3).

LEMMA A.1: *If the predictable process R satisfies condition* (3.2) *for some constant* $p \in [1, \infty)$, *then the process defined by*

$$I_t(R) := \int_0^t R_s\, dM_s = R_\tau \mathbf{1}_{\{\tau \le t\}} - \int_0^{\tau \wedge t} R_s \lambda_s\, ds$$

is a uniformly integrable martingale with respect to $(\mathbb{F}, P)$.

PROOF: The given process being a local martingale, all there is to prove is that it is uniformly integrable. Using the definition of the pure jump martingale M in conjunction with Davis' inequality we have that

$$E\left[\sup_{t\ge 0} |I_t(R)|\right] \le C \cdot E\left[\mathbf{1}_{\{\tau<\infty\}} |R_\tau|\right] \le C \cdot E\left[\sup_{t\ge 0} |R_t|\right]$$

holds for some nonnegative constant C, and the desired result now follows from condition (3.2). *Q.E.D.*

The next lemma establishes the martingale property of the single jump process Z defined by (3.6) and is the basis for the definition of the absolutely continuous probability measure P'.

LEMMA A.2: *Assume that condition* (2.3) *holds and fix some nonrandom time* $T \leq \vartheta$. *Then the nonnegative process defined by*

$$Z_t := \mathbf{1}_{\{\tau > t\wedge T\}} \Lambda_{t\wedge T} = \mathbf{1}_{\{\tau > t\wedge T\}} \exp\left(\int_0^{t\wedge T} \lambda_s \, ds\right)$$

is a uniformly integrable $(\mathbb{F}, P)$-*martingale*, *which is almost surely strictly positive on the stochastic interval* $[\![0, \tau[\![$.

PROOF: Using Itô's lemma for processes with jumps in conjunction with the definition of the pure jump martingale M, we have that

$$-dZ_t = \mathbf{1}_{\{t<T\}} Z_{t-} \, dM_t = \mathbf{1}_{\{t<T\}} Z_{t-} \big(d\mathbf{1}_{\{\tau \leq t\}} + \mathbf{1}_{\{\tau > t\}} \lambda_t \, dt\big).$$

It follows that Z is a $(\mathbb{F}, P)$-local martingale. Furthermore, due to the integrability condition (2.3), the maximal function $\sup_{t\geq 0} |Z_t|$ is integrable under the probability measure P. Hence, we conclude that the process Z is a uniformly integrable martingale. *Q.E.D.*

The following lemma establishes a form of Bayes' rule for conditional expectations under two absolutely continuous probability measures and will be used repeatedly in the proof of our main results.

LEMMA A.3: *Assume that the probability measure* Q *is absolutely continuous with respect to* P *and denote by* $(L_t)_{t\geq 0}$ *its density process*. *Then the conditional expectations formula*

$$E[L_\infty H | \mathcal{F}_t] = L_t E^Q[H | \mathcal{Q}_t]$$

holds for all Q-*integrable random variables*. *Here* $E^Q[\cdot]$ *denotes the expectation operator under the probability measure* Q *and the filtration* $\mathbb{Q} := (\mathcal{Q}_t)_{t\geq 0}$ *is the augmentation of* $\mathbb{F}$ *by the null sets of* Q.

PROOF: Let H be a random variable satisfying the conditions of the statement, fix an arbitrary $t \geq 0$, and define

$$C := L_t E^Q[H | \mathcal{Q}_t] = L_t E^Q[H | \mathcal{F}_t],$$

where the second equality follows from the definition of the augmented filtration $\mathbb{Q} = (\mathcal{Q}_t)_{t\geq 0}$. The random variable C being $\mathcal{F}_t$-measurable, all there is to establish in order to complete the proof of the lemma is that the identity $E[\mathbf{1}_{\{A\}} C] = E[\mathbf{1}_{\{A\}} L_T H]$ holds for any event $A \in \mathcal{F}_t$. Using the definition of the random variable C, we obtain

$$\begin{aligned}
E[\mathbf{1}_{\{A\}} C] &= E\big[\mathbf{1}_{\{A\}} L_t E^Q[H | \mathcal{F}_t]\big] \\
&= E\big[\mathbf{1}_{\{A\}} E[L_\infty | \mathcal{F}_t] E^Q[H | \mathcal{F}_t]\big] \\
&= E\big[\mathbf{1}_{\{A\}} L_\infty E^Q[H | \mathcal{F}_t]\big] = E^Q\big[\mathbf{1}_{\{A\}} E^Q[H | \mathcal{F}_t]\big] \\
&= E^Q[\mathbf{1}_{\{A\}} H] = E[\mathbf{1}_{\{A\}} L_\infty H],
\end{aligned}$$

where the second equality follows from the $(\mathbb{F}, P)$-martingale property of the process L; the third and fifth equalities follow from the law of iterated expectations and the fourth and last equalities follow from the definition of the density process. *Q.E.D.*

APPENDIX B: PROOFS

This appendix contains all the proofs of the results in the main part of the paper. We start with the proof of our main result, Theorem 1.

PROOF OF THEOREM 1: From Lemma A.2 we have that the nonnegative process Z is a uniformly integrable martingale and that P' is a well defined probability measure that is absolutely continuous with respect to P. Hence, it follows from Lemma A.3 that

$$Z_t E'[H|\mathcal{F}_t'] = \mathbf{1}_{\{\tau>t\}} \Lambda_t E'[H|\mathcal{F}_t'] = E[Z_T H|\mathcal{F}_t] \tag{B.1}$$

holds for all P'-integrable random variables. Let $t < T$ be an arbitrary but fixed time and denote by S' the process on the right-hand side of (3.5). Using the definition of Z and (3.3) we obtain

$$\begin{aligned} S_t &= E\left[\frac{B_t X}{B_T}\mathbf{1}_{\{\tau>T\}} + \int_t^T \mathbf{1}_{\{\tau>s\}} \frac{B_t R_s}{B_s} \lambda_s \, ds \middle| \mathcal{F}_t\right] \\ &= E\left[\frac{B_t X}{\Lambda_T B_T} Z_T + \int_t^T \frac{B_t R_s}{\Lambda_s B_s} Z_s \lambda_s \, ds \middle| \mathcal{F}_t\right] \\ &= E\left[\frac{B_t X}{\Lambda_T B_T} Z_T + \int_t^T \frac{B_t R_s}{\Lambda_s B_s} Z_T \lambda_s \, ds \middle| \mathcal{F}_t\right] \\ &= Z_t E'\left[\frac{B_t X}{\Lambda_T B_T} + \int_t^T \frac{B_t R_s}{\Lambda_s B_s} \lambda_s \, ds \middle| \mathcal{F}_t'\right] = S_t'. \end{aligned}$$

Here, the third equality follows from the law of iterated expectations and the $(\mathbb{F}, P)$-martingale property of the process Z, while the fourth follows from conditions (2.3) and (3.2), the definition of the probability measure P', Hölder's inequality, and (B.1). *Q.E.D.*

PROOF OF LEMMA 1: Since Z is a locally bounded process by construction, we have that the quadratic covariation process $\langle L, Z\rangle$ exists and satisfies

$$\langle L, Z\rangle_t = \int_0^t Z_{s-} \, d\langle L, M\rangle_s.$$

The results in the statement are now straightforward consequences of the Girsanov theorem for absolutely continuous probability measures, which was established by Lenglart (1977, Theorem 2 and 3). Hence, we omit the details. *Q.E.D.*

PROOF OF PROPOSITION 1: Let $h_t := E[H|\mathcal{F}_t]$ be the uniformly integrable martingale associated with the random variable

$$H := \frac{X}{\Lambda_T B_T} + \int_0^T \frac{\lambda_s R_s}{\Lambda_s B_s} \, ds.$$

Using the definition of the process Λ in conjunction with (3.2) and Doob's maximal inequality, it is easily seen that h is an $\mathcal{H}^p$ martingale under the probability measure P. On the other hand, combining (2.3) with Lemma A.2, we have that the process Z is an $\mathcal{H}^q$ martingale under the probability measure P and it thus follows from Theorem 10.39 in He, Wang, and Yan (1992) that the process

$$L_t := h_t Z_t - [h, Z]_t = h_t Z_t + \mathbf{1}_{\{\tau \le T\}} \Lambda_\tau \Delta h_\tau$$

is a uniformly integrable martingale under the probability measure P. Coming back to the proof itself, let S denote the ex-dividend price process of the security. Using the result of Theorem 1, we have that

$$\begin{aligned}\frac{S_t}{\Lambda_t B_t} + \int_0^t \mathbf{1}_{\{\tau>t\}} \frac{\lambda_s R_s}{\Lambda_s B_s}\, ds &= \mathbf{1}_{\{\tau>t\}} E'[H|\mathcal{F}'_t] \\ &= E[h_T Z_T|\mathcal{F}_t] = E\big[L_T - \mathbf{1}_{\{\tau\le T\}}\Lambda_\tau \Delta h_\tau|\mathcal{F}_t\big] \\ &= L_t - E\big[\mathbf{1}_{\{\tau\le T\}}\Lambda_\tau \Delta h_\tau|\mathcal{F}_t\big] \\ &= h_t Z_t - \mathbf{1}_{\{\tau>t\}} E\big[\mathbf{1}_{\{\tau\le T\}}\Lambda_\tau \Delta h_\tau|\mathcal{F}_t\big]\end{aligned}$$

holds for all $t < T$, where the second equality follows from (B.1), while the third and fourth equalities follow from the definition and martingale property of L. Using the definition of the process V in conjunction with the fact that $\Lambda_\tau \Delta h_\tau = B_\tau \Delta V_\tau$ gives the second equation in the statement and our proof is complete. *Q.E.D.*

PROOF OF THEOREM 2: The first part of the statement follows directly from the results of Antonelli (1993). In order to establish the second part, consider the adapted process defined by $S' = V'$ on the stochastic interval $[\![0, \tau \wedge T[\![$ and zero otherwise. Using (3.8), we find

$$\begin{aligned}S'_t &= \mathbf{1}_{\{\tau>t\}} E'\left[\frac{\Lambda_t B_t X}{\Lambda_T B_T} + \int_t^T \frac{\Lambda_t B_t}{\Lambda_s B_s} R_s(V'_{s-})\lambda_s\, ds \Big| \mathcal{F}'_t\right] \\ &= \mathbf{1}_{\{\tau>t\}} E'\left[\frac{\Lambda_t B_t X}{\Lambda_T B_T} + \int_t^T \frac{\Lambda_t B_t}{\Lambda_s B_s} R_s(V'_{s-})\lambda_s\, ds \Big| \mathcal{F}'_t\right] \\ &= E\left[\frac{B_t X}{B_T}\mathbf{1}_{\{\tau>T\}} + \int_t^T \frac{B_t R_s(S'_{s-})}{B_s}\mathbf{1}_{\{\tau>s\}}\lambda_s\, ds \Big| \mathcal{F}_t\right],\end{aligned} \tag{B.2}$$

where the second equality follows from the fact that $\{\tau \le T\}$ is a null set of the probability measure P' and the third equality follows from (B.1), the definition of the process Z, and the fact that, by assumption, we have $R(0) = 0$ almost everywhere. In order to complete our proof it is now sufficient to show that the local martingale

$$I_t := \int_0^t \frac{R_s(S'_{s-})}{B_s}\, dM_s$$

is in fact a uniformly integrable $(\mathbb{F}, P)$-martingale since this will imply that S' satisfies the fundamental valuation formula (3.1). To establish this martingale property we argue as follows. Using the nonnegativity of the processes r and λ in conjunction with equation (B.2) and the uniform Lipschitz condition (3.7), we have that

$$|S'_t| \le E\left[\mathbf{1}_{\{\tau>T\}}|X| + \int_0^T \mathbf{1}_{\{\tau>s\}} k|V'_{s-}|\, ds \Big| \mathcal{F}_t\right]$$

holds almost everywhere for some nonnegative constant k. Applying Davis' inequality and using the above expression in conjunction with the law of iterated expectations, we obtain that

$$\begin{aligned}E\left[\sup_{t\ge0}|I_t|\right] &\le CE\big[\mathbf{1}_{\{\tau\le T\}}|S'_{\tau-}|\big] \le CE[|S'_{\tau-}|] \\ &\le CE\left[\mathbf{1}_{\{\tau>T\}}|X| + \int_0^T \mathbf{1}_{\{\tau>s\}} k|V'_{s-}|\, ds\right] \\ &\le CE[|X|] + CE'\left[\int_0^T k|V'_{s-}|\, ds\right]\end{aligned}$$

holds for some nonnegative constant C where the equality follows from the definition of S' and the last inequality follows from the definition of the probability measure P'. The first part of the statement and (3.2) then imply that the right-hand side of the above string of inequalities is finite and the desired martingale property follows. *Q.E.D.*

PROOF OF PROPOSITION 2: In order to establish the desired result we start by solving the recursive equation (3.8) associated with the given default payoff process. Consider the $\mathbb{F}'$-adapted process defined by

$$V'_t := E'\left[e^{-\int_t^T \delta_s\lambda_s\,ds}\frac{B_tX}{B_T}\Big|\mathcal{F}'_t\right].$$

As is easily seen, this process is uniformly integrable under P' and has the property that the process

$$L_t := E'\left[e^{-\int_0^T \delta_s\lambda_s\,ds}\frac{X}{B_T}\Big|\mathcal{F}'_t\right] = e^{-\int_0^t \delta_s\lambda_s\,ds}\frac{V'_t}{B_t}$$

is uniformly integrable under P'. Using these facts in conjunction with Itô's product rule and the nonnegativity of $(1-\delta)\lambda$, we obtain that the process

$$\frac{V'_t}{\Lambda_tB_t} + \int_0^t \frac{\lambda_s(1-\delta_s)V'_s}{\Lambda_sB_s}\,ds = \int_0^t e^{-\int_0^s (1-\delta_u)\lambda_u\,du}\,dL_u$$

is a uniformly integrable martingale under the probability measure P' and it follows that V' solves the recursive equation (3.8) with the given default payoff process. The remaining claim in the statement now follows from an argument similar to that used in the second part of the proof of Theorem 2; we omit the details. *Q.E.D.*

PROOF OF PROPOSITION 3: Starting from the same space $(\Omega, \mathcal{F}, \mathbb{F})$ as before, we take as given in our construction a probability measure Q and an $\mathbb{F}$-stopping time τ such that:

(i) the filtration $\mathbb{F}$ is complete with respect to the null sets of the probability measure Q;

(ii) the stopping time τ is totally inaccessible and has constant intensity $\ell > 0$ under the probability measure Q.

As in our main result, Theorem 1, let us define an absolutely continuous probability measure by setting

$$\frac{dQ'}{dQ}\bigg|_{\mathcal{F}_t} = \mathbf{1}_{\{\tau>t\wedge T\}}e^{\ell(t\wedge T)},$$

and denote by $\mathbb{Q}' := (\mathcal{Q}_t)_{t\geq0}$ the corresponding augmented filtration. Consider the recursive stochastic integral equation

$$U_t = E_{Q'}\left[\frac{\Lambda_tB_tX}{\Lambda_TB_T} + \int_t^T \frac{\Lambda_tB_t}{\Lambda_sB_s}\lambda_s(U_{s-})R_s(U_{s-})\,ds\,\Big|\mathcal{Q}'_t\right],$$

where the nonnegative, increasing, and adapted process Λ is defined as in (3.4) with $\lambda = \lambda(U_-)$. Using the assumptions of the statement in conjunction with the results of Antonelli (1993), we have that there exists a unique bounded solution $(U'_t)_{t\geq0}$ to this recursive equation in the space of $\mathbb{Q}'$-adapted processes.

Note that $\lambda(U_-)$ is *not* the intensity of the default time under the probability measure Q. Nevertheless, with all of this in place, we can now define a new probability measure under which the default time does admit $\lambda(U_-)$ as its intensity process. To this end, consider the probability measure defined by

$$\frac{dP}{dQ}\bigg|_{\mathcal{F}_t} = Y_t := \left[1 + \mathbf{1}_{\{\tau\leq t\}}\frac{\lambda_\tau(U_{\tau-})-\ell}{\ell}\right]\frac{e^{\ell(\tau\wedge t)}}{\Lambda_{\tau\wedge t}}.$$

Applying Itô's product rule and using the boundedness of λ, we deduce that the process Y is a strictly positive $(\mathbb{F}, Q)$-martingale. It follows that P is an equivalent probability measure. Now consider the bounded process defined by $S = U$ on the stochastic interval $[\![0, \tau \wedge T[\![$ and zero otherwise. Using this notation and applying Girsanov's theorem, we obtain that the stopping time τ has intensity

$$\mathbf{1}_{\{\tau>t\}}\lambda_t(U_{t-}) = \mathbf{1}_{\{\tau>t\}}\lambda_t(S_{t-})$$

under the probability measure P and all there remains to establish in order to complete the proof is that the candidate security price process satisfies the fundamental valuation equation (3.1).

To this end, let P' be the absolutely continuous probability measure defined by (3.6) with $\lambda = \lambda(U_-)$ and denote by $\mathbb{F}' := (\mathcal{F}'_t)_{t\geq 0}$ the corresponding augmented filtration. Using the definition of the probability measures Q' and P, we obtain that

$$\begin{aligned}\frac{dP'}{dQ}\bigg|_{\mathcal{F}_t} &= \left(\frac{dP'}{dP}\bigg|_{\mathcal{F}_t}\right)\cdot\left(\frac{dP}{dQ}\bigg|_{\mathcal{F}_t}\right) = \mathbf{1}_{\{\tau>t\wedge T\}}\Lambda_{t\wedge T}\cdot Y_t \\ &= \mathbf{1}_{\{\tau>t\wedge T\}}e^{\ell(t\wedge T)} = \frac{dQ'}{dQ}\bigg|_{\mathcal{F}_t}\end{aligned}$$

holds almost everywhere and conclude that the probability measures P' and Q' coincide. As a result, the corresponding augmented filtrations must also coincide and we have that the process $(U_t)_{t\geq 0}$ is the unique bounded and $\mathbb{F}'$-adapted solution to the recursive equation

$$U_t = E'\left[\frac{\Lambda_t B_t X}{\Lambda_T B_T} + \int_t^T \frac{\Lambda_t B_t}{\Lambda_s B_s}\lambda_s(U_{s-})R_s(U_{s-})\,ds\bigg|\mathcal{F}'_t\right].$$

The fact that the candidate price process satisfies the fundamental valuation (3.1) now follows from an argument similar to that used in the second part of the proof of Theorem 2. For the sake of brevity we omit the details. *Q.E.D.*

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