Online appendix to:

Credit market frictions and capital structure dynamics

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This document provides the proofs of all the results in the main text. To avoid confusions the numbering of equations and figures is continued from the text.

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A Equilibria with a fixed default threshold

Fix a default threshold $X_d(1) = 1/Z_d$ and denote by τ_d the corresponding default time. The scale invariance of the geometric Brownian motion and the linearity of the payoffs to the firm's stakeholders imply that the equilibrium value functions are homogeneous of degree one with respect to the firm's current coupon and cash flows. That is, there are functions $d(Z|Z_d), e(Z|Z_d), v(Z|Z_d)$ such that

$$
\begin{aligned}
D(X, C|\mathbf{P}) &= Xd(Z|Z_d), \\
E(X, C|\mathbf{P}) &= Xe(Z|Z_d),
\end{aligned}
$$

and

$$V(X, C|\mathbf{P}) = E(X, C|\mathbf{P}) + D(X, C|\mathbf{P}) = Xv(Z|Z_d)$$

where $Z = C/X$. A direct calculation using well-known properties of geometric Brownian motion shows that the reduced form debt value is given by

$$d(Z|Z_d) = (Z/r) - (Z_d/r)(1 - r\phi/Z_d)(Z_d/Z)^{\beta}.$$

With a fixed default policy, maximizing the equity value is equivalent to maximizing firm value. It turns out that, for technical reasons, firm value maximization problem is easier to deal with. For this reason, whenever the fixed default policy case is considered in the Appendix, we will always study the latter problem. By definition, the equilibrium firm value is the solution to the stochastic control problem defined by

$$V(X, C|\mathbf{P}) = \sup_{a \in \mathbb{A}} E\left[\int_0^{\tau_d} e^{-rs}((1-\tau)X_s + \tau C_{s-})ds + \tilde{H}(a_s, X_s, C_{s-}|\mathbf{P})dN_s) + e^{-r\tau_d}\phi X_{\tau_d}\right] \tag{9}$$

where we have set

$$\tilde{H}(a, X, C|\mathbf{P}) = X\tilde{h}(a, Z|Z_d) = -\eta(V(X, aC|\mathbf{P}) - V(X, C|\mathbf{P})) - (1-\eta)qD(X, aC|\mathbf{P}).$$

The following result follows by direct calculation and allows to recast the equilibrium problem in terms of a single state variable.

Lemma A.1 *The process $Z_t = C_t/X_t$ evolves according to*

$$dZ_t = -\mu Z_{t-}dt - \sigma Z_{t-}d\hat{B}_t + Z_{t-}(a_t - 1)dN_t \tag{10}$$

where the process $\hat{B}_t$ is a standard one dimensional Brownian motion under the equivalent probability measure defined by

$$\hat{P}(A) = E\left[e^{-\mu t}\frac{X_t}{X_0}1_{\{A\}}\right], \qquad \forall A \subseteq \mathscr{F}_t$$

Consequently, (9) is equivalent to

$$v(Z|Z_d) = \sup_{a\in\mathbb{A}} \hat{E}\left[\int_0^{\tau_d} e^{-(r-\mu)s}((1-\tau+\tau Z_{s-})ds + h(a_s, Z_{s-}|Z_d)dN_s) + e^{-(r-\mu)\tau_d}\phi\right]$$

and any solution to this equation satisfies the inequalities

$$0 \leq v(Z|Z_d) \leq \frac{1-\tau(1-Z_d)}{r-\mu}.$$

Now, standard dynamic programming arguments imply

Lemma A.2 *Let τ_N denote the first jump of the Poisson process and define*

$$\mathscr{P}(v)(Z) = \max_{a\geq 1}(1-\eta)(v(aZ) - qd(aZ|Z_d)) + \eta v(Z).$$

Then the function $v(Z|Z_d)$ is the unique Borel-measurable, bounded function satisfying

$$\begin{aligned} v(Z|Z_d) = \hat{E}\bigg[\int_0^{\tau_d\wedge\tau_N} e^{-(r-\mu)t}\,(1-\tau+\tau Z_t)\,dt + 1_{\{\tau_d\leq\tau_N\}}e^{-(r-\mu)\tau_d}\phi \\ + 1_{\{\tau_d>\tau_N\}}e^{-(r-\mu)\tau_N}\mathscr{P}(v(\cdot|Z_d))(Z_{\tau_N-})\bigg] \end{aligned} \tag{11}$$

We then have the following result.

Lemma A.3 *The transformation that maps a function v into the right-hand side of equation (11) is a contraction in the space $L_\infty[0, Z_d]$ of essentially bounded measurable functions and has a unique fixed point that belongs to the space $C[0, Z_d]$ of continuous functions on $[0, Z_d]$. Consequently, $v(Z|Z_d)$ is this unique fixed point.*

Proof. Let $A(v)$ denote the operator in the statement. Using the fact that τ_N is independent of the Brownian motion and exponentially distributed with parameter λ we obtain

$$A(v)(Z) = \hat{E}\left[\int_0^{\tau_d} e^{-(r-\mu+\lambda)t}\left(1 - \tau + \tau Z_t^0 + \lambda \mathscr{P}(v)(Z_s^0)\right) dt + e^{-(r-\mu+\lambda)\tau_d}\phi\right].$$

where the nonnegative process Z_t^0 evolves according to equation (10) with $a \equiv 1$. For any pair of continuous functions (v_1, v_2) and any $z \in [0, Z_d)$, let

$$a_i = \arg\max_{a \geq 1}\left(v_i(az) - qd(az)1_{\{a>1\}}\right)$$

and assume for simplicity that $a_i > 1$. Then, we have

$$\begin{aligned}
\mathscr{P}(v_1)(z) - \mathscr{P}(v_2)(z) &= (1-\eta)(v_1(a_1 z) - qd(a_1 z)) \\
&\quad - (1-\eta)(v_2(a_2 z) - qd(a_2 z)) + \eta(v_1(z) - v_2(z)) \\
&\leq (1-\eta)(v_1(a_1 z) - qd(a_1 z)) \\
&\quad - (1-\eta)(v_2(a_1 z) - qd(a_1 z)) + \eta\|v_1 - v_2\|_{C[0,z_d]} \\
&= (1-\eta)(v_1(a_1 z) - v_2(a_1 z)) + \eta\|v_1 - v_2\|_{C[0,z_d]} \\
&\leq \|v_1 - v_2\|_{C[0,z_d]}.
\end{aligned}$$

and interchanging the roles of v_1 and v_2 we get that

$$|\mathscr{P}(v_1)(z) - \mathscr{P}(v_2)(z)| \leq \|v_1 - v_2\|_{C[0,Z_d]}.$$

This immediately implies that

$$\|A(v_1) - A(v_2)\|_{C[0,Z_d]} \leq \frac{\lambda\|v_1 - v_2\|_{C[0,Z_d]}}{r - \mu + \lambda}.$$

and the desired result now follows from the fact that $r - \mu > 0$ by assumption. ■

Lemma A.4 *The map $A(v)$ is monotone increasing in v, and monotone decreasing in r, η, ω and q for any nonnegative function v.*

Proof. To prove monotonicity in v it suffices to show that $\mathscr{P}$ is increasing in v. This is obvious because $\eta \in (0,1)$. Monotonicity in r and q is also clear. To prove monotonicity in η, it suffices to show that the operator $\mathscr{P}$ is monotone decreasing in η. Fix $z \geq 0$ and consider

$$G(\eta) = \max_{a \geq 1} \left\{ \eta v(z) + (1-\eta)(v(az) - qd(az) 1_{\{a>1\}}) \right\}$$

If the maximum is attained for some $a > 1$ then we clearly must have $v(az) - qd(az) > v(z)$. This in turn implies that we have

$$G(\eta) = \max\{v(z), k(\eta)\}$$

with

$$k(\eta) = \max_{a > 1} \left\{ \eta v(z) + (1-\eta)(v(az) - qd(az)) 1_{\{v(az) - qd(az) > v(z)\}} \right\}$$

and the desired result follows by noting that $k(\eta)$ is monotone decreasing. ■

Lemma A.5 *The firm value function $v(Z)$ is monotone increasing in λ, μ and monotone decreasing in q, η, r and ω. As a result, the equity value function $e(Z)$ is monotone increasing in λ and monotone decreasing in q and η.*

Proof. Pick an arbitrary bounded function $v_0 \in C[0, Z_d]$ and denote by A^n the n-th iteration of A so that $v = \lim_n A^n(v_0)$. Let α be a parameter with respect to which the operator A is increasing in the sense that

$$A(v_0; \alpha_1) < A(v_0; \alpha_2), \qquad \forall \alpha_1 < \alpha_2\ ,\ \forall v_0 \in C[0, Z_d]$$

Since A is increasing in v, a simple induction argument implies that we have

$$A^n(v_0; \alpha_1) < A^n(v_0; \alpha_2), \qquad \forall \alpha_1 < \alpha_2, n \geq 1.$$

Sending $n \to \infty$ shows that v is increasing in α and monotonicity in all parameters except λ now follows from Lemma A.4. Finally, monotonicity in λ follows from that in η because the firm value function depends on λ only through $\lambda(1-\eta)$.

The remaining claims follow from the first part of the statement and the fact that $e(Z) = v(Z) - d(Z)$ where the function $d(Z)$ is independent from λ, q and η. ■

Lemma A.3 constructs the value function as a continuous fixed point of a non-linear map. We will now show that it is in fact in $C^2[0, Z_d]$ and solves the corresponding HJB equation. To this end, we will first need the following lemma. Let

$$\mathscr{L} f(Z) \equiv -\mu Z f'(Z) + \frac{1}{2}\sigma^2 Z^2 f''(Z)$$

be the continuous part of the $\hat{P}-$generator of the state variable and denote by $\psi < 0 < 1 < \psi_1$ the solutions to the quadratic equation $Q(x; r + \lambda(1-\eta)) = 0$

Lemma A.6 *Let $f(Z)$ be a bounded and Borel measurable function. The unique bounded solution to*

$$(r - \mu + \lambda)Y(Z) = \mathscr{L}Y(Z) + f(Z), \qquad Z \in [0, Z_d),$$

such that $Y(Z_d) = \phi$ is explicitly given by

$$Y(Z) = y_1\, Z^{1-\psi} - \frac{Z^{1-\psi_1}}{\sigma^2/2}\int_0^Z \frac{f(x)}{\psi_1 - \psi} x^{\psi_1 - 2} dx - \frac{Z^{1-\psi}}{\sigma^2/2}\int_Z^{Z_d} \frac{f(x)}{\psi_1 - \psi} x^{\psi - 2}\, dx$$

with

$$y_1 = Z_d^{\psi - 1}\left(\frac{Z_d^{1-\psi_1}}{\sigma^2/2}\int_0^{Z_d} \frac{f(x)}{\psi_1 - \psi} x^{\psi_1 - 2} dx - \phi\right)$$

In particular, the derivative $Y'(Z_d)$ depends continuously on Z_d and $f(Z)$ in the $L_\infty[0, Z_d]-$topology.

This existence and uniqueness result immediately allows us to establish the required regularity of the firm value function.

Lemma A.7 *Let*

$$\mathscr{O}(v)(Z) = \max_{a\geq 1}\big(v(aZ) - qd(aZ)1_{\{a>1\}} - v(Z)\big) \tag{12}$$

For a fixed default threshold the equilibrium firm value function $v(Z|Z_d)$ is the unique $C^2[0,Z_d]$ solution to the HJB equation

$$(r-\mu)v(Z) = \mathscr{L}v(Z) + 1 - \tau + \tau Z + \lambda(1-\eta)\mathscr{O}(v)(Z) \tag{13}$$

with boundary condition $v(Z_d|Z_d) = \phi$.

Proof. Let $Y \in C^2[0,Z_d]$ be the unique bounded solution to

$$(r-\mu+\lambda)Y(Z) = \mathscr{L}Y(Z) + 1 - \tau + \tau Z + \lambda\mathscr{P}(v(\cdot|Z_d))(Z), \tag{14}$$

such that $Y(Z_d) = \phi$ whose existence is provided by Lemma A.6. Then, standard arguments based on Itô's lemma imply that

$$Y(Z) = E_0\left[\int_0^{\tau_d} e^{-(r-\mu+\lambda)t}\left(1-\tau+\tau Z_t^0 + \lambda\mathscr{P}(v)(Z_{t-}^0)\right)dt + e^{-(r-\mu+\lambda)\tau_d}\phi\right]$$

and it now follows from Lemma A.2 that $Y(Z) = v(Z|Z_d)$. Given this identity, a direct calculation implies that (14) is equivalent to (13). ■

Lemma A.8 *Suppose that $\mathscr{A} : C[0,Z_d] \to C[0,Z_d]$ is a contraction which is monotone in the sense that $v_1 \leq v_2$ implies $\mathscr{A}v_1 \leq \mathscr{A}v_2$, and denote by $v \in C[0,Z_d]$ its unique fixed point. If $w \in C[0,Z_d]$ is such that $w \leq \mathscr{A}w$ then $w \leq v$. Similarly, if $w \geq \mathscr{A}w$ then $w \geq v$.*

Proof. Monotonicity of A together with $w \geq Aw$ implies $w \geq A^n w$ for any $n \geq 1$. Therefore, $w \geq \lim_{n\to\infty} A^n w = v$ and the claim follows. ■

The following lemma directly implies the results of Propositions 2 and 4 and will be of repeated use in what follows. Let

$$Z_d^{\mathrm{nr}} \equiv \frac{\beta-1}{\beta}\frac{r}{r-\mu}$$

denote the optimal default threshold that would prevail in a model where the firm is not allowed to restructure its debt.

Lemma A.9 *Let $\hat{v}(Z|Z_d)$ be the value of a firm that never restructures its debt and defaults at the stopping time τ_d. Then $v(Z|Z_d) \geq \hat{v}(Z|Z_d) > \phi$ for all $Z \in [0, Z_d)$ and we have*

1. *If $q \geq \tau$ then $v(Z|Z_d) \equiv \hat{v}(Z|Z_d)$, the unique equilibrium threshold is given by Z_d^{nr} and the equity value function satisfies $e_Z(Z_d|Z_d) > 0$ for all $Z_d > Z_d^{\mathrm{nr}}$;*

2. *If $q < \tau$ then $v(Z|Z_d^{\mathrm{nr}}) \not\equiv \hat{v}(Z|Z_d^{\mathrm{nr}})$ and the equity value function satisfies $e_Z(Z_d|Z_d) < 0$ for all $Z_d < Z_d^{\mathrm{nr}}$. In particular, we have $Z_d \geq Z_d^{\mathrm{nr}} > 1$ in any equilibrium.*

Proof. Since $v(Z|Z_d)$ is the value function of a firm following an optimal policy, it dominates the sub-optimal policy of never restructuring. The value of such a firm is

$$\hat{v}(Z|Z_d) = \phi_0 + \frac{\tau Z}{r} - \left(\phi_0 \omega + \tau Z_d / r\right) \left(Z/Z_d\right)^{1-\beta} \tag{15}$$

and satisfies

$$\hat{v}(Z|Z_d) - \phi = \omega \phi_0 \left[1 - (Z/Z_d)^{1-\beta}\right] + \frac{\tau}{r} Z \left[1 - (Z/Z_d)^{-\beta}\right] > 0\,. \tag{16}$$

for all $Z < Z_d$ since $\beta < 0$. Let $\tilde{v}(Z|Z_d) \equiv \hat{v}(Z|Z_d) - q d(Z|Z_d)$ so that

$$\tilde{v}'(Z|Z_d) = \frac{\tau - q}{r} \left[1 - (1-\beta)(Z_d/Z)^{\beta}\right] - \phi_0(\omega + q(1-\omega))(1-\beta) Z_d^{-1} (Z_d/Z)^{\beta}. \tag{17}$$

and assume first that $q \geq \tau$. To prove that $v(Z|Z_d) \equiv \hat{v}(Z|Z_d)$ we need to show that

$$\mathscr{O}(\hat{v}(\cdot|Z_d)) \equiv 0. \tag{18}$$

We have $\tilde{v}'(0|Z_d) = (\tau - q)/r \leq 0$ and since

$$\tilde{v}''(Z|Z_d) = \beta(\beta-1) Z_d^{\beta-1} Z^{-\beta-1} \left(\frac{q-\tau}{r} Z_d - \phi_0(\omega + q(1-\omega))\right)$$

does not change sign we have that the function $\tilde{v}(Z|Z_d)$ is either convex, or concave and decreasing. If $\tilde{v}(Z|Z_d)$ is decreasing then (18) clearly holds. On the other hand, if $\tilde{v}(Z|Z_d)$

is convex then (16) implies that we have

$$\max_{y\in(Z,Z_d]} \tilde{v}(y|Z_d) = \max\{\tilde{v}(Z|Z_d)\,,\,\tilde{v}(Z_d|Z_d)\} = \max\{\tilde{v}(Z|Z_d)\,,\,(1-q)\phi\} < \hat{v}(Z|Z_d)$$

and (18) follows. To complete the proof of the first part, set $q = 1$ in (17) to obtain

$$e_Z(Z_d|Z_d) = \frac{\tau\,-1}{r}\beta - (1-\beta)Z_d^{-1}\phi_0.$$

This shows that $e_Z(Z_d|Z_d)$ is positive for $Z_d > Z_d^{\mathrm{nr}}$ and negative for $Z_d < Z_d^{\mathrm{nr}}$ and implies that the desired result. Consequently, $e(Z|Z_d^{\mathrm{nr}})$ is C^1 and satisfies the HJB equation

$$\begin{aligned} 0 = \max\{ &-e(Z|Z_d^{\mathrm{nr}}), \\ &-(r-\mu)e(Z|Z_d^{\mathrm{nr}}) + \mathscr{L}e(Z|Z_d^{\mathrm{nr}}) \\ &+(1-\tau)(1-Z) + \lambda\,(1-\eta)\mathscr{O}(e(Z|Z_d^{\mathrm{nr}}) + d(Z|Z_d^{\mathrm{nr}}))\} \end{aligned}$$

Standard verification results for optimal stopping problems (see, e.g., Dayanik and Karatzas (2003)) combined with Lemma A.7 implies that Z_d^{nr} is the optimal default boundary.

Let now $q < \tau$ and suppose on the contrary that $v(Z|Z_d) \equiv \hat{v}(Z|Z_d)$. To reach a contradiction, it suffices to show that $\mathscr{O}(\hat{v}) \not\equiv 0$. By (15) we have that

$$\tilde{v}(Z|Z_d) = \phi_0 + \frac{\tau - q}{r} - \tilde{a}Z^{1-\beta}\,, \tag{19}$$

where

$$\tilde{a} = -a - (q/r)(1 - r\phi/Z_d)\,Z_d^{\beta} = \frac{\tau - q}{r}Z_d^{\beta} + \phi_0(\omega + q(1-\omega))Z_d^{\beta-1} > 0.$$

It follows that the function $\tilde{v}(Z|Z_d)$ is concave and therefore attains a global maximum at the unique point Z_o such that

$$\tilde{v}_Z(Z_o|Z_d) = 0 \Longleftrightarrow \frac{(\tau - q)Z_o}{r} = \tilde{a}\,(1-\beta)\,Z_o^{1-\beta}\,.$$

Substituting this identity into (19) gives

$$\max_{y\geq 0} \tilde{v}(y|Z_d) = \tilde{v}(Z_o|Z_d) = \phi_0 + \frac{(\tau - q)Z_o}{r} - \frac{(\tau - q)Z_o}{r(1-\beta)} > \phi_0 = \hat{v}(0|Z_d)$$

and it follows that $\mathscr{O}(\hat{v})(0) > 0$ which is a contradiction. To prove the remaining claims in the statement we will use the fact that by definition $e(Z|Z_d) \geq \hat{e}(Z|Z_d)$. By the first part of the statement we have $\hat{e}_Z(Z|Z_d) < 0$ for all $Z_d < Z_d^{\mathrm{nr}}$ and it follows that

$$e(Z|Z_d) > \hat{e}(Z|Z_d) > \epsilon(Z_d - Z)$$

for some $\epsilon > 0$ in a left neighborhood of $Z_d < Z_d^{\mathrm{nr}}$ since $e(Z_d|Z_d) = 0$. This immediately implies that $e_Z(Z_d|Z_d) < 0$ for all $Z_d < Z_d^{\mathrm{nr}}$ which is what had to be proved. ■

To establish the existence of Markov perfect equilibria in barrier strategies, we will need an auxiliary construction.

Let $\tilde{e}(Z|\bar{Z}_d)$ be the equity value for a firm whose debt-holders price the debt using $\tilde{d}(Z)$, whereas the firm actually defaults when $Z = \bar{Z}_d$. We will prove several properties of this function. The first is provided in the following lemma.

Lemma A.10 *$\tilde{e}(Z|\bar{Z}_d)$ is C^2 in Z on $[0, \bar{Z}_d]$ and $e_Z(\bar{Z}_d|\bar{Z}_d)$ is continuous in $Z_d, \bar{Z}_d$.*

Proof. The claim follows from Lemma A.6 by the same arguments as Lemma A.7. ■

Lemma A.11 *We have $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d) < 0$ for $\bar{Z}_d \leq Z_d^{\mathrm{nr}}$.*

Proof. Since the optimal equity value function dominates the equity value value of a firm that never restructures, we have that $\tilde{e}(Z|\bar{Z}_d) >$ is positive for all $Z \leq \bar{Z}_d \leq Z_d^{\mathrm{nr}}$ and therefore $\tilde{e}_Z(Z_d|Z_d) < 0$ for all $Z_d < Z_d^{\mathrm{nr}}$. ■

Lemma A.12 *If $\bar{Z}_d^1 > \bar{Z}_d^2$ and $\tilde{e}(\bar{Z}_d^2|\bar{Z}_d^1) \geq 0$. Then, $\tilde{e}(Z|\bar{Z}_d^1) \geq \tilde{e}(Z|\bar{Z}_d^2)$ for all $Z \leq \bar{Z}_d^2$.*

Proof. Let $\tilde{d}(Z)$ be the function that the debt-holders use to value debt. Denote by $\tau_{d,i}$ the first time that the process Z_t hits $\bar{Z}_d^i$ and observe that $\tau_{d,1} > \tau_{d,2}$. Then, it follows directly by standard dynamic programming arguments that

$$\begin{aligned}
e(Z|Z_d^1) &= \hat{E}\bigg[\int_0^{\tau_{d,2}\wedge\tau_N} e^{-\rho t}(1-\tau)(1-Z_t)dt + 1_{\{\tau_{d,2}\le\tau_N\}}e^{-\rho\tau_{d,2}}\tilde{e}(\bar{Z}_d^2|\bar{Z}_d^1) \\
&\quad + 1_{\{\tau_{d,2}>\tau_N\}} \max_{a\in[1,\bar{Z}_d^1/Z_{\tau_N-}]} e^{-\rho\tau_N}\big[(1-\eta)\big(\tilde{v}(aZ_{\tau_N-}|Z_d^1) - q1_{\{a>1\}}\tilde{d}(aZ_{\tau_N-}) - \tilde{d}(Z_{\tau_N-})\big) \\
&\quad + \eta e(Z_{\tau_N-}|Z_d^1)\big)\bigg] \\
&\geq \hat{E}\bigg[\int_0^{\tau_{d,2}\wedge\tau_N} e^{-\rho t}(1-\tau)(1-Z_t)dt \qquad (20) \\
&\quad + 1_{\{\tau_{d,2}>\tau_N\}} \max_{a\in[1,\bar{Z}_d^2/Z_{\tau_N-}]} e^{-\rho\tau_N}\big((1-\eta)\big(\tilde{v}(aZ_{\tau_N-}|Z_d^1) - q1_{\{a>1\}}\tilde{d}(aZ_{\tau_N-}) - \tilde{d}(Z_{\tau_N-})\big) \\
&\quad + \eta e(Z_{\tau_N-}|Z_d^1)\big)\bigg]
\end{aligned}$$

for all $Z\in[0,\bar{Z}_d^2]$ where $\rho = r-\mu$ and $\tilde{v}(z|Z_d) = \tilde{e}(z,|Z_d) - \tilde{d}(z)$ denotes the corresponding firm value function. Note that we only take the maximum over $a\in[1,\bar{Z}_d^1/Z_{\tau_N-}]$ because, by assumption, the firm always defaults when $Z\geq\bar{Z}_d^1$. Denote the map on the right-hand side of (20) by $\mathscr{A}$. The same arguments as in the proof of Lemma A.3 imply that the operator $\mathscr{A}$ is a monotone contraction and the required assertion now follows from Lemma A.8 since we have $\tilde{e}(Z|\bar{Z}_d^1)\geq\mathscr{A}\tilde{e}(Z|\bar{Z}_d^1)$, and $\tilde{e}(Z|\bar{Z}_d^2)=\mathscr{A}\tilde{e}(Z|\bar{Z}_d^2)$ by the above. ■

Lemma A.13 *Fix an arbitrary $\bar{Z}_d>0$ and suppose that $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d)<0$. Then, $\tilde{e}(Z|\bar{Z}_d)$ is monotone increasing in $\bar{Z}_d$ for $\bar{Z}_d$ in a left neighborhood of $\bar{Z}_d$. Similarly, if $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d)>0$ then $\tilde{e}(Z|\bar{Z}_d)$ is monotone decreasing in $\bar{Z}_d$ for $\bar{Z}_d$ in a left neighborhood of $\bar{Z}_d$.*

Proof. The first claim follows directly from Lemma A.12 because, by assumption $\tilde{e}(Z_d^2|Z_d^1)>\tilde{e}(Z_d^1|Z_d^1)=0$ for any $Z_d^1>Z_d^2$ that are sufficiently close to $\bar{Z}_d$. The proof of the second claim is analogous. ■

The following result is a direct consequence of Lemma A.13.

Lemma A.14 *If $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d)<0$ for all $\bar{Z}_d>0$ let $Z_d^*=\infty$. Otherwise, set*

$$Z_d^* \equiv \min\{Z_d > Z_d^{\text{nr}} : \tilde{e}_Z(Z_d|Z_d) = 0\}$$

Then, $\tilde{e}(Z|Z_d^*) > 0$ *for all* $Z \le Z_d^*$.

Proof. By Lemma A.10, $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d)$ is continuous and therefore, by Lemma A.11, $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d) < 0$ for all $\bar{Z}_d < Z_d^*$. By Lemma A.13, $\tilde{e}(Z|\bar{Z}_d)$ is monotone increasing in $\bar{Z}_d \in [0, Z_d^*)$ and therefore $\tilde{e}(Z|Z_d^*) > 0$ for all $Z < Z_d^*$ and $e_Z(Z_d^*|Z_d^*) = 0$. ■

Lemma A.15 *We have*

$$(1-\tau)(1-Z) + \max_{a\ge 1}\left((1-q\mathbf{1}_{a>1})\tilde{d}(aZ) - \tilde{d}(Z)\right) \;<\; 0$$

for all $Z > Z_d^*$.

Proof. We have

$$\begin{aligned}
0.5\sigma^2\tilde{e}_{ZZ}(Z_d^*|Z_d^*) \;&=\; (r-\mu+\lambda(1-\eta))\tilde{e}(Z_d^*|Z_d^*) + \mu Z_d^*\tilde{e}_Z(Z_d^*|Z_d^*) - (1-\tau)(1-Z_d^*) \\
&- \max_{a\ge 1}\left(\tilde{e}(aZ_d^*|Z_d^*) + (1-q\mathbf{1}_{a>1})\tilde{d}(aZ_d^*) - \tilde{d}(Z_d^*)\right) \\
&=\; -\left((1-\tau)(1-Z_d^*) + \max_{a\ge 1}\left((1-q\mathbf{1}_{a>1})\tilde{d}(aZ_d^*) - \tilde{d}(Z_d^*)\right)\right)
\end{aligned} \tag{21}$$

because $\tilde{e}(aZ_d^*|Z_d^*) = 0$ for all $a \ge 1$. Since $\tilde{e}(Z|Z_d^*) > 0$ for all $Z \in [0, Z_d^*)$ and $\tilde{e}(Z_d^*|Z_d^*) = \tilde{e}_Z(Z_d^*|Z_d^*) = 0$, we get that $\tilde{e}_{ZZ}(Z_d^*|Z_d^*) \ge 0$. Consequently,

$$(1-\tau)(1-Z_d^*) + \max_{a\ge 1}\left((1-q\mathbf{1}_{a>1})\tilde{d}(aZ_d^*) - \tilde{d}(Z_d^*)\right) \;\le\; 0$$

and the claim follows from the fact that, by assumption, $(1-\tau)(1-Z) + \max_{a\ge 1}((1 - q\mathbf{1}_{a>1})\tilde{d}(aZ) - \tilde{d}(Z))$ is monotone decreasing in Z. ■

Proof of Proposition 1. To prove the result, it suffices to show that $\tilde{e} = \tilde{e}(Z|Z_d^*)$ is the value function of the firm. Standard verification results for optimal stopping (see, e.g., Dayanik and Karatzas (2003)) combined with the arguments from the proof of Lemma A.7 imply that it suffices to verify that $\tilde{e}(Z|Z_d^*)$ satisfies the HJB equation

$$\begin{aligned}
0 = \max\{&-\tilde{e}(Z), \\
&-(r-\mu+\lambda(1-\eta))\tilde{e}(Z) + \mathscr{L}\tilde{e}(Z) \\
&+(1-\tau)(1-Z) + \lambda(1-\eta)\max_{a\ge 1}\left(\tilde{e}(aZ) + (1-q\mathbf{1}_{a>1})\tilde{d}(aZ) - \tilde{d}(Z)\right)\}
\end{aligned}$$

The same arguments as in the proof of Lemma A.7 imply that

$$
\begin{aligned}
0 &= -(r-\mu+\lambda(1-\eta))\tilde{e}(Z) + \mathscr{L}\tilde{e}(Z) \\
&\quad + (1-\tau)(1-Z) + \lambda(1-\eta)\max_{a\geq 1}(\tilde{e}(aZ) + (1-q\mathbf{1}_{a>1})\tilde{d}(aZ) - \tilde{d}(Z))
\end{aligned}
$$

for all $Z \in [0, Z_d^*]$ and by Lemma A.14 we have that $\tilde{e}(Z) \geq 0$ for all $Z \geq 0$. On the other hand, for $Z > Z_d^*$ we have $\tilde{e}(Z) = 0$ and it thus follows from Lemma A.15 that

$$
\begin{aligned}
&-(r-\mu+\lambda(1-\eta))\tilde{e}(Z) + \mathscr{L}\tilde{e}(Z) \\
&+ (1-\tau)(1-Z) + \lambda\,(1-\eta)\max_{a\geq 1}\,(\tilde{e}(aZ) + (1-q\mathbf{1}_{a>1})\tilde{d}(aZ) - \tilde{d}(Z)) \\
&= (1-\tau)(1-Z) + \lambda\,(1-\eta)\max_{a\geq 1}\,(\tilde{e}(aZ) + (1-q\mathbf{1}_{a>1})\tilde{d}(aZ) - \tilde{d}(Z)) < 0
\end{aligned}
$$

which completes the proof. ■

Lemma A.16 *We have $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d) < 0$ for all $\bar{Z}_d > Z_d^*$. Hence, Z_d^* is the unique solution $\bar{Z}_d$ to $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d) = 0$.*

Proof. Suppose the contrary. Then, there exists a $\bar{Z}_d > Z_d^*$ such that $\tilde{e}_Z(\bar{Z}_d|\bar{Z}_d) = 0$. By Lemma A.15 and (21), we have

$$
0.5\sigma^2\tilde{e}_{ZZ}(\bar{Z}_d|\bar{Z}_d) \;=\; -((1-\tau)(1-\bar{Z}_d) + \max_{a\geq 1}\,((1-q\mathbf{1}_{a>1})\tilde{d}(a\bar{Z}_d) - d(\bar{Z}_d))) \;>\; 0\,.
$$

Therefore, $\tilde{e}(Z|\bar{Z}_d) > 0 = \tilde{e}(Z|Z_d^*)$ for Z sufficiently close to $\bar{Z}_d$. But this is impossible because, by Proposition 1, defaulting at Z_d^* is optimal. ■

Finally, the next result shows that the debt function for a barrier default policy does satisfy the required monotonicity condition.

Lemma A.17 *The function*

$$
(1-\tau)(1-Z) + \max_{a\geq 1}\,((1-q\mathbf{1}_{a>1})d(aZ|Z_d) - d(Z|Z_d))
$$

is strictly monotone decreasing in Z.

Proof. Clearly, it suffices to show that $\max_{a\geq 1}\left((1-q\mathbf{1}_{a>1})d(aZ|Z_d)-d(Z|Z_d)\right)$ is non-increasing in Z. A direct calculation shows that $d(Z|Z_d)$ is either increasing in Z or is convex and attains a maximum at some $Z_m < Z_d$. In the first case,

$$\max_{a\geq 1}\left((1-q\mathbf{1}_{a>1})d(aZ|Z_d)-d(Z|Z_d)\right) \;=\; \max\{0,(1-q)d(Z_d|Z_d)-d(Z|Z_d)\}$$

is obviously non-increasing. In the second case,

$$\max_{a\geq 1}\left((1-q\mathbf{1}_{a>1})d(aZ|Z_d)-d(Z|Z_d)\right) \;=\; \max\{0,(1-q)d(Z_m|Z_d)-d(Z|Z_d)\}$$

is also non-increasing. ■

The next result proves Proposition 3. Namely, it shows that, given a fixed barrier default policy, the optimal restructuring policy is of barrier type.

Lemma A.18 *Either restructuring is not optimal or there exist* $0 < Z_u \leq Z_o < Z_d$ *such that*

$$Z_o = \operatorname*{argmax}_{Z\in[Z_u,Z_d]}\{v(Z|Z_d)-q\,d(Z|Z_d)\}$$

and

$$\mathscr{O}(v(\cdot|Z_d))(Z) = 1_{\{Z<Z_u\}}\{v(Z_o|Z_d)-qd(Z_o|Z_d)-v(Z|Z_d)\}\ .$$

Proof. Assume that restructuring is optimal and let $q>0$. The case $q=0$ can be treated similarly. To simplify the notation we fix the default threshold Z_d and write $v(Z)=v(Z|Z_d)$ and $d(Z)=d(Z|Z_d)$. Since it is optimal for the firm to restructure its capital structure at some point we know that the operator $\mathscr{O}(v)$ is not zero and it follows that

$$Z_u \equiv \max\{Z\geq 0 : \mathscr{O}(v)(Z)>0\}$$

is well-defined and smaller or equal to Z_d. Furthermore, by continuity we have

$$\mathscr{O}(v)(Z_u)=0 \Longleftrightarrow v(Z_u)=\max_{y\geq Z_u}\{v(y)-qd(y)\}>0.$$

Now consider the higher threshold defined by

$$Z_o = \min\{y \geq Z_u : v(Z_u) = v(y) - qd(y)\}. \tag{22}$$

By Lemma A.9 we have $v(Z_d) = \phi < v(Z_u) = v(Z_o) - qd(Z_o) < v(Z_o)$ and therefore $Z_u < Z_o < Z_d$ since issuance costs are strictly positive. This in turn implies that the point Z_o is a local maximum of the function $\bar{v}(Z) = v(Z) - qd(Z)$ and since $v(Z_u) \leq v(Z)$ for $Z \in [Z_u, Z_o]$ by definition of the threshold Z_u we necessarily have that $v'(Z_u) \geq 0$.

Before carrying on with the rest of the proof we start by proving that $\bar{v}(Z)$ does not have admit any local maximum such that $\bar{v}(Z) > v(Z_u)$ on the interval $[0, Z_o]$. Suppose the contrary and let $\bar{Z}_n < Z_o$ denote the location of the largest such local maximum. Since the point Z_o is a local maximum of $\bar{v}(Z)$ this implies, as illustrated by the left panel of Figure A, that the function $\bar{v}(Z)$ achieves a local minimum at some point $\bar{Z}_m \in [\bar{Z}_n, Z_o]$ such that

$$\bar{v}(\bar{Z}_m) < \bar{v}(Z_o) = v(Z_u) < \bar{v}(\bar{Z}_n). \tag{23}$$

This in turn implies that we have $\mathscr{O}(v)(\bar{Z}_n) = 0$ and combining this with the fact that the functions d and v solve

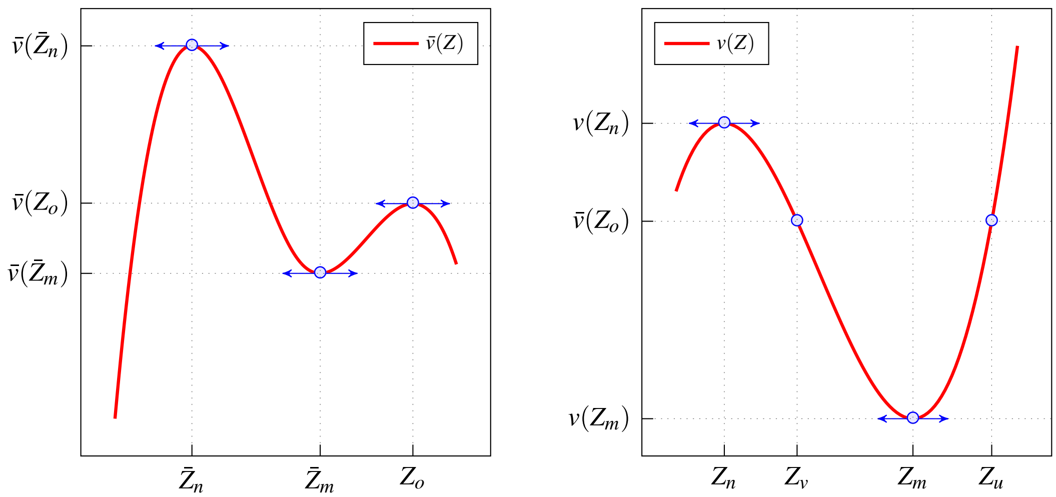
$$\begin{aligned}
(r-\mu)d(Z) &= \mathscr{L}d(Z) + Z \\
(r-\mu)v(Z) &= \mathscr{L}v(Z) + 1 - \tau(1-Z) + \lambda(1-\eta)\mathscr{O}(v)(Z)
\end{aligned}$$

we finally obtain

$$\begin{aligned}
(r-\mu)\bar{v}(Z_o) > (r-\mu)\bar{v}(\bar{Z}_m) & \\
&= \mathscr{L}v(\bar{Z}_m) + 1 - \tau + (\tau - q)\bar{Z}_m + \lambda(1-\eta)\mathscr{O}(v)(\bar{Z}_m) \\
&\geq \mathscr{L}v(\bar{Z}_m) + 1 - \tau + (\tau - q)\bar{Z}_m \\
&\geq 1 - \tau + (\tau - q)\bar{Z}_m > 1 - \tau + (\tau - q)\bar{Z}_n \\
&\geq \mathscr{L}v(\bar{Z}_n) + 1 - \tau + (\tau - q)\bar{Z}_n \\
&= \mathscr{L}v(\bar{Z}_n) + 1 - \tau + (\tau - q)\bar{Z}_n + \lambda(1-\eta)\mathscr{O}(v)(\bar{Z}_n) = (r-\mu)\bar{v}(Z_n)
\end{aligned}$$

where the second inequality follows from the nonnegativity of $\mathscr{O}$, and the third and fifth

Figure 4: Shape of the functions $\bar{v}(Z)$ and $v(Z)$ in the proof of Lemma A.18



inequalities follow from the fact that $v'(Z) = 0$ and $v''(Z) \leq 0$ (resp. ≥ 0) at a local maximum (resp. local minimum). This contradicts equation (23) and therefore establishes our claim regarding the local maxima of the function $\bar{v}(Z)$.

To complete the proof we now need to establish that $v(Z) \leq v(Z_u)$ on $[0, Z_u]$. Suppose that this is not the case, let

$$Z_v = \max\{Z \leq Z_u : v(Z) = v(Z_u) = v(Z_o) - qd(Z_o)\}$$

and assume for simplicity that $Z_v < Z_u$ so that the function $v(Z)$ reaches a local minimum at some point $Z_m \in [Z_v, Z_u]$.[1] As a first step towards a contradiction we claim that the function $v(Z)$ is monotone decreasing on $[0, Z_v]$. If not then as illustrated by the right panel of Figure 4 there is a point $Z_n \in [0, Z_v]$ at which the function $v(Z)$ achieves a local maximum such that

$$v(Z_n) > v(Z_v) = v(Z_u) = v(Z_o) - qd(Z_o) = \max_{y \geq Z_n} \bar{v}(y) \tag{24}$$

where the last equality follows from the first part of the proof. This immediately implies that we have $\mathscr{O}(v)(Z_n) = 0$ and combining this property with the same arguments as in the

[1]When the point Z_u is a local minimum of the function $v(Z)$ we have $Z_v = Z_u$. This case is completely analogous, up to small modifications.

first part of the proof then gives

$$\begin{aligned}(r-\mu)v(Z_m) &= \mathscr{L}v(Z_m)+1-\tau(1-Z_m)+\lambda(1-\eta)\mathscr{O}(v)(Z_m)\\ &\geq \mathscr{L}v(Z_m)+1-\tau(1-Z_m)\\ &\geq 1-\tau(1-Z_m)\\ &> \ 1-\tau(1-Z_n)\\ &\geq \mathscr{L}v(Z_n)+1-\tau(1-Z_n)\\ &= \mathscr{L}v(Z_n)+1-\tau(1-Z_n)+\lambda(1-\eta)\mathscr{O}(v)(Z_n)=(r-\mu)v(Z_n)\end{aligned}$$

which contradicts equation (24) and therefore establishes our claim regarding the monotonicity of $v(Z)$ on the interval $[0, Z_v]$. Combining this property with the fact that $v(0)=\phi_0$ we immediately get that $\phi_0 > v(Z)$ on $(0, Z_o)$ but this is impossible since $v(Z)\geq \hat{v}(Z)\geq \phi_0$ in a right neighborhood of zero by Lemma A.9. ■

Proof of Proposition 5. Let $d(Z)=d(Z|Z_d),\ e(Z)=e(Z|Z_d),\ v(Z)=v(Z|Z_d)$. In terms of the Z variable, we need to show that

$$e(Z)\ >\ e((1-a)Z)-\frac{a}{1-a}d((1-a)Z)$$

for all $a\in(0,1)$. Let us show that the right-hand side is monotone decreasing in a. Differentiating with respect to a, we obtain that we need to show the inequality

$$\frac{1}{1-a}(d(x)-xd'(x))+v'(x)x\ >\ 0$$

with $x=(1-a)Z$. A direct calculation implies that $d(x)/x$ is monotone decreasing and hence $d(x)-xd'(x)>0$. Thus, it suffices to show that

$$\chi(x)\ \equiv\ d(x)-xd'(x)+v'(x)x\ >\ 0\ \Leftrightarrow\ \frac{d(x)}{x}+e'(x)\ >\ 0\,.$$

First we note that, by the above $d(x)/x>d'(x)$ and $v'(x)\geq 0$ for all $x<Z_u$. Hence, for $x<Z_u$, we have

$$\frac{d(x)}{x}+e'(x)>d'(x)+e'(x)=v'(x)\geq 0.$$

For $x > Z_u$, a direct calculation implies that the function $y(x) = e'(x)$ satisfies

$$\frac{1}{2}\sigma^2 x^2 y''(x) + (\sigma^2 - \mu)y'(x) - (\rho + \mu)y(x) - (1 - \tau) = 0$$

whereas the function $w(x) = d(x)/x$ satisfies

$$\frac{1}{2}\sigma^2 x^2 w''(x) + (\sigma^2 - \mu)w'(x) - (\rho + \mu)w(x) + 1 = 0$$

Hence, $\chi(x) = y(x) + w(x)$ solves

$$\frac{1}{2}\sigma^2 x^2 \chi''(x) + (\sigma^2 - \mu)\chi'(x) - (\rho + \mu)\chi(x) + \tau = 0.$$

Furthermore, $e'(Z_d) = 0$ and hence $\chi(Z_d) > 0$. Suppose that $\chi(x)$ is not positive. Then, since $\chi(Z_u) > 0$ by the above, it must have a negative local minimum for some $x_* \in (Z_u, Z_d)$. In that point, $\chi''(x_*) > 0 = \chi'(x_*)$ and therefore

$$0 \geq (\rho + \mu)\chi(x_*) = \frac{1}{2}\sigma^2 x^2 \chi''(x_*) + (\sigma^2 - \mu)\chi'(x_*) + \tau > \tau,$$

which is a contradiction. ■

Proof of Proposition 6. Proposition 6 follows directly from Lemmas A.18 and A.16. ■

B The case $q = 0$ and the general existence result

Proof of Proposition 7. Let $\lambda^* = \lambda(1 - \eta)$. It follows from equation (13) and Lemma A.18 that there are constants a_1, a_3, a_4 such that

$$v(Z|Z_d) = v_s(Z, Z_o, a_1, a_3, a_4; q) \equiv \frac{1 - \tau + \lambda^*(v(Z_o|Z_d) - qd(Z_o|Z_d))}{r - \mu + \lambda^*} + \frac{\tau Z}{r + \lambda^*} + a_1 Z^{1-\psi}$$

for all $Z \in [0, Z_u]$, and

$$v(Z|Z_d) = v_{ns}(Z, a_3, a_4) \equiv \phi_0 + \frac{\tau Z}{r} + a_3 Z^{1-\beta} + a_4 Z^{1-\alpha}$$

for all $Z \in [Z_u, Z_d)$. Since $q = 0$ it follows immediately from (22) that we have $Z_o = Z_u$. Evaluating the first of the above identities at the point $Z = Z_o$ gives

$$v(Z_o|Z_d) = \frac{1-\tau+\lambda^* v(Z_o)}{r-\mu+\lambda^*} + \frac{\tau Z_o}{r+\lambda^*} + a_1 Z_o^{1-\psi}$$

and solving this equation for $v(Z_o)$ we obtain

$$v(Z_o|Z_d) = \frac{r-\mu+\lambda^*}{r-\mu}\left(\frac{1-\tau}{r-\mu+\lambda^*} + \frac{\tau Z_o}{r+\lambda^*} + a_1 Z_o^{1-\psi}\right).$$

Therefore, the value matching condition at the point Z_o can be written as

$$\phi_0 + \frac{\tau Z_o}{r} + a_3 Z_o^{1-\beta} + a_4 Z_o^{1-\alpha} = \frac{r-\mu+\lambda^*}{r-\mu}\left(\frac{1-\tau}{r-\mu+\lambda^*} + \frac{\tau Z_o}{r+\lambda^*} + a_1 Z_o^{1-\psi}\right).$$

which is equivalent to

$$\frac{r-\mu+\lambda^*}{r-\mu}\left(\frac{\tau\mu\lambda^* Z_o}{r(r+\lambda^*)(r-\mu+\lambda^*)} + a_1 Z_o^{1-\psi}\right) = a_3 Z_o^{1-\beta} + a_4 Z_o^{1-\alpha}.$$

The facts that $Z_o = Z_u$ and that $v(Z|Z_d)$ is continuously differentiable with $v(Z_d) - \phi = v'(Z_o) = 0$ jointly imply the remaining free constants are determined by

$$\begin{aligned}
\frac{\tau Z_o}{r+\lambda^*} + a_1(1-\psi)Z_o^{1-\psi} &= 0,\\
\frac{\tau Z_o}{r} + a_3(1-\beta)Z_o^{1-\beta} + a_4(1-\alpha)Z_o^{1-\alpha} &= 0,\\
\omega\phi_0 + \frac{\tau Z_d}{r} + a_3 Z_d^{1-\beta} + a_4 Z_d^{1-\alpha} &= 0.
\end{aligned}$$

Combining these equations shows that we have

$$\tau/r = \frac{a_3 Z_o^{-\beta}}{\kappa_1(\lambda)} = \frac{a_4 Z_o^{-\alpha}}{\kappa_2(\lambda)} \tag{25}$$

$$f(J) = 1 + \kappa_1(\lambda)J^{-\beta} + \kappa_2(\lambda)J^{-\alpha} = -(r\omega\phi_0/\tau)Z_d^{-1} \tag{26}$$

where the function f is defined as in the text and we have set $J = Z_d/Z_o$. Note that since $\kappa_1(\lambda) < 0 < \kappa_2(\lambda)$ we have that $a_3 \leq 0$ and $a_4 \geq 0$. In order to calculate the equilibrium, it remains to impose the smooth pasting condition which now takes the form

$$\tau + \tau(1-\beta)\kappa_1(\lambda)J^{-\beta} + \tau(1-\alpha)\kappa_2(\lambda)J^{-\alpha} = \beta + r(1-\beta)(1-\omega)\phi_0/Z_d.$$

Substituting the value for Z_d we get the required equation (3.3) for J. Thus, if an equilibrium exists, it is given by the expressions from Proposition 3. It remains to show that the corresponding equation has a solution if and only if $\tau < \tau^*$ and that this solution is unique.

Since $\kappa_1(\lambda) < 0 < \kappa_2(\lambda)$ we have that f is decreasing. Therefore, the default threshold in (26) is positive if and only if $J > J_0$ where J_0 is the unique solution to (3.3). Let $g(y)$ be as in the text. A direct calculation shows that $g(\infty) = -\infty$ and that

$$g(J_0) - \beta/r > 0 \Longleftrightarrow \tau < \tau^*$$

Thus, existence follows from the intermediate value theorem. To prove uniqueness, it suffices to show that g is decreasing for $J \geq J_0$. This is obvious if $1-\beta+\omega(\beta-\alpha) > 0$. Otherwise, g increases up to the point J_* where its derivative vanishes and decreases afterwards. Therefore, it suffices to show that we have $g(J_*) > 0$ but this follows from the fact that

$$g(J_*) = 1 + \left(1 + \frac{\alpha(1-\beta-\alpha\omega)(\alpha-\beta)}{\beta(\beta-1)}\right)\kappa_2(\lambda)J_*^{-\alpha} > 0 \tag{27}$$

since $\alpha\omega + \beta < \alpha + \beta = 1 - 2\mu/\sigma^2 < 1$. ■

In order to prove the general existence result of Theorem 1 for the model with search we will also need the following standard lemma.

Lemma B.1 *Suppose that the function $f(Z,x)$ is continuous, monotone decreasing in x and satisfies $f(Z_1,x) > 0 > f(Z_2,x)$. Then, the minimal solution $Z_0(x) \in [Z_1,Z_2]$ to the equation $f(Z,x) = 0$ is monotone decreasing in x.*

Proof. Let $A(x) \equiv \{Z \in [Z_1,Z_2] : f(Z,x) \leq 0\}$. Then, the set $A(x)$ is a compact set and it is monotone increasing in x in the inclusion order. Therefore, $Z_0(x) = \min\{A(x)\}$ is monotone decreasing. ■

Now, to prove uniqueness we will need the following generic non-degeneracy result. Let

$$\mathscr{C} = (\omega, \lambda, \eta, \mu, r - \mu, \tau, \theta) \in \mathbb{R}^6_+$$

denote the vector of model parameters and say that the vector $\mathscr{C}$ is admissible if its components are such that $\tau < \tau^*$ and $r > \mu$.

Lemma B.2 *Consider the system*

$$\begin{aligned}
F_1(Z_u, Z_o, Z_d, a_1, a_3, a_4; q) &\equiv v_s(Z_u, Z_o, a_1, a_3, a_4; q) - v_{ns}(Z_u, a_3, a_4) = 0,\\
F_2(Z_u, Z_o, Z_d, a_1, a_3, a_4; q) &\equiv v_s'(Z_u, Z_o, a_1, a_3, a_4; q) - v_{ns}'(Z_u, a_3, a_4) = 0,\\
F_3(Z_u, Z_o, Z_d, a_1, a_3, a_4; q) &\equiv v_{ns}'(Z_o, a_3, a_4) - q\, d'(Z_o|Z_d) = 0,\\
F_4(Z_u, Z_o, Z_d, a_1, a_3, a_4; q) &\equiv v_s(Z_u, Z_o, a_1, a_3, a_4; q) - v_{ns}(Z_o, a_3, a_4) + qd(Z_o|Z_d) = 0,\\
F_5(Z_u, Z_o, Z_d, a_1, a_3, a_4; q) &\equiv v_{ns}(Z_d, a_3, a_4) - d(Z_d|Z_d) = 0,\\
F_6(Z_u, Z_o, Z_d, a_1, a_3, a_4; q) &\equiv v_{ns}'(Z_d, a_3, a_4) - d'(Z_d|Z_d) = 0.
\end{aligned}$$

Denote by $J(\mathscr{C})$ the unique solution to (3.3), define $z_o(\mathscr{C})$ and $z_d(\mathscr{C})$ by (25), (26) and let

$$\begin{aligned}
\tilde{a}_1(\mathscr{C}) &= -\tau z_o(\mathscr{C})/((r + \lambda^*)(1 - \psi)),\\
\tilde{a}_3(\mathscr{C}) &= (\tau/r)\kappa_1(\lambda) z_o(\mathscr{C})^\beta,\\
\tilde{a}_4(\mathscr{C}) &= (\tau/r)\kappa_2(\lambda) z_o(\mathscr{C})^\alpha.
\end{aligned}$$

Suppose that there exists an admissible $\mathscr{C}$ such that

$$\mathscr{J} F(z_o(\mathscr{C}), z_o(\mathscr{C}), z_d(\mathscr{C}), \tilde{a}_1(\mathscr{C}), \tilde{a}_3(\mathscr{C}), \tilde{a}_4(\mathscr{C}); 0) \neq 0\,.$$

where $\mathscr{J}$ denotes the Jacobian operator. Then, for Lebesque almost every admissible $\mathscr{C}$ there exists an open neighborhood

$$\mathscr{B} \supseteq (z_o(\mathscr{C}), z_o(\mathscr{C}), z_d(\mathscr{C}), \tilde{a}_1(\mathscr{C}), \tilde{a}_3(\mathscr{C}), \tilde{a}_4(\mathscr{C}))$$

and an $\epsilon > 0$ such that, for all $q \in [0, \epsilon)$, there exists a unique Markov perfect equilibrium in barrier strategies whose parameters satisfy $(Z_u, Z_o, Z_d, a_1, a_3, a_4; q) \in \mathscr{B}$.

Proof. The function $\mathscr{J}F(z_o(\mathscr{C}), z_o(\mathscr{C}), z_d(\mathscr{C}), \tilde{a}_1(\mathscr{C}), \tilde{a}_3(\mathscr{C}), \tilde{a}_4(\mathscr{C}); 0)$ is clearly real analytic in $\mathscr{C}$. Therefore, if it is not identically zero, it is non-zero for almost every $\mathscr{C}$. The last claim follows then from the implicit function theorem. ■

Proof of Theorem 1. Uniqueness of equilibrium in a small neighborhood of the $q = 0$ equilibrium for the case when q is small follows from Lemma B.2. Continuous dependence of the equity value and its derivative on all model parameters follow from Lemma A.6.

Suppose that $\tau < \tau^*$. In this case it follows from Proposition 7 and its proof that the exists a unique equilibrium default barrier $Z^*_{d,0}$ such that

$$e_Z(Z^*_{d,0}|Z^*_{d,0})\Big|_{q=0} = 0 < e_Z(Z_d|Z_d)\Big|_{q=0}, \qquad \forall Z_d > Z^*_{d,0}.$$

Consequently, $e(Z|Z_d)$ is negative for $Z < Z_d$ in a left neighborhood of Z_d when $q = 0$ and since equity value decreases with the issuance cost parameter we conclude that $e(Z|Z_d)$ is negative for $Z < Z_d$ in a left neighborhood of Z_d and all $q > 0$. Since $e(Z_d|Z_d) = 0$ this in turn implies

$$e_Z(Z_d|Z_d) > 0, \qquad \forall Z > Z^*_{d,0}.$$

On the other hand, Lemma A.9 shows that

$$e_Z(Z_d|Z_d) < 0, \qquad \forall Z_d < Z_d^{\mathrm{nr}},$$

and it now follows from the intermediate value theorem that there exists at least one $Z^*_d \in [Z_d^{\mathrm{nr}}, Z^*_{d,0}]$ for which $e_Z(Z_d|Z_d) = 0$. The proof of Proposition 6 implies that Z^*_d is an equilibrium default strategy.

Since the required monotonicity follows from Lemmas A.5 and B.1 it now only remains to show that when $\tau < \tau^*$ there exists an $\epsilon > 0$ such that for all $q < \epsilon$ there exists a unique equilibrium in barrier strategies. Suppose the contrary. Then, there exists a sequence $q_n \downarrow 0$ such that for each n there exist at least two equilibria $Z^1_d(n) < Z^2_d(n)$. By Lemma B.2, we cannot have that both $Z^1_d(n)$ and $Z^2_d(n)$ converge to $Z^*_{d,0}$ and the above argument show that $Z^1_d(n), Z^2_d(n) \le Z^*_{d,0}$. Therefore, there exists a subsequence of equilibrium default thresholds that converges to some $Z^c_d < Z^*_{d,0}$ and by continuity (see Lemma A.6) we have

$e_Z(Z_d^c|Z_d^c)|_{q=0} = 0$ which is impossible because there exists a unique equilibrium when $q = 0$. ■

Existence for the case $\tau > \tau^*$. In this case we have $e_Z(Z_d|Z_d)|_{q=0} < 0$ for all $Z_d > 0$ by the proof of Proposition 7. On the other hand, Lemma A.9 guarantees that

$$e_Z(Z_d|Z_d)\Big|_{q=\tau} > 0, \qquad \forall Z_d > Z_d^{\mathrm{nr}},$$

and by continuity the same is true for q sufficiently close to τ. Therefore, the cutoff level of the issuance costs parameter defined by

$$q^* \equiv \inf\left\{q > 0 : \sup_{Z_d > 0} e_Z(Z_d|Z_d) > 0\right\}$$

satisfies $q^* < \tau$ and the fact that an equilibrium exists if and only if $q > q^*$ now follows by the same argument as in the proof of Theorem 1. ■

Proof of Proposition 8. The monotonicity of the default threshold in η, λ and q follows directly from the corresponding monotonicity of the equity value function for a fixed default threshold, see Lemma A.5.

To establish the other claims it suffices to consider the case $q = 0$ since the case of small q follows from the implicit function theorem and Lemma B.2. Recall that

$$g(y) = \beta\omega - \frac{\beta\omega}{\tau} + (1-\beta)\left[1 + \kappa_1(\lambda)\, y^{-\beta}\right] + \left[(\beta-\alpha)\,\omega + 1 - \beta)\right]\kappa_2(\lambda)\, y^{-\alpha}.$$

As shown above (equation (27)) we have that the function $g(y)$ is monotone decreasing in y for $y > J_0$ and monotone decreasing in τ. Therefore, by the implicit function theorem, so is the unique solution $J(\tau)$ to the equation $g(J;\tau) = 0$. Define

$$\gamma \;\equiv\; ((\beta-\alpha)\omega + (1-\beta))\,.$$

Differentiating $g(J;\tau) = 0$ and solving we get

$$J'_\tau \;=\; \frac{\beta\omega\tau^{-2}}{\beta(1-\beta)\kappa_1 J^{-\beta-1} + \alpha\gamma\kappa_2 J^{-\alpha-1}}\,.$$

The function $f(J)$ is decreasing in J and its coefficients are independent of τ and f is decreasing. Therefore, differentiating the identity

$$Z_d \;=\; \frac{r\omega\frac{\tau^{-1}-1}{r-\mu}}{-f(J(\tau))},$$

we get that Z_d is monotone increasing in τ if and only if

$$\begin{aligned}(\beta(\beta-1)\kappa_1 J^{-\beta-1} - \alpha\gamma\kappa_2 J^{-\alpha-1})(1+\kappa_1 J^{-\beta}+\kappa_2 J^{-\alpha}) \\ < (\beta\kappa_1 J^{-\beta-1}+\alpha\kappa_2 J^{-\alpha-1})(1-\tau^{-1})\beta\omega.\end{aligned} \tag{28}$$

The inequality $Z_d > Z_d^{\mathrm{nr}}$ established in Lemma A.9 can be rewritten as

$$\frac{r\omega\frac{\tau^{-1}-1}{r-\mu}}{-f(J(\tau))} \;>\; \frac{\beta-1}{\beta}\frac{r}{r-\mu},$$

or, equivalently,

$$(\tau^{-1}-1)\omega > -f(J(\tau))\frac{\beta-1}{\beta}.$$

Substituting this into (28) shows that it suffices to establish

$$(\beta(1-\beta)\kappa_1 J^{-\beta-1}+\alpha\gamma\kappa_2 J^{-\alpha-1}) < (\beta\kappa_1 J^{-\beta-1}+\alpha\kappa_2 J^{-\alpha-1})(1-\beta)\,,$$

which follows because $\beta < 0 < \alpha$ and therefore $\gamma < 1-\beta$. We conclude that Z_d is monotone increasing in τ and it follows that $X_d(1) = Z_d^{-1}$ is decreasing in τ. Since the function $J(\tau)$ is decreasing in τ by the above this further implies that $T^* = Z_d/J(\tau)$ and $X_u(1) = 1/T^*$ are respectively increasing and decreasing in τ.

It remains to prove monotonicity in λ. Here, the situation is more complicated because both $f(y)$ and $g(y)$ depend explicitly on λ through the coefficients $\kappa_{1,2}(\lambda)$. A direct (albeit tedious) calculation using the fact that

$$\psi = \frac{-(\mu-\sigma^2/2)-\sqrt{(\mu-\sigma^2/2)^2+2\sigma^2(r+\lambda^*)}}{\sigma^2}$$

is monotone decreasing in λ shows that $\kappa_1(\lambda)$ is monotone increasing in λ. Differentiating the identity $g(J;\lambda)=0$, we get

$$J'_\lambda \;=\; \frac{(1-\beta)J^{-\beta}+\gamma\frac{1-\beta}{\alpha-1}J^{-\alpha}}{\beta(1-\beta)\kappa_1 J^{-\beta-1}+\alpha\gamma\kappa_2 J^{-\alpha-1}}\kappa_1'(\lambda)>0$$

and it follows that

$$\frac{1}{T^* X_d(1)}=\frac{X_u(1)}{X_d(1)}=J$$

is increasing in λ. On the other hand we have

$$\begin{aligned}\frac{1}{\kappa_1'(\lambda)}\frac{d}{d\lambda}f(\lambda,J(\lambda)) &= J^{-\beta}+\frac{1-\beta}{\alpha-1}J^{-\alpha}\\ &\quad -(\kappa_1\beta J^{-\beta-1}+\kappa_2\alpha J^{-\alpha-1})\frac{(1-\beta)J^{-\beta}+\gamma\frac{1-\beta}{\alpha-1}J^{-\alpha}}{\beta(1-\beta)\kappa_1 J^{-\beta-1}+\alpha\gamma\kappa_2 J^{-\alpha-1}}.\end{aligned}$$

Therefore, the inequality $\frac{d}{d\lambda}f(\lambda,J(\lambda))>0$ is equivalent to

$$0<\alpha\gamma\kappa_2+\frac{1-\beta}{\alpha-1}\beta(1-\beta)\kappa_1-\kappa_1\beta\gamma\frac{1-\beta}{\alpha-1}-\kappa_2\alpha(1-\beta)$$

which in turn can be rewritten as

$$\alpha+(\alpha-\beta)(1-\beta)\kappa_1(\lambda)<0.$$

Since $\kappa_1(\lambda)$ is increasing in λ, it suffices to verify this inequality for $\lambda=\infty$. A direct calculation based on the quadratic equation satisfied by α and β implies that

$$\alpha+(\alpha-\beta)(1-\beta)\kappa_1(\infty)=0,$$

and the claim follows. Thus, Z_d is monotone increasing in λ. In order to prove that the target $T^*=1/X_u(1)$ is decreasing in λ we need to differentiate $Jf(J;\lambda)$ and show that the derivative is positive. A direct calculation shows that this condition is equivalent to

$$(\beta-\alpha)\omega(\alpha\kappa_2-\frac{1-\beta}{\alpha-1}\beta\kappa_1)J^{-\alpha-\beta-1} \tag{29}$$

$$+((1-\beta)J^{-\beta}+\gamma\frac{1-\beta}{\alpha-1}J^{-\alpha})(1+\kappa_1 J^{-\beta}+\kappa_2 J^{-\alpha})<0 \tag{30}$$

While (29) is nonnegative by the above, (30) is always strictly negative because $f(J) < 0$. Since the expression is linear in κ_1 (because κ_2 is linear in κ_1), it suffices to establish the inequality for the extreme cases $\lambda = 0$ and $\lambda = \infty$. The first one follows by direct calculation. The second one follows because (29) converges to zero as $\lambda \to \infty$.

Since the value function and all policies depend on λ and η only through the product $\lambda(1-\eta)$, monotonicity in λ is equivalent to opposite monotonicity in η. ■

C Restructuring with existing creditors

In this appendix we study the model in which the firm can raise funds by contacting either outside or inside creditors. Instead of assuming as in the main text that the cost of collective action is proportional to the firm's coupon level prior to restructuring we allow here for more general costs given by $X\,\nu(C/X)$ for some function $\nu(Z)$ that satisfies the following condition:

Assumption 1 *The function*

$$\mathscr{L}\nu(Z) - (r - \mu + \lambda^*)\nu(Z) - \tau Z$$

is monotone decreasing in Z.

As we show below, this assumption guarantees that barrier restructuring strategies are optimal. This assumption trivially holds if $\nu(Z) = \epsilon Z$ as in the text but many other cases can also be considered. In particular, we note that no monotonicity conditions on the function $\nu(Z)$ itself need to be imposed for the validity of this assumption.

Fix an arbitrary default threshold $X_{\mathbf{b}d}(1) = 1/Z_d$ and denote by $\mathbf{P} = \mathbf{P}(X_{\mathbf{b}d}(1))$ the associated equilibrium strategy. With this notation we have that the corresponding equilibrium firm value is given by

$$V_{\mathbf{b}}(X, C|\mathbf{P}) = \sup_{b \in \mathbb{B}(\mathbf{P})} E\Bigg[\int_0^{\tau_d} e^{-rt}((1-\tau)X_t + \tau C_{t-})dt + e^{-(r-\mu)\tau_d}\phi X_{\tau_d} \\ + \sum_{k=1}^{\infty} 1_{\{\tau_k \le \tau_d\}} e^{-r\tau_k} H_{\mathbf{b}}(\tau_k, b_{\tau_k}, X_{\tau_k}, C_{\tau_k -}|\mathbf{P})\Bigg]$$

where

$$
\begin{aligned}
\bar{H}_{\mathbf{b}}(\tau, b', X, C|\mathbf{P}) = -qD(X, b'C|\mathbf{P}) - 1_{\{\tau \in \mathscr{N}\}}\eta S(b', X, C|\mathbf{P}) \\
- 1_{\{\tau \notin \mathscr{N}\}}[(1-\theta)X\nu(X/C) + \theta S(b', X, C|\mathbf{P})]
\end{aligned}
$$

represents the cash flow from restructuring and $\mathscr{N}$ is the set of jump times of the Poisson process that governs meetings between the firm and outside investors. Let also

$$
\begin{aligned}
\mathscr{I}(v)(Z) &= \max_{a\geq 1}\{v(aZ) - qd(aZ|Z_d) - v(Z)\} \\
\mathscr{E}(v)(Z) &= \max_{a\geq 1}\{v(aZ) - qd(aZ|Z_d) - v(Z) - \nu(Z)\}
\end{aligned}
$$

Our first result in this section follows from standard dynamic programming arguments.

Lemma C.1 *If $v_{\mathbf{b}}(Z|Z_d)$ is a bounded and Borel measurable function such that*

$$
\begin{aligned}
v_{\mathbf{b}}(Z|Z_d) = \sup_{\tau\in\mathbb{S}} \ \hat{E}\Bigg[&\int_0^{\tau\wedge\tau^N\wedge\tau_d} e^{-(r-\mu)s}(1-\tau+\tau Z_{s-})ds + 1_{\{\tau_d\leq\tau\wedge\tau^N\}}e^{-(r-\mu)\tau_d}\phi \\
&+ 1_{\{\tau<\tau^N\wedge\tau^d\}}\, e^{-(r-\mu)\tau}((1-\theta)\mathscr{E}(v_{\mathbf{b}}(\cdot|Z_d))(Z_{\tau-}) + v_{\mathbf{b}}(Z_{\tau-})) \\
&+ 1_{\{\tau^N<\tau\wedge\tau^d\}}\, e^{-(r-\mu)\tau^N}((1-\eta)\mathscr{I}(v_{\mathbf{b}}(\cdot|Z_d))(Z_{\tau^N-}) + v_{\mathbf{b}}(Z_{\tau^N-}))\Bigg].
\end{aligned}
$$

then $V_{\mathbf{b}}(X, C|Z_d) = Xv_{\mathbf{b}}(C/X|Z_d)$.

As a result of Lemma C.1, our problem reduces to that of finding a bounded solution to the dynamic programming equation. Note that it is a priori not obvious that such a solution exists. In particular, the contraction mapping techniques that we used in the model with search cannot be directly applied here due to the possibility of contacting creditors at all times, and so new methods need to be developed. We start with a standard lemma for solving optimal stopping problems.

Lemma C.2 *Let $\varphi(Z) \in C[0, Z_d]$ be a bounded function and $\xi(Z)$ a bounded, Borel measurable function. Suppose that a bounded function $y(Z)$ on $[0, Z_d]$ with $y(Z_d) = \phi$ is such that there exists a threshold $\bar{Z}_{\mathbf{b}u} < Z_d$ with the following properties*

1. *The function $y(Z)$ is C^1 and piecewise C^2 on $[0, Z_d)$.*

2. On $[\bar{Z}_{\mathbf{b}u}, Z_d]$ the function $y(Z)$ satisfies

$$(r-\mu+\lambda^*)y(Z) = \mathscr{L}y(Z) + \xi(Z).$$

3. On $[0, Z_d)$ the function $y(Z)$ satisfies $y(Z) \geq \varphi(Z)$.

4. On $[0, \bar{Z}_{\mathbf{b}u}]$ the function $y(Z)$ satisfies $y(Z) = \varphi(Z)$ and

$$(r-\mu+\lambda^*)y(Z) \geq \mathscr{L}y(Z) + \xi(Z).$$

Then the function $y(Z)$ is given by

$$y(Z) = \sup_{\tau\in\mathbb{S}} \hat{E}\left[\int_0^{\tau\wedge\tau_d} e^{-(r-\mu+\lambda^*)s}\xi(Z_s^0)ds + e^{-(r-\mu+\lambda^*)\tau\wedge\tau_d}\left(1_{\{\tau_d\leq\tau\}}\phi + 1_{\{\tau<\tau_d\}}\varphi(Z_\tau^0)\right)\right]$$

where the process Z_t^0 evolves according to (10) with $a \equiv 1$.

To find a solution to our problem, we will approximate the optimal stopping problem by a problem in which the firm can only contact existing creditors at the jump times of an independent Poisson process with intensity $\Lambda > 0$ and then let this intensity increase to infinity. The following proposition describes this auxiliary problem.

Proposition C.3 *Fix a default threshold $Z_d > Z_d^{\mathrm{nr}}$ and let $\rho(\Lambda) \equiv r-\mu+\lambda+\Lambda$. Then the dynamic programming equation:*

$$\begin{aligned} v^\Lambda(Z) = \hat{E}\Bigg[\int_0^{\tau_d} e^{-\rho(\Lambda)t}\big((1-\tau+\tau Z_t^0) \\ + \Lambda((1-\theta)\mathscr{E}(v^\Lambda)(Z_t^0) + v^\Lambda(Z_t^0)) \\ + \lambda((1-\eta)\mathscr{I}(v^\Lambda)(Z_t^0) + v^\Lambda(Z_t^0)\big)dt + e^{-\rho(\Lambda)\tau_d}\phi\Bigg]. \end{aligned} \tag{31}$$

admits a unique solution that belongs to $C^2[0, Z_d]$ and the corresponding optimal restructuring policy is a barrier policy that is characterized by thresholds $\bar{Z}_{\mathbf{b}u}(\Lambda) < Z_{\mathbf{b}u}(\Lambda) < Z_{\mathbf{b}o}(\Lambda) < Z_d$.

The proof of the above proposition will be based on a sequence of lemmas. The same argument as in the model with search implies that the following is true.

Lemma C.4 *The unique solution to* (31) *is* $C^2[0, Z_d]$ *and satisfies*

$$(r-\mu)v^\Lambda(Z) = \mathscr{L}v^\Lambda(Z) + 1 - \tau + \tau Z + \lambda(1-\eta)\mathscr{O}(v^\Lambda)(Z) + \Lambda(1-\theta)\mathscr{O}_{\mathbf{b}}(v^\Lambda)(Z)$$

where the operators on the right are defined by

$$\mathscr{O}_{\mathbf{b}}(v)(Z) \equiv \max_{a\geq 1}\big(v(aZ) - q1_{\{a>1\}}d(aZ|Z_d) - \nu(Z) - v(Z)\big)^+$$

and equation (12).

Lemma C.5 *There are thresholds* $Z_{\mathbf{b}u}(\Lambda) < Z_{\mathbf{b}o}(\Lambda) < Z_d$ *such that*

$$\mathscr{O}(v^\Lambda)(Z) = 1_{\{Z\leq Z_{\mathbf{b}u}\}}\big(v^\Lambda(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d) - v^\Lambda(Z)\big)$$

Proof. Assume for simplicity that restructuring with new creditors is optimal and that $q > 0$. It follows that

$$Z_{\mathbf{b}u} \equiv \max\{Z : \mathscr{O}(v^\Lambda) > 0\} < Z_d\,.$$

is well-defined and is smaller than or equal to Z_d. Furthermore, by continuity, we have

$$\mathscr{O}(v^\Lambda)(Z_{\mathbf{b}u}) = 0 \;\Leftrightarrow\; v^\Lambda(Z_{\mathbf{b}u}) = \max_{y\geq Z_{\mathbf{b}u}}(v^\Lambda(y) - qd(y))\,.$$

Now consider the higher threshold defined by

$$Z_{\mathbf{b}o} \equiv \min\{y \geq Z_{\mathbf{b}u} :\; v^\Lambda(Z_{\mathbf{b}u}) = v^\Lambda(y) - qd(y)\}\,.$$

By the same argument as in the proof of Lemma A.9, we have $v^\Lambda(\phi) < v^\Lambda(Z_{\mathbf{b}u}) = v^\Lambda(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d) < v^\Lambda(Z_{\mathbf{b}o})$ and therefore $Z_{\mathbf{b}u} < Z_{\mathbf{b}o} < Z_d$ since issuance costs are strictly positive. This in turn implies that the point $Z_{\mathbf{b}o}$ is a local maximum of the function $v^\Lambda(y) - qd(y)$.

To complete the proof, we need to show that for $Z \leq Z_{\mathbf{b}u}$, we have $v^\Lambda(Z) \leq v^\Lambda(Z_{\mathbf{b}u})$. Indeed, in that case,

$$\max_{y\geq Z}(v^\Lambda(y) - qd(y)) \;\leq\; \max_{y\geq Z} v^\Lambda(y) \;\leq\; v^\Lambda(Z_{\mathbf{b}u}) = v^\Lambda(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o})$$

and, consequently,

$$\mathscr{O}(v^\Lambda)(Z) = 1_{\{Z \le Z_{\mathbf{b}u}\}} \left(v^\Lambda(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d) - v^\Lambda(Z) \right) .$$

Suppose that this is not the case. Let

$$Z_v = \max\{Z \le Z_{\mathbf{b}u} : v^\Lambda(Z) = v^\Lambda(Z_{\mathbf{b}u}) = v(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o})\}$$

and assume for simplicity that $Z_v < Z_{\mathbf{b}u}$ so that the function $v^\Lambda(Z)$ reaches a local minimum at some point $Z_m \in [Z_v, Z_{\mathbf{b}u}]$.[2] As a first step towards a contradiction we claim that the function $v^\Lambda(Z)$ is monotone decreasing on $[0, Z_v]$. If not then as illustrated by the right panel of Figure A there is a point $Z_n \in [0, Z_v]$ at which the function $v^\Lambda(Z)$ achieves a local maximum such that

$$v^\Lambda(Z_n) > v^\Lambda(Z_v) = v^\Lambda(Z_{\mathbf{b}u}) = v^\Lambda(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}) \;=\; \max_{y \ge Z_v}(v^\Lambda(y) - qd(y)) . \tag{32}$$

Furthermore, by the definition of Z_n, $v^\Lambda(Z_n)$ is monotone decreasing on $[Z_n, Z_v]$ and hence, for all $y \in [Z_n, Z_v]$, we have

$$v^\Lambda(Z_n) \ge v^\Lambda(y) \ge v^\Lambda(y) - qd(y) . \tag{33}$$

Combining (32) and (33), we get

$$v^\Lambda(Z_n) \;\ge\; \max_{y \ge Z_n}(v^\Lambda(y) - qd(y)) .$$

This immediately implies that we have $\mathscr{O}(v^\Lambda)(Z_n) = 0$. Furthermore, by definition,

$$0 \le \; \mathscr{O}^\Lambda(v^\Lambda) \;\le\; \mathscr{O}(v^\Lambda)$$

[2]When the point $Z_{\mathbf{b}u}$ is a local minimum of the function $v^\Lambda(Z)$ we have $Z_v = Z_{\mathbf{b}u}$. This case is completely analogous, up to small modifications.

and hence $\mathscr{O}^{\Lambda}(v^{\Lambda})(Z_n) = 0$. Therefore,

$$
\begin{aligned}
(r-\mu)v^{\Lambda}(Z_m) &= \mathscr{L}v^{\Lambda}(Z_m) + 1 \\
&\quad - \tau(1-Z_m) + \lambda(1-\eta)\mathscr{O}(v)(Z_m) + \Lambda(1-\theta)\mathscr{O}(v^{\Lambda})(Z_m) \\
&\geq \mathscr{L}v^{\Lambda}(Z_m) + 1 - \tau(1-Z_m) \\
&\geq 1 - \tau(1-Z_m) > 1 - \tau(1-Z_n) \\
&\geq \mathscr{L}v^{\Lambda}(Z_n) + 1 - \tau(1-Z_n) \\
&= \mathscr{L}v^{\Lambda}(Z_n) + 1 - \tau(1-Z_n) = (r-\mu)v^{\Lambda}(Z_n)
\end{aligned}
$$

which contradicts equation (32) and establishes our claim regarding the monotonicity of the function $v^{\Lambda}(Z)$ in the interval $[0, Z_v]$. Therefore, (32) and the same argument as in (33) implies that this property with the fact that $\mathscr{O}(v)(Z) = 0$ for all $Z \leq Z_v$. Consequently,

$$
(r-\mu)v^{\Lambda}(Z) = \mathscr{L}v^{\Lambda}(Z) + 1 - \tau(1-Z)
$$

on $[0, Z_v]$ and therefore

$$
v^{\Lambda}(Z) \;=\; \frac{1-\tau}{r-\mu} + \frac{\tau Z}{r} + a_1 Z^{1-\beta} + a_2 Z^{1-\alpha}
$$

for some $a_1, a_2 \in \mathbb{R}$. Since v^{Λ} is bounded, we have $a_2 = 0$ and therefore $v^{\Lambda}(0) = \phi_0$. Since $v^{\Lambda}(Z)$ is decreasing on $[0, Z_v]$, we immediately get that $\phi_0 > v^{\Lambda}(Z)$ on that interval. But this is impossible since

$$
v^{\Lambda}(Z) \geq \hat{v}^{\Lambda}(Z) \geq \phi_0
$$

in a right neighborhood of zero by Lemma A.9. ■

Lemma C.6 *There is a threshold $\bar{Z}_{\mathbf{b}u}(\Lambda) < Z_{\mathbf{b}u}(\Lambda)$ such that*

$$
\mathscr{O}_{\mathbf{b}}(v^{\Lambda})(Z) = 1_{\{Z \leq \bar{Z}_{\mathbf{b}u}\}} \left(v^{\Lambda}(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d) - v^{\Lambda}(Z) - \nu(Z) \right)
$$

Proof. Since $\mathscr{O}_{\mathbf{b}}(v^\Lambda)(Z) < \mathscr{O}(v^\Lambda)(Z)$, we have that the threshold

$$\bar{Z}_{\mathbf{b}u} \equiv \sup\{Z > 0 : \mathscr{O}_{\mathbf{b}}(v^\Lambda)(Z) > 0\}$$

is well defined and satisfies $\bar{Z}_{\mathbf{b}u} < Z_{\mathbf{b}u}$. By continuity, we have

$$v^\Lambda(\bar{Z}_{\mathbf{b}u}) + \nu(\bar{Z}_{\mathbf{b}u}) \;=\; v^\Lambda(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d)\,.$$

Suppose that the claim of the statement is not true. Then, there exists a $Z_v < \bar{Z}_{\mathbf{b}u}$ such that $v^\Lambda(Z_v) + \nu(Z_v) = v^\Lambda(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d)$. Let us show that $W(Z) \equiv v^\Lambda(Z) + \nu(Z)$ is monotone decreasing on $[0, Z_v]$. Indeed, suppose the contrary. Let Z_n be the largest local maximum of $W(Z)$ on $[0, Z_v]$. Let also

$$\lambda^* \equiv \lambda(1-\eta), \Lambda^* \equiv \Lambda(1-\theta)\,.$$

Then, defining

$$\zeta(Z) \equiv \mathscr{L}\nu(Z) - (r - \mu + \lambda^*)\nu(Z),$$

we have

$$(r - \mu + \lambda^*)W(Z) = -\zeta(Z) + \mathscr{L}W(Z) + 1 - \tau + \tau Z + \lambda^*(\mathscr{O}(v^\Lambda) + v^\Lambda) + \Lambda^*\mathscr{O}_{\mathbf{b}}(v^\Lambda).$$

Since $W(Z_v) = W(\bar{Z}_{\mathbf{b}u}) > W(Z)$ for all $Z \in (Z_v, \bar{Z}_{\mathbf{b}u})$, $W(Z)$ also has the largest local minimum at some $Z_m \in (Z_v, \bar{Z}_{\mathbf{b}u})$. Therefore, using the fact that, by assumption, $\tau Z - \zeta(Z)$

is monotone increasing, we get

$$\begin{aligned}(r-\mu+\lambda^*)W(Z_m) &= \mathscr{L}W(Z_m)-\zeta(Z_m)+1-\tau(1-Z_m)\\ &\quad + \lambda^*(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d))+\Lambda^*\mathscr{O}^\Lambda(v^\Lambda)(Z_m)\\ &\geq -\zeta(Z_m)+1-\tau(1-Z_m)+\lambda^*(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d))+\Lambda^*\mathscr{O}^\Lambda(v^\Lambda)(Z_m)\\ &\geq -\zeta(Z_m)+1-\tau(1-Z_m)+\lambda^*(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d))\\ &> -\zeta(Z_n)+1-\tau(1-Z_n)+\lambda^*(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d))\\ &\geq \mathscr{L}W(Z_n)-\zeta(Z_n)+1-\tau(1-Z_n)+\lambda^*(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d))\\ &= (r-\mu+\lambda^*)W(Z_n)\end{aligned}$$

which is a contradiction. Thus, it has to be that $W(Z)$ is monotone decreasing on $[0, Z_v]$ and still has a local minimum at Z_m, so that

$$W(0) \geq W(Z_m) \geq \frac{-\zeta(Z_m)+(1-\tau+\tau Z_m)+\lambda^*\left(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d)\right)}{r-\mu+\lambda^*}.$$

Since $\mathscr{O}_{\mathbf{b}}(v^\Lambda)=0$ for $Z \leq Z_v$, we have

$$\frac{1}{2}\sigma^2 Z^2\, v^\Lambda_{ZZ}(Z)-\mu Z_n v^\Lambda_Z(Z)+(1-\tau+\tau Z)+\lambda^*\left(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d)\right)=(r-\mu+\lambda^*)v^\Lambda(Z)$$

in that interval. A direct calculation implies that

$$v^\Lambda(0)=\frac{1-\tau+\lambda^*(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d))}{r-\mu+\lambda^*}$$

Therefore,

$$\begin{aligned}W(0) &= \nu(0)+\frac{1-\tau+\lambda^*(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d))}{r-\mu+\lambda^*}\\ &\leq \frac{-\zeta(Z_m)+(1-\tau+\tau Z_m)+\lambda^*\left(v^\Lambda(Z_{\mathbf{b}o})-qd(Z_{\mathbf{b}o}|Z_d)\right)}{r-\mu+\lambda^*} \leq W(Z_m)\end{aligned}$$

because

$$(r-\mu+\lambda^*)\,\nu(0)=\tau\ \cdot 0-\zeta(0)\leq \tau Z_m-\zeta(Z_m)$$

since, by assumption, $\tau Z - \zeta(Z)$ is increasing. This is a contradiction, and the claim follows. ■

Lemma C.7 *As $\Lambda \to \infty$ the thresholds $\bar{Z}_{\mathbf{b}u}$, $Z_{\mathbf{b}u}$, $Z_{\mathbf{b}o}$ converge to some finite limits and $v^{\Lambda}(Z)$ converges uniformly to a function $v_{\mathbf{b}}(Z|Z_d)$ which satisfies (C.1). In particular, the optimal stopping time is the first time that the state variable is enter the interval $[0, \bar{Z}_{\mathbf{b}u}]$.*

Proof. To prove this result we will show that $v^{\Lambda}(Z)$ converges to a function $v^{\mathbf{b}}(Z)$ that satisfies the conditions of Lemma C.2. Let us first discuss convergence. Since the interval of interest $[0, Z_d]$ is compact, we can always pick a subsequence Λ_n such that

$$(\bar{Z}_{\mathbf{b}u}(\Lambda_n), Z_{\mathbf{b}u}(\Lambda_n), Z_{\mathbf{b}o}(\Lambda_n)) \longrightarrow (\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o})$$

for some constants

$$\bar{Z}_{\mathbf{b}u} \leq Z_{\mathbf{b}u} \leq Z_{\mathbf{b}o} \leq Z_d.$$

The same arguments as in the proof of Lemma A.5 imply that $v^{\Lambda}(Z)$ is increasing as a function of Λ. In particular, $\max_{y \geq 0}(v^{\Lambda}(y) - qd(y|Z_d))$ is increasing in Λ and it follows that $Z_{\mathbf{b}o}(\Lambda_n)$ cannot converge to the exogenously fixed default threshold Z_d.

The fact that the function $v^{\Lambda}(Z)$ converges on the interval $[\bar{Z}_{\mathbf{b}u}, Z_d]$ to a limit $v_{\mathbf{b}}(Z|Z_d)$ that solves the same equation as the function $v(Z|Z_d)$ follows directly from Lemma A.6 and the fact that the function $v^{\Lambda}(Z)$ is increasing in Λ and bounded from above. A direct calculation based on Lemma A.6 implies that only the function but also its derivative converges. On the interval $[0, \bar{Z}_{\mathbf{b}u}]$ we define the limiting function by

$$v_{\mathbf{b}}(Z) = \lim_{n \to \infty} (v^{\Lambda_n}(Z_{\mathbf{b}o}(\Lambda_n)) - qd(Z_{\mathbf{b}o}(\Lambda_n)|Z_d)) - \nu(Z)\,.$$

By definition of the threshold $\bar{Z}_{\mathbf{b}u}$ we have that $v_{\mathbf{b}}(Z)$ is continuous at the point $\bar{Z}_{\mathbf{b}u}$ and to complete the proof we will now sequentially verify that the limiting functions satisfies the conditions of Lemma C.2 with

$$\xi(Z) = 1 - \tau(1 - Z) + \lambda^*(v_{\mathbf{b}}(Z) + \mathscr{O}(v_{\mathbf{b}})(Z))$$

and

$$\varphi(Z) = \max_{a\geq 1}(v_{\mathbf{b}}(aZ) - qd(aZ|Z_d)1_{\{a>1\}} - \nu(Z)) = v_{\mathbf{b}}(Z) + \mathscr{O}_{\mathbf{b}}(v_{\mathbf{b}})(Z)$$

To prove that the limiting function is C^1 consider the function

$$W(Z) = W^{\Lambda}(Z) \equiv v^{\Lambda}(Z) + \nu(Z)$$

and observe that since the derivative of $v^{\Lambda}(Z)$ converges to that of $v_{\mathbf{b}}(Z)$ for $Z \geq \bar{Z}_{\mathbf{b}u}$ by the above it suffices to prove that

$$\lim_{\Lambda\to\infty} W_Z^{\Lambda}(\bar{Z}_{\mathbf{b}u}(\Lambda)) = 0. \tag{34}$$

By definition of the restructuring threshold $\bar{Z}_{\mathbf{b}u}(\Lambda)$ we have

$$\begin{aligned}\mathscr{O}_{\mathbf{b}}(v^{\Lambda})(Z) = \mathscr{O}(v^{\Lambda})(Z) - \nu(Z) &= v(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d) - v^{\Lambda}(Z) - \nu(Z)\\ &= W(\bar{Z}_{\mathbf{b}u}(\Lambda)) - W(Z)\end{aligned}$$

for all $Z \leq \bar{Z}_{\mathbf{b}u}(\Lambda)$ and it now follows from Lemma C.4 that over this region the function solves the ordinary differential equation

$$(r - \mu + \Lambda^*)W(Z) = \mathscr{L}W(Z) - \vartheta(Z) + 1 - \tau + \Lambda^* W(\bar{Z}_{\mathbf{b}u}(\Lambda))$$

where

$$\vartheta(Z) = \mathscr{L}\nu(Z) - (r - \mu + \lambda^*)\nu(Z) - \tau Z$$

is a decreasing function by Assumption 1 and $\Lambda^* = \lambda^* + \Lambda(1-\theta)$. Define $\gamma < 0 < 1 < \gamma_1$ to be the solutions to $Q(x, r + \Lambda^*) = 0$. With this notation it follows from a slight modification of Lemma A.6 that the solution is explicitly given by

$$W(Z) = W(0) + y_1 Z^{1-\gamma} + y_2 Z^{1-\gamma_1} + \frac{2}{\sigma^2 Z}\int_Z^{\bar{Z}_{\mathbf{b}u}(\Lambda)} \left[\left(\frac{x}{Z}\right)^{\gamma-2} - \left(\frac{x}{Z}\right)^{\gamma_1-2}\right]\frac{\vartheta(x)dx}{\gamma_1-\gamma}$$

for some constants y_1 and y_2 where

$$W(0) = \frac{1-\tau-\vartheta(0)+\Lambda^* W(\bar{Z}_{\mathbf{b}u}(\Lambda))}{r-\mu+\Lambda^*}.$$

Let us first determine the constant y_2 using the fact the function is bounded at the origin. Since the function $\vartheta(Z)$ is decreasing we have

$$\left|\frac{2}{\sigma^2 Z}\int_Z^{\bar{Z}_{\mathbf{b}u}(\Lambda)}\left(\frac{x}{Z}\right)^{\gamma-2}\frac{\vartheta(x)dx}{\gamma_1-\gamma}\right| \leq \frac{K_0}{Z}\int_Z^{\bar{Z}_{\mathbf{b}u}(\Lambda)}\left(\frac{x}{Z}\right)^{\gamma-2}dx \leq K_1$$

for some constants K_0 and $K_1 > 0$. Thus, we only need to take care of the terms with negative exponent and it follows that

$$y_2 = \frac{2}{\sigma^2}\int_0^{\bar{Z}_{\mathbf{b}u}(\Lambda)}\frac{x^{\gamma_1-2}\vartheta(x)dx}{\gamma_1-\gamma}$$

where the integral does not explode at $x = 0$ because by definition $\gamma_1 > 1$. Using this constant we can rewrite the solution as

$$W(Z) = W(0) + y_1 Z^{1-\gamma} + \frac{2}{\sigma^2 Z}\left[\int_0^Z \left(\frac{x}{Z}\right)^{\gamma_1-2}\frac{\vartheta(x)dx}{\gamma_1-\gamma} + \int_Z^{\bar{Z}_{\mathbf{b}u}(\Lambda)}\left(\frac{x}{Z}\right)^{\gamma-2}\frac{\vartheta(x)dx}{\gamma_1-\gamma}\right]. \quad (35)$$

and the remaining constant is now determined by requiring that the solution be continuous at the upper boundary point:

$$W(\bar{Z}_{\mathbf{b}u}(\Lambda)) = W(0) + y_1 \bar{Z}_{\mathbf{b}u}(\Lambda)^{1-\gamma} + \frac{2}{\sigma^2 \bar{Z}_{\mathbf{b}u}(\Lambda)}\int_0^{\bar{Z}_{\mathbf{b}u}(\Lambda)}\left(\frac{x}{\bar{Z}_{\mathbf{b}u}(\Lambda)}\right)^{\gamma_1-2}\frac{\vartheta(x)dx}{\gamma_1-\gamma}.$$

Solving this equation, substituting the solution into (35) and differentiating the resulting expression at the upper boundary point gives

$$W_Z(\bar{Z}_{\mathbf{b}u}(\Lambda)) = (1-\gamma)\frac{W(\bar{Z}_{\mathbf{b}u}(\Lambda)) - W(0)}{\bar{Z}_{\mathbf{b}u}(\Lambda)} - \frac{2}{(\sigma \bar{Z}_{\mathbf{b}u}(\Lambda))^2}\int_0^{\bar{Z}_{\mathbf{b}u}(\Lambda)}\left(\frac{x}{\bar{Z}_{\mathbf{b}u}(\Lambda)}\right)^{\gamma_1-2}\vartheta(x)dx.$$

A direct calculation shows that the constants $(1-\gamma)/(r-\mu+\Lambda^*)$ converge to zero as Λ^* goes to infinity, Therefore, since $W(Z) = W^\Lambda(Z)$ is bounded as a function of Λ and the restructuring threshold converges to a finite number we obtain

$$\lim_{\Lambda\to\infty}(1-\gamma)\frac{W(\bar{Z}_{\mathbf{b}u}(\Lambda))-W(0)}{\bar{Z}_{\mathbf{b}u}(\Lambda)}=\lim_{\Lambda\to\infty}\frac{(1-\gamma)(\tau-1+\vartheta(0)+(r-\mu)W(\bar{Z}_{\mathbf{b}u}(\Lambda)))}{(r-\mu+\Lambda^*)\bar{Z}_{\mathbf{b}u}(\Lambda)}=0.$$

On the other hand, since γ_1 diverges to infinity as Λ increases we have that $(x/Z)^{\gamma_1-2}$ converges to zero for all $x<Z$ and it now follows from the dominated convergence theorem that

$$\lim_{\Lambda\to\infty}\frac{2}{(\sigma\bar{Z}_{\mathbf{b}u}(\Lambda))^2}\int_0^{\bar{Z}_{\mathbf{b}u}(\Lambda)}\left(\frac{x}{\bar{Z}_{\mathbf{b}u}(\Lambda)}\right)^{\gamma_1-2}\vartheta(x)dx=0.$$

This shows that (34) holds and completes the verification of condition 1. The validity of conditions 2 and 3 follows directly from the above arguments. To establish the validity of condition 4 we need to show that the quantity

$$C(Z)=\mathscr{L}\varphi(Z)-(r-\mu+\lambda^*)\varphi(Z)+\xi(Z)=1-\tau-\vartheta(Z)-(r-\mu)W(\bar{Z}_{\mathbf{b}u})$$

is non positive for all $Z\leq\bar{Z}_{\mathbf{b}u}$ and since $\vartheta(Z)$ is decreasing it suffices to check that this property holds at the upper boundary point. The above result implies that $W'(\bar{Z}_{\mathbf{b}u})=0$ and since the function $W(Z)$ cannot be decreasing to the left of $\bar{Z}_{\mathbf{b}u}$ we have

$$\mathscr{L}W(\bar{Z}_{\mathbf{b}u})=\frac{1}{2}(\bar{Z}_{\mathbf{b}u})^2W''(\bar{Z}_{\mathbf{b}u})\geq 0$$

Combining this with the definition of the function $W(Z)$ and the fact that

$$(r-\mu)v_{\mathbf{b}}(\bar{Z}_{\mathbf{b}u})=\mathscr{L}v_{\mathbf{b}}(\bar{Z}_{\mathbf{b}u})+1-\tau(1-\bar{Z}_{\mathbf{b}u})+\lambda^*\mathscr{O}(v_{\mathbf{b}})(\bar{Z}_{\mathbf{b}u})$$

then gives

$$\begin{aligned}C(\bar{Z}_{\mathbf{b}u})&=1-\tau(1-\bar{Z}_{\mathbf{b}u})-\vartheta(\bar{Z}_{\mathbf{b}u})-(r-\mu)W(\bar{Z}_{\mathbf{b}u})\\&\leq 1-\tau(1-\bar{Z}_{\mathbf{b}u})-\vartheta(\bar{Z}_{\mathbf{b}u})-(r-\mu)W(\bar{Z}_{\mathbf{b}u})+\mathscr{L}W(\bar{Z}_{\mathbf{b}u})=0\end{aligned}$$

and completes the proof. ■

Recall that the functions v_{ns} and v_s are defined by

$$\begin{aligned}
v_{ns}(Z) &= v_{ns}(Z, a_3, a_4) \equiv a_3 Z^{1-\beta} + a_4 Z^{1-\alpha} + \phi_0 + \frac{\tau Z}{r} \\
v_s(Z) &= v_s(Z, Z_{\mathbf{b}o}, a_1, a_2, a_3, a_4; q) \\
&\equiv a_1 Z^{1-\psi} + a_2 Z^{1-\psi_1} + \frac{\tau Z}{r} + \frac{1 - \tau + \lambda^*(v_{ns}(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d))}{r - \mu + \lambda^*}.
\end{aligned}$$

Solving

$$v_s(Z_{\mathbf{b}u}) - v_{ns}(Z_{\mathbf{b}u}) = v_s'(Z_{\mathbf{b}u}) - v_{ns}'(Z_{\mathbf{b}u}) = 0$$

for a_1, a_2 gives

$$a_1 \;=\; A_1(Z_{\mathbf{b}u}, a_3, a_4; q) \;,\; a_2 \;=\; A_2(Z_{\mathbf{b}u}, a_3, a_4; q)\,.$$

The following lemma establishes the local uniqueness of the Markov perfect equilibrium in barrier strategies and constitutes the direct counterpart of Lemma B.2 for the model in which the firm can issue debt to inside creditors.

Lemma C.8 *Let $\nu(Z) = \epsilon Z$. Consider the following system*

$$\begin{aligned}
&F_{1\mathbf{b}}(\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o}, Z_{\mathbf{b}d}, a_3, a_4; q) \equiv v_{ns}(Z_d) - d(Z_d|Z_d) = 0 \\
&F_{2\mathbf{b}}(\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o}, Z_{\mathbf{b}d}, a_3, a_4; q) \equiv v_{ns}'(Z_d) - d'(Z_d|Z_d) = 0 \\
&F_{3\mathbf{b}}(\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o}, Z_{\mathbf{b}d}, a_3, a_4; q) \equiv v_{ns}'(Z_{\mathbf{b}o}) - qd'(Z_{\mathbf{b}o}|Z_d) = 0 \\
&F_{4\mathbf{b}}(\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o}, Z_{\mathbf{b}d}, a_3, a_4; q) \equiv v_{ns}(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d) - v_{ns}(Z_{\mathbf{b}u}) = 0 \\
&F_{7\mathbf{b}}(\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o}, Z_{\mathbf{b}d}, a_3, a_4; q) \equiv v_s(\bar{Z}_{\mathbf{b}u}) + \nu(\bar{Z}_{\mathbf{b}u}) - (v_{ns}(Z_{\mathbf{b}o}) - qd(Z_{\mathbf{b}o}|Z_d)) = 0 \\
&F_{8\mathbf{b}}(\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o}, Z_{\mathbf{b}d}, a_1, a_2, a_3, a_4; q) \equiv v_s'(\bar{Z}_{\mathbf{b}u}) + \nu'(\bar{Z}_{\mathbf{b}u}) = 0.
\end{aligned}$$

Denote by $J(\mathscr{C})$ the unique solution to (3.3), define $z_o(\mathscr{C})$ and $z_d(\mathscr{C})$ by (25), (26) and let

$$\begin{aligned}
\tilde{a}_3(\mathscr{C}) &= (\tau/r)\kappa_1(\infty) z_o(\mathscr{C})^{\beta}, \\
\tilde{a}_4(\mathscr{C}) &= (\tau/r)\kappa_2(\infty) z_o(\mathscr{C})^{\alpha}.
\end{aligned}$$

Suppose that there exists an admissible $\mathscr{C}$ such that

$$\mathscr{J} F_{\mathbf{b}}(z_o(\mathscr{C}), z_o(\mathscr{C}), z_o(\mathscr{C}), z_d(\mathscr{C}), \tilde{a}_3(\mathscr{C}), \tilde{a}_4(\mathscr{C}); 0) \neq 0\,.$$

where $\mathscr{J}$ denotes the Jacobian operator. Then, for Lebesque almost every admissible $\mathscr{C}$ there exists an open neighborhood

$$\mathscr{B}_{\mathbf{b}} \supseteq (z_o(\mathscr{C}), z_o(\mathscr{C}), z_o(\mathscr{C}), z_d(\mathscr{C}), \tilde{a}_1(\mathscr{C}), \tilde{a}_2(\mathscr{C}), \tilde{a}_3(\mathscr{C}), \tilde{a}_4(\mathscr{C}))$$

and a $\delta > 0$ such that, for all $q, \epsilon \in [0, \delta)$, there exists a unique Markov perfect equilibrium in barrier strategies whose parameters satisfy $(\bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}u}, Z_{\mathbf{b}o}, Z_{\mathbf{b}d}, a_3, a_4) \in \mathscr{B}_{\mathbf{b}}$.

Lemma C.9 *For a fixed default threshold the equity value function is decreasing in q and ϵ. As a result, (8) has a solution for any $q, \epsilon > 0$ whenever it has a solution for $q = \epsilon = 0$.*

Proof. Monotonicity of the equity value function for each finite Λ is proved by the same argument as in the proof of Lemma A.5. Then, Lemma C.7 implies the required monotonicity for the true equity value. By the above, for $\tau < \tau^*(\infty)$, equity value satisfies

$$\frac{\partial}{\partial X} E_{\mathbf{b}}(X, C|\mathbf{P}(X_d(1)))\Big|_{q=\epsilon=0, X=X_d(C)} < 0,$$

for any constant $X_d(1) < X^*_{d,0}(1)$. Thus, for any $X_d(1) < X^*_{d,0}(1)$, equity value is negative in a right neighborhood of $X_d(C)$ when $q = 0$. Since equity value is monotone decreasing in the issuance cost q, ϵ, it follows that

$$E_{\mathbf{b}}(X, C|\mathbf{P}(X_d(1))) \leq 0$$

for any $q > 0$ in a right neighborhood of $X_d(C)$. This in turn implies that for any $X_d(1) < X^*_{d,0}(1)$ and $q, \epsilon > 0$, we have

$$\frac{\partial}{\partial X} E_{\mathbf{b}}(X, C|\mathbf{P}(X_d(1)))\Big|_{X=X_d(C)} < 0.$$

This fact, together with condition (6) and the intermediate value theorem, implies that the smooth pasting condition has a solution in $\big(X^*_{d,0}(C), X^S_d(C)\big)$ for any $q, \epsilon > 0$ when

$\tau < \tau^*(\infty)$. Furthermore, standard implicit function type arguments imply that this solution is unique when q is sufficiently small. ■

Proof of Theorem 2. The proof of Theorem 2 is analogous to that of Theorem 1 and follows directly from Proposition 9, Lemma C.8 and Lemma C.9. ■

Proof of Proposition 10. The proof is completely analogous to that of Proposition 8. Indeed, in the limit when $q, \epsilon \to 0$, firm value converges to that in the search model with an infinite intensity, and all three thresholds $Z_{\mathbf{b}u}, \bar{Z}_{\mathbf{b}u}, Z_{\mathbf{b}o}$ collapse to one. Monotonicity of default threshold with respect to η, λ follows directly from the proof of Lemma C.9, because, for fixed default threshold, the value of the firm is decreasing in η and increasing in λ. ■

D Restructuring probabilities

In order to compute the restructuring probabilities associated with the Markov perfect equilibria in the three models let

$$\tau(y) \equiv \inf\{t \geq 0 : X_t = y\}$$

denote the first time that the cash flow process reaches $y \geq 0$ and define a nonnegative bounded function by setting

$$F(x, T; y, z) \equiv P\left[\tau(z) \leq T \wedge \tau(y) | X_0 = x\right]$$

The probability of restructuring before time T is therefore given by $F(x, T; X_{d0}(1), X_{u0}(1))$ for the frictionless model, and by $F(x, T; X_{d\mathbf{b}}(1), X_{u\mathbf{b}}(1))$ for the model in which the firm bargains with current creditors. The following lemma provides an expression for the function F which can be easily approximated numerically.

Lemma D.1 *For $0 < y < z$ and $x \in (y, z)$ we have that*

$$\begin{aligned} F(x, T; z, y) = &\sum_{n=0}^{\infty} \left[\Phi\left(\frac{b_n - \nu T}{\sqrt{T}}\right) - e^{2\nu b_n} \Phi\left(\frac{-b_n - \nu T}{\sqrt{T}}\right)\right] \\ - &\sum_{n=0}^{\infty} \left[\Phi\left(\frac{a_n - \nu T}{\sqrt{T}}\right) - e^{2\nu a_n} \Phi\left(\frac{-a_n - \nu T}{\sqrt{T}}\right)\right] \end{aligned}$$

where the sequence $(a_n, b_n)_{n\geq 1}$ *is defined by*

$$a_n = a_n(x,y,z) \equiv \log(z/x)^{\frac{1}{\sigma}} + \log(z/y)^{\frac{2n}{\sigma}},$$
$$b_n = b_n(x,y,z) \equiv a_n + \log(x/y)^{\frac{2}{\sigma}} = \log(zx/y^2)^{\frac{1}{\sigma}} + \log(z/y)^{\frac{2n}{\sigma}},$$

the function $\Phi : \mathbb{R} \to (0,1)$ *is the cumulative distribution function of a standard Gaussian random variable, and we have set* $\nu \equiv m/\sigma - \sigma/2$.

Proof. See Borodin and Salminen (2002). ■

In the search model, restructuring occurs the first time that the firm meets creditors while the cash flow shock is above the search boundary $X_u^*(1)$. Therefore, it follows from standard results on Poisson point processes (see e.g. Brémaud (1981)) that the associated probability of restructuring can be computed as

$$1 - G(x, T; X_d^*(1), X_u^*(1))$$

where

$$G(x,T;y,z) \equiv E\left[e^{-\lambda \int_0^{T\wedge\tau(y)} 1_{\{X_s\geq z\}} ds} \middle| X_0 = x \right].$$

In order to derive a numerical approximation for this function we start by computing its Laplace transform with respect to the time parameter. To facilitate the presentation let $\Theta = \Theta(q) < 0$, and $\Psi = \Psi(q) \geq 0$ denote the roots of $Q(x;q) = 0$ where the function Q is defined as in the main text.

Lemma D.2 *For* $0 < y < z$ *and* $x \geq y$ *we have that the Laplace transform*

$$\hat{G}(x,\phi;y,z) \equiv \int_0^\infty e^{-\phi t} G(x,t;y,z)dt.$$

is explicitly given by

$$\hat{G}(x,\phi;y,z) = 1_{\{y\leq x\leq z\}} \hat{G}_b(x,\phi;y,z) + 1_{\{x\geq z\}} \hat{G}_a(x,\phi;y,z)$$

where the functions $\hat{G}_a$ and $\hat{G}_b$ are defined by

$$\hat{G}_a(x,\phi;y,z) \equiv \frac{1}{\phi+\lambda}\left[1+(x/z)^{\Theta(\phi+\lambda)}\,\frac{\lambda}{\phi}\frac{A(\phi;y,z)}{B(\phi;y,z)}\right],$$

$$\hat{G}_b(x,\phi;y,z) \equiv \frac{1}{\phi}\left[1+\frac{\lambda}{\phi+\lambda}\frac{\Theta(\phi+\lambda)}{B(\phi;y,z)}\left(x^{\Psi(\phi)}y^{\Theta(\phi)}-x^{\Theta(\phi)}y^{\Psi(\phi)}\right)\right],$$

with

$$A(\phi;y,z) \equiv z^{\Psi(\phi)}y^{\Theta(\phi)}\Psi(\phi)-z^{\Theta(\phi)}y^{\Psi(\phi)}\Theta(\phi),$$

$$B(\phi;y,z) \equiv z^{\Theta(\phi)}y^{\Psi(\phi)}\left(\Theta(\phi+\lambda)-\Theta(\phi)\right)+z^{\Psi(\phi)}y^{\Theta(\phi)}\left(\Psi(\phi)-\Theta(\phi+\lambda)\right).$$

Proof. Using the boundedness of the function $G(x,t;y,z)$ together with an application of Fubini's theorem we deduce that

$$\hat{G}(x,\phi;y,z) = E\left[\int_0^\infty e^{-\int_0^{t\wedge\tau(y)}(\phi+\lambda(X_s;z))ds}dt\,\middle|\,X_0=x\right].$$

where we have set

$$\lambda(x;z) = \lambda 1_{\{x\geq z\}}.$$

Therefore, it follows from Theorem 4.9 in Karatzas and Shreve (1991) that $\hat{G}(x)\equiv\hat{G}(x,\phi;y,z)$ is the unique bounded and piecewise C^2 solution to

$$mx\hat{G}'(x)+\frac{1}{2}\sigma^2x^2\hat{G}''(x)+1 = (\phi+\lambda(x;z))\hat{G}(x), \qquad x>y,$$

subject to the boundary condition

$$\lim_{x\downarrow y}\hat{G}(x) = 1/\phi.$$

The general solution to this second order ODE is given by

$$\hat{G}(x) = 1_{\{y\leq x\leq z\}}\hat{G}_b(x)+1_{\{x\geq z\}}\hat{G}_a(x)$$

where

$$\hat{G}_b(x) \equiv 1/\phi + C_1 x^{\Theta(\phi)} + C_2 x^{\Psi(\phi)},$$
$$\hat{G}_a(x) \equiv 1/(\phi+\lambda) + C_3 x^{\Theta(\phi+\lambda)} + C_4 x^{\Psi(\phi+\lambda)}$$

for some constants $(C_i)_{i=1}^4$ to be determined. Since the solution has to remain bounded as the state increases, it must be that $C_4 = 0$. In addition, the boundary condition at $x = y$ and the smoothness of the solution require that

$$\lim_{x \downarrow y} \hat{G}_b(x) = 1/\phi,$$
$$\lim_{x \downarrow z} \hat{G}_b(x) = \lim_{x \uparrow z} \hat{G}_a(z),$$
$$\lim_{x \downarrow z} \hat{G}_b'(z) = \lim_{x \uparrow z} \hat{G}_a'(z).$$

Solving this system of three equations for the remaining constants, plugging the solution into the definition of the functions $(\hat{G}_b, \hat{G}_a)$ and simplifying the result gives the desired result. ■

To obtain the probability of restructuring before a fixed date we need to invert the Laplace transform. Unfortunately, due to the complex dependence of the transformed function on the transform parameter, this cannot be carried out in closed form. To circumvent this difficulty, we follows Abate and Whitt (1995) and approximate the original function as

$$G(x,T) \approx \sum_{k=0}^{m} \binom{m}{k} \frac{e^{A/2}}{2^{1+m}T} \left[\hat{G}\left(x, \frac{A}{2T}\right) + 2 \sum_{\ell=1}^{n+k} (-1)^\ell \Re \hat{G}\left(x, \frac{A}{2T} + \ell \frac{i\pi}{T}\right) \right]$$

where (m, n, A) are constants that control the accuracy of the approximation and we have suppressed the dependence on the thresholds to simplify the notation. In our numerical calculations we use the values

$$m = 11, \quad n = 15, \quad A = 8 \log 10,$$

suggested by Abate and Whitt (1995) to obtain an accuracy of the order of 10^{-8} and verify that the results we obtain are insensitive to that choice.